Complexity of 4-Manifolds

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CONTENTS

We define and study a notion of complexity for smooth, closed and orientable 4-manifolds. This notion, based on the theory of Turaev shadows, represents the 4-dimensional analogue of Matveev's complexity of 3-manifolds. We classify complexity-0 and complexity-1 4-manifolds and provide examples of manifolds of higher complexity.

1. INTRODUCTION

A natural notion of complexity of a PL n-dimensional manifold is the minimal number of highest-dimensional simplices in a triangulation of the manifold. Such a complexity is an integer-valued function and is finite (for each $k \geq 0$ there are only finitely many manifolds whose complexity is less than or equal to k). In order to find all the n-manifolds of complexity k, one has to identify all the possible ways of gluing k copies of the n-simplex such that the link of each point is an $(n-1)$-sphere. Hence, producing lists of low-complexity n-manifolds can be a difficult task if $n \geq 3$ because of the sphere-recognition problem. In dimension 3, S. Matveev [Matveev 90] defined an alternative notion of complexity that for "most" 3-manifolds is equivalent to the one defined above. Matveev's complexity is based on a combinatorial description of 3-manifolds by means of 2-polyhedra (their "spines"), and it turns out to be strictly related to the topological properties of the manifolds: for instance, it is additive under connected sums and is finite when restricted to irreducible manifolds. Its combinatorial nature makes it a computable invariant: using the stratification of the set of 3-manifolds induced by Matveev's complexity it is possible to produce a list of 3-manifolds up to complexity 10 by means of computer-based computations [Martelli and Petronio 04]. Moreover, the techniques and tools [Matveev 03] that have been set up to study the topology of 3-manifolds in order to produce these lists have allowed the creation of computer programs that "recognize" 3-manifolds [Matveev 05].

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On the other side, the existence of "exotic" spaces makes the smooth topology of 4-manifolds an intriguing and still rather mysterious subject. It is our hope that a combinatorial "complexity-based" approach could produce new examples of 4-manifolds sufficiently "simple" to be studied directly. Hence, we define a notion of complexity of 4-manifolds based on the theory of Turaev's shadows [Turaev 94] that represents an analogue in dimension 4 of Matveev's complexity. Roughly speaking, shadows of 4-manifolds can be viewed as simple polyhedra equipped with integer colorings on the regions that can be canonically thickened to smooth (or, equivalently, PL) 4-manifolds. To clarify the reasons why we use shadows instead of triangulations in order to define a complexity of closed 4-manifolds, let us note that a triangulation contains a full description of a handle decomposition of the manifold itself, while it is known that the union of the handles of index up to 2 is sufficient to reconstruct the manifold. Hence, in a sense, the information contained in a triangulation is redundant; in contrast, it can be shown that a shadow of a 4-manifold encodes combinatorially only the union of handles of index up to 2. Moreover, as in the 3-dimensional case with Matveev's complexity, it is straightforward to prove that shadow complexity is subadditive under the connected sum, while the same is not obvious a priori for triangulation-based complexity.

In Section 2 we recall the basic definitions and results on shadows that we will need later; no new results are proved in that section. In Section 3, we introduce two notions of complexity of a closed 4-manifold X: the *complexity*, denoted by $c(X)$, and the *special complexity*, denoted by $c^{\mathrm{sp}}(X)$. The former represents the direct analogue of Matveev's complexity and has the drawback that infinitely many 4-manifolds have complexity 0: this is related to the problem of restricting to irreducible (in a smooth sense!) manifolds; further comments on these aspects will be provided in Section 3. On the other hand, special complexity, obtained by restricting the set of shadows used to encode 4-manifolds, turns out to be finite. In particular we prove the following theorem:

Theorem 1.1. *If a closed, smooth and orientable 4-manifold X has 0 special complexity then X is diffeomorphic to one of the following manifolds: S^4, $\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2$, $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$, $S^2\times S^2$, $\mathbb{CP}^2\#\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2\#\overline{\mathbb{CP}}^2$. Moreover, there are no closed 4-manifolds with special complexity 1.*

It is interesting to stress that because of Freedman's theorem, from the point of view of classification up to homeomorphism, the above statement is not surprising, since low-complexity special polyhedra carry only low-rank homology. Thus the nontrivial content of the above result is given by the fact that it proves that no exotic versions of the complexity-0 manifolds exist in complexity 1. In Section 3.3 we provide examples of higher complexity manifolds and estimate the complexity of the elliptic surfaces $E(n)$. We then use this to provide upper estimates for the answer to the following question:

Question 1.2. What is the minimal (special) complexity of a pair of homeomorphic but nondiffeomorphic 4-manifolds with/without boundary, where the complexity of a pair is defined as the maximum between the complexities of the two manifolds composing the pair?

2. A CRASH COURSE ON SHADOWS OF 4-MANIFOLDS

In this section we recall the basic definition and results about shadows; no new results are proven. For a more detailed account, see [Turaev 94] and [Costantino 05].

2.1 Simple Polyhedra

Definition 2.1. A simple polyhedron P is a 2-dimensional CW complex whose local models are those depicted in Figure 1; the set of points whose neighborhoods have models of the two rightmost types is a 4-valent graph, called the *singular set* of the polyhedron and denoted by $\mathrm{sing}(P)$. The connected components of $P-\mathrm{sing}(P)$ are the *regions* of P. A simple polyhedron whose regions are all disks is called *special*.[1] The *complexity* of a simple polyhedron P, denoted by $c(P)$, is the number of its vertices.

Standard polyhedra can be described in a combinatorial way by decomposing them into the blocks of Figure 1. One can always "build up" a special polyhedron

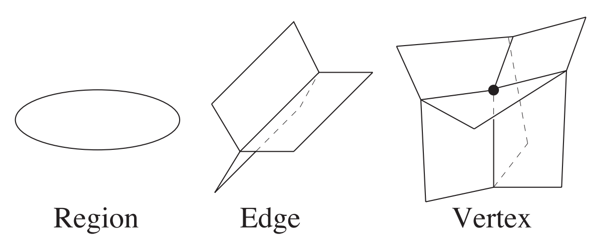

FIGURE 1. The three local models of a simple polyhedron.

[1]According to our definition a polyhedron can be special even if it does not contain any vertices.

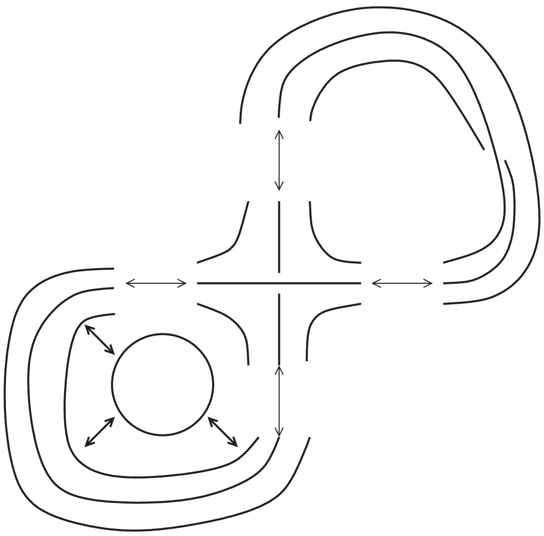

FIGURE 2. How to build up a special polyhedron from its local models.

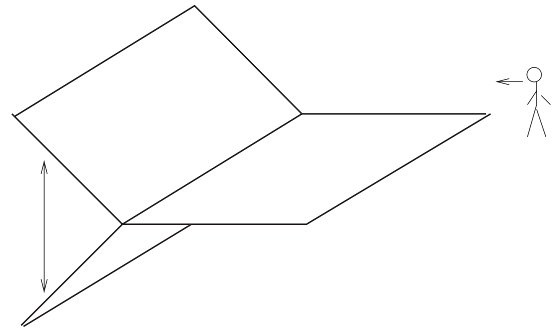

FIGURE 3. The picture sketches the position of the polyhedron in a 3-dimensional slice of the ambient 4-manifold. The direction indicated by the vertical double arrow is that along which the two regions touching the horizontal one get separated.

as exemplified in Figure 2: the central block corresponds to the rightmost block of Figure 1, the curved blocks to the central one of Figure 1, and the regions are disks glued along the boundary of the resulting polyhedron. The resulting diagram unambiguously defines the initial special polyhedron, but different diagrams could encode the same polyhedron. In Figures 7 and 8, we draw all the possible special polyhedra having at most one vertex (in the figures, disks are to be glued along the boundary components of the polyhedra in order to get special polyhedra).

2.2 Decorations on Polyhedra

We describe now the basic decorations we need in order to thicken a special polyhedron P to a 4-manifold. For a more detailed account see [Turaev 94]. Let us denote by $\mathbb{Z}/2$ the group of integer multiples of $\frac{1}{2}$. There are two canonical *colorings* on the regions of P, i.e., assignments of elements of $\mathbb{Z}_2$ or $\mathbb{Z}/2$, the second depending on a flat embedding of P in an oriented 4-manifold:

The $\mathbb{Z}_2$-gleam of P, constructed as follows. Let D be an (open) region of P, and $\overline{D}$ the natural compactification of the surface represented by D. The embedding of D in P extends to a map $i : \overline{D} \to P$ that is injective in $\text{int}(\overline{D})$ and locally injective on $\partial\overline{D}$, and that sends $\partial\overline{D}$ into $\text{sing}(P)$. Using i we can "pull back" a small open neighborhood of D in P and construct a simple polyhedron $N(D)$ collapsing on $\overline{D}$ and such that i extends as a local homeomorphism $i' : N(D) \to P$ whose image is contained in a small neighborhood of the closure of D in P. When i is an embedding of $\overline{D}$ in P, then $N(D)$ turns out to be homeomorphic to a neighborhood of D in P and i' is its embedding in P. In general, $N(D)$ has the following structure: each component of $\partial\overline{D}$ is glued to the core of an annulus or of a Möbius strip, and some small disks are glued along half of their boundary on segments that are properly embedded in these annuli or strips and cut their cores once transversally. We define the $\mathbb{Z}_2$-gleam of D in P as the reduction modulo 2 of the number of Möbius strips used to construct $N(D)$. This coloring depends only on the combinatorial structure of P.

The gleam of P, constructed as follows. Let us now suppose that P is flat in an oriented 4-manifold M, with D, $\overline{D}$, and $i : \overline{D} \to P$ as above. Pulling back a small neighborhood of $i(N(D))$ in M through i, we obtain a 4-dimensional oriented neighborhood B of $N(D)$, over which we fix an auxiliary Riemannian metric. Since $N(D)$ is locally flat in B, $N(D) - D$ defines a line normal to $\overline{D}$ in B along $\partial\overline{D}$ and hence a section of the projectivized normal bundle of $\overline{D}$ (see Figure 3). Let then $\text{gl}(D)$ be equal to $\frac{1}{2}$ times the obstruction to extend this section to the whole of $\overline{D}$; such an obstruction is an element of $H^2(\overline{D}, \partial\overline{D}; \pi_1(S^1))$, which is canonically identified with $\mathbb{Z}$ since B is oriented. Note that the gleam of a region is an integer if and only if its $\mathbb{Z}_2$-gleam is zero.

Using the fact that the $\mathbb{Z}_2$-gleam is always defined, Turaev [Turaev 94] generalized the notion of gleam to nonembedded polyhedra as follows:

Definition 2.2. (Gleam.) A *gleam* on a simple polyhedron P is a coloring on the regions of P with values in $\mathbb{Z}/2$ such that the color of a region is an integer if and only if its $\mathbb{Z}_2$-gleam is zero.

2.3 The Canonical Thickening Procedure

We now describe how any simple polyhedron equipped with gleams (P, gl) can be canonically thickened to a smooth 4-manifold collapsing on it. From now on, all the 4-manifolds will be smooth, compact, and orientable, and all the polyhedra will be flatly embedded unless explicitly stated otherwise. Let P' be the regular neighborhood of $\mathrm{sing}(P)$ in P; when P is special, P' is obtained by puncturing P once along each region. To thicken P, carry out the following steps:

1. Thicken P' to a (possibly nonorientable) 3-manifold L collapsing on it.
2. Thicken L to an oriented 4-manifold H made up of only 0- and 1-handles.
3. Glue suitable blocks to H corresponding to the regions of P.

Step 1. To thicken P' and get L, glue copies of the two rightmost blocks of Figure 4 according to the combinatorics of P'. The result is a pair (L, P') where P' is a properly embedded copy of P' in L.

Step 2. To thicken L and get H, one takes the total space of the determinant fiber bundle of L: for instance if L is orientable, $H = L \times [-1, 1]$. More in general, fix an arbitrary orientation on each of the blocks of Figure 4 and glue their products with $[-1, 1]$ using the attaching maps of L along the blocks $\times\{0\}$ and gluing the fibers by multiplying them by -1 if (and only if) the gluing maps between the 3-dimensional blocks are orientation-preserving.

The resulting manifold H is canonically oriented (it admits an orientation-reversing diffeomorphism), collapses over P' (which is properly embedded in it), and so in particular is made of 0- and 1-handles. Moreover, $\partial P'$ is a link in ∂H and has a canonical framing induced by its regular neighborhood in $\partial L \subset \partial H$. Indeed, such a neighborhood is a union of bands collapsing on $\partial P'$, so that any curve running parallel to a component c of $\partial P'$ can be described by an integer if the neighborhood of c in ∂L is an annulus and by a half-integer otherwise.

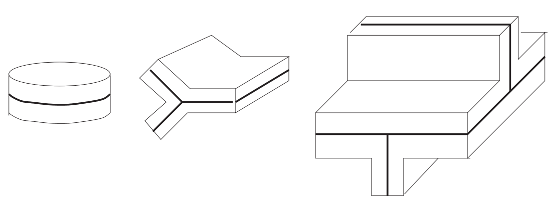

FIGURE 4. In this picture we show the blocks used to thicken a polyhedron to a 3-manifold.

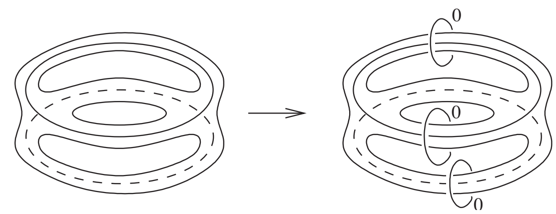

FIGURE 5. How to pass from a diagram of P' to a surgery presentation of $(\partial M_{(P',\varnothing)}, \partial P')$.

Step 3. We associate with each region R_i the block $R_i \times D^2$ (if R_i is not orientable one chooses the twisted disk bundle over R_i, which is unique since R_i collapses on a graph). Then we glue it along $\partial R_i \times D^2$ on ∂H, by sending $\partial R_i \times \{0\}$ into the corresponding component of $\partial P'$. The gluing map is then completely described once one determines how many twists the image of the framing $\partial R_i \times \{1\}$ performs with respect to the framing of $\partial P' \subset \partial H$, and this is specified by the gleam of R_i.

The above thickening procedure proves part of the following theorem:

Theorem 2.3. [Turaev 94] *Let P be a simple polyhedron and P' the regular neighborhood of* $\mathrm{sing}(P)$ *in P. It is possible to canonically thicken P' to an oriented* 4-*manifold $M_{(P',\varnothing)}$ composed of* 0- *and* 1-*handles in which P' is locally flat and properly embedded. If P is equipped with gleams* gl, *it is possible to extend the thickening to P in a canonical way, obtaining a flat embedding in an oriented* 4-*manifold $M_{(P,\mathrm{gl})}$ collapsing on P. Moreover, if P is embedded in a* 4-*manifold M, and* gl *is the gleam induced on P by its embedding (see Section 2.2), then $M_{(P,\mathrm{gl})}$ is diffeomorphic to a neighborhood of P in M.*

Example 2.4. If P is a spine of an orientable 3-manifold N, its modulo-2 gleam is zero. By performing the construction above, using as gleam on P the 0 gleam over all the regions, we get the manifold $N \times [-1, 1]$.

Example 2.5. To construct a surgery presentation of the pair $(\partial M_{(P',\varnothing)}, \partial P')$ it is sufficient to start from a diagram of P constructed as explained in Section 2.1, choose a maximal tree T in $\mathrm{sing}(P) = \mathrm{sing}(P')$, and encircle with 0-framed meridians all the three-tuples of strands running over edges not belonging to the tree (see Figure 5). It is remarkable that the choice of the over/under crossings in the construction does not affect the resulting pair. If P is special, $\partial M_{(P,\mathrm{gl})}$ is then obtained by integral Dehn surgery over the pair thus constructed.

Remark 2.6. All the 4-manifolds obtained by thickening the polyhedra equipped with gleams as in Theorem 2.3 are 4-handlebodies. That is, they admit a handle decomposition containing no handles of index greater than 2. It can be shown that the reverse also holds: any 4-handlebody can be obtained by applying Theorem 2.3 to a suitable polyhedron equipped with gleams (see [Costantino 04]).

Definition 2.7. (Shadows of 4-handlebodies.) A polyhedron equipped with gleam (P, gl) is a *shadow* of a 4-manifold M if M is diffeomorphic to the thickening $M_{(P,\mathrm{gl})}$ of (P, gl) obtained through Theorem 2.3.

2.4 Shadows of Closed 4-Manifolds

By Remark 2.6, shadows can be used to describe combinatorially only a subset of all the smooth 4-manifolds other than closed ones. To obviate this apparent weakness of the theory, let us recall the following result due to Laudenbach and Poenaru [Laudenbach and Poenaru 72]:

Theorem 2.8. *Let M be an oriented, smooth, and compact 4-manifold with boundary equal to S^3 or to a connected sum of copies of $S^2 \times S^1$. Then, up to diffeomorphism, there is only one closed, smooth, and oriented 4-manifold obtained by "closing M," that is, by attaching to M some 3- and 4-handles.*

Roughly speaking, the above result states that when a manifold is "closable," then it is so in a unique way. This allows us to describe all the closed 4-manifolds by means of polyhedra with gleams: given a closed manifold equipped with an arbitrary handle decomposition, then by considering the union of all handles of index strictly less than 3, we get a new manifold M that admits a shadow and can be described combinatorially as explained in Section 2.3. The initial manifold can be then uniquely recovered from M because of Theorem 2.8. With a slight abuse of notation, we then give the following definition:

Definition 2.9. (Shadows of closed manifolds.) A polyhedron with gleams (P, gl) is a *shadow* of a closed 4-manifold X if and only if X can be obtained by attaching 3- and 4-handles to the 4-manifold $M_{(P,\mathrm{gl})}$ obtained from P through the reconstruction map of Theorem 2.3.

Hence a necessary and sufficient condition for a pair (P, gl) to be a shadow of a closed 4-manifold is that $\partial M_{(P,\mathrm{gl})}$ be S^3 or a connected sum of copies of $S^2 \times S^1$.

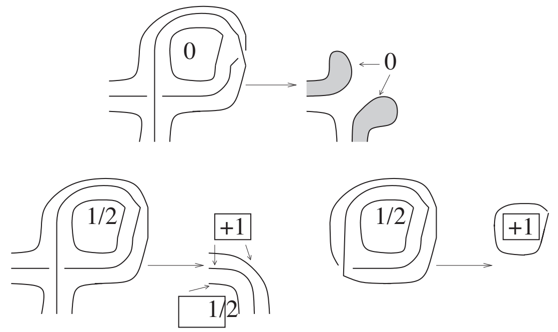

FIGURE 6. Two simplifying tricks for removing a vertex.

We will often use the three simplifying moves of Figure 6, whose effect on a polyhedron P is to produce a simpler polyhedron (possibly with boundary and hence retractible on some subpolyhedron) whose thickening is diffeomorphic to $M_{(P,\mathrm{gl})}$. The first one produces a region with boundary, which can then be collapsed. The remaining two regions are locally capped with two 0-gleam disks. The other two delete a region with gleam $\pm\frac{1}{2}$, changing the gleams of the neighboring regions by the amounts prescribed in the rectangular boxes. The idea of the proof of the fact that these moves do not change the 4-thickening of the polyhedra is based on the observation that the local patterns to which the moves apply may be viewed as the mapping cylinders of the projections onto a disk $D^2 \subset B^4$ of some strands of links contained in $S^3 = \partial B^4$; stated using Turaev's notation [Turaev 94], these patterns are the *shadow projections in D^2 of the strands of the links.* Then the first move corresponds to the application of the second Reidemeister move to the strands, while the other two are instances of the first Reidemeister move.

3. COMPLEXITY OF 4-MANIFOLDS

We begin with a definition:

Definition 3.1. (Complexity of closed manifolds.) Let X be a closed, orientable, smooth 4-manifold. The *complexity* of X, denoted by $c(X)$, is the least number of vertices in a shadow of X.

This definition is quite natural and represents the straightforward translation to the 4-dimensional case of Matveev's complexity of 3-manifolds [Matveev 90], based on spines. One of its fundamental properties is indeed shared by this notion:

Proposition 3.2. *Complexity is subadditive under connected sums. That is, if X_1 and X_2 are closed 4-manifolds, then $c(X_1 \# X_2) \leq c(X_1) + c(X_2)$.*

Proof: Let (P_i, gl_i), $i = 1, 2$, be shadows of X_i. Connecting them through an arc whose endpoints are in the interior of two regions and then "pushing our fingers along the arc," we produce a new (connected) shadow, called $P_1 + P_2$ by Turaev [Turaev 94] and containing $c(P_1) + c(P_2)$ vertices. It is not difficult to guess what the gleam of P should be and to prove then that $M_{(P,\mathrm{gl})} = M_{(P_1,\mathrm{gl}_1)} \#_\partial M_{(P_2,\mathrm{gl}_2)}$ (where $\#_\partial$ is the boundary-connected sum), so that closing $M_{(P,\mathrm{gl})}$ produces $X_1 \# X_2$. $\square$

We will prove that $\mathbb{CP}^2$ has complexity 0. In fact, it can be easily proved that any product $F \times S^2$ or $F \tilde{\times} S^2$ with F an orientable surface or $F \tilde{\times} S^2$ with F nonorientable has complexity 0. As a consequence, the following holds:

Corollary 3.3. *There are infinitely many nondiffeomorphic 4-manifolds of complexity 0.*

Remark 3.4. The fact that $c(\mathbb{CP}^2)$ is equal to 0 also implies that complexity cannot be additive under the connected sum. Indeed, for each closed 4-manifold X, there exists an integer k such that $X \# k\mathbb{CP}^2$ is diffeomorphic to $n\mathbb{CP}^2 \# m\overline{\mathbb{CP}}^2$ for some n, m, and $c(n\mathbb{CP}^2 \# m\overline{\mathbb{CP}}^2) = 0$.

We stress here that the nonfiniteness described above is common to Matveev's complexity. Indeed, in dimension 3, there are infinitely many manifolds having 0 complexity, e.g., any connected sum of $L(3, 1)$ with itself. The main problem is that in dimension 3 it makes sense to restrict to irreducible 3-manifolds, while in dimension 4, smooth irreducibility is a property that is not yet completely understood (see [Stipsicz 97]). In order to keep complexity finite, the proof of Proposition 3.2 suggests that one should restrict to special polyhedra. In dimension 3, this is a consequence of restricting to irreducible manifolds, so we ask the following question:

Question 3.5. What is the class of 4-manifolds admitting a minimal shadow that is special?

Even if one restricts to special shadows, it is not obvious that there is only a finite number of 4-manifolds having a fixed complexity. Indeed, a priori, there could exist infinitely many gleams on the same polyhedron P such that $\partial M_{(P,\mathrm{gl})} = S^3 \#_k S^2 \times S^1$, and this is indeed the case! But fortunately, the following remarkable result of B. Martelli [Martelli 04] ensures finiteness of complexity on special polyhedra:

Theorem 3.6. *Let N and N' be two closed 3-manifolds and $L \subset N$ a framed link. Up to diffeomorphism, there exist only finitely many cobordisms from N to N' constructed by gluing 2-handles to N along L.*

We stress here that the above result does not claim that there are finitely many slopes on L over which surgery produces N'. It claims only finiteness of the resulting 4-cobordisms.

Corollary 3.7. *Let P be a special polyhedron and P' the polyhedron obtained by puncturing P once over each region. Furthermore, let $(N, L) \doteq (\partial M_{(P',\varnothing)}, \partial P')$. Then there are only finitely many closed 4-manifolds admitting a shadow whose underlying polyhedron is P.*

In what follows, we restrict to special shadows of 4-manifolds and classify all the 4-manifolds admitting a special shadow with at most one vertex.

Definition 3.8. (Special complexity.) Let X be a closed and oriented 4-manifold. The *special complexity* of X, denoted by $c^{\mathrm{sp}}(X)$ is the least number of vertices of a special shadow of X.

Theorem 3.9. *If a closed 4-manifold X has a shadow with k vertices and r regions that are not disks and whose total Euler characteristic is e, then $c^{\mathrm{sp}}(X) \leq k + 2(r + 2e)$. Moreover, the following hold:*

(i) *For each integer k there exists only a finite number of smooth, closed, and oriented 4-manifolds having special complexity less than or equal to k.*

(ii) *If X_1 and X_2 are closed, oriented 4-manifolds, then $c^{\mathrm{sp}}(X_1 \# X_2) \leq c^{\mathrm{sp}}(X_1) + c^{\mathrm{sp}}(X_2) + 4$.*

(iii) *$c^{\mathrm{sp}}(\overline{X}) = c^{\mathrm{sp}}(X)$, where $\overline{X}$ is X with the opposite orientation.*

Proof: The first statement is a standard fact. It suffices to apply some local modifications called "$(0 \to 2)$-moves" to the initial shadow in order to split the nondisk regions into disks. Each of these moves creates two vertices, and the total number of these moves is bounded above by $r + 2e$. Fact (i) is a direct consequence of Corollary 3.7

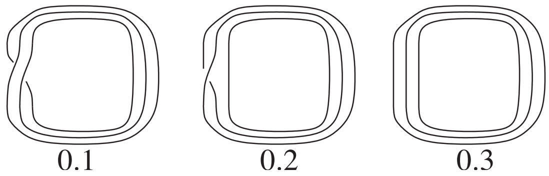

FIGURE 7. The three complexity-0 special polyhedra.

and of the fact that there are only finitely many special polyhedra with no more than k vertices. To prove fact (ii), it is sufficient to repeat the construction of the proof of Proposition 3.2 and add two “lune moves,” producing four new vertices, to ensure that the final polyhedron is special. Fact (iii) is a direct consequence of the fact that if (P, gl) is a shadow of M, then $(P, -\mathrm{gl})$ is a shadow of $\overline{M}$. □

The main reason that it is interesting to restrict to special shadows is that the number of special polyhedra with fewer than k vertices is finite for every k. In particular, Figures 7 and 8 summarize respectively the complexity-0 and complexity-1 special polyhedra.

Theorem 3.10. *The only closed, smooth 4-manifolds having 0 special complexity are S^4, $\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2$, $S^2 \times S^2$, $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, $\mathbb{CP}^2 \# \mathbb{CP}^2$, and $\overline{\mathbb{CP}}^2 \# \overline{\mathbb{CP}}^2$. Moreover, there are no manifolds with special complexity 1.*

Remark 3.11. If one restricts to simply connected manifolds viewed up to homeomorphism, it is not surprising that there are no new ones of complexity 1: indeed, by Freedman’s theorem, these manifolds are classified up to homeomorphism by their self-intersection form. Hence, since the second homology of a shadow of a 4-manifold surjects onto the second homology of the 4-manifold, and the maximal second Betti number of a special-complexity 1-polyhedron is 4, the possible intersection forms obtainable using complexity-1 polyhedra are those already carried by complexity-0 manifolds. What is interesting is that no exotic structure on complexity-0 manifolds has been found in complexity one.

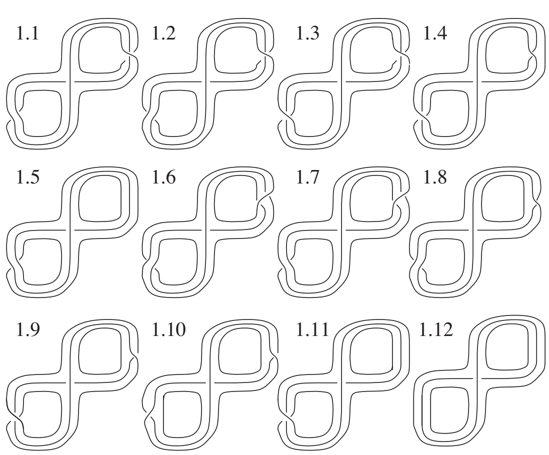

FIGURE 8. The twelve complexity-1 special polyhedra.

To prove Theorem 3.10, for each polyhedron P of Figures 7 and 8, we find all the gleams such that $\partial M_{(P,\mathrm{gl})}$ is $S^3 \#_k S^2 \times S^1$ for some $k \geq 0$. Then, for each of these gleams, we prove that the closed 4-manifold obtained by closing $M_{(P,\mathrm{gl})}$ belongs to the list above. To do that, we use a series of results ranging from classical topology to hyperbolic geometry to quantum topology. The next subsection is devoted to recalling these results, suitably adapted to our needs.

3.1 Useful Tools

The following proposition collects some classical results:

Proposition 3.12. *Let (P, gl) be a polyhedron, let (R_i, gl_i), $i = i, \ldots, n$, be its regions equipped with gleams and oriented arbitrarily, and let $M = M_{(P,\mathrm{gl})}$. The following hold:*

(i) $H_*(P, \mathbb{Z}) \cong H_*(M, \mathbb{Z})$, $\pi_*(P, x_0) \cong \pi_*(M, x_0)$, *for each base point* $x_0 \in P$.

(ii) *If* $H_2(M; \mathbb{Z}) = 0$ *and* $\mathrm{tors}(H_1(M)) \neq 0$, *then* $\partial M \neq S^3 \#_k S^2 \times S^1$.

(iii) *If* $H_2(M; \mathbb{Z}) = 0$ *and* $H_1(M; \mathbb{Z})$ *is free, then* $H_1(\partial M; \mathbb{Z}) \cong H_1(M; \mathbb{Z})$.

(iv) *Each element of* $H_2(M; \mathbb{Z})$ *can be represented in a unique way as a sum* $\sum_i k_i R_i$, *with* $k_i \in \mathbb{Z}$.

(v) *Given a basis* $c^1, \ldots, c^k$ *of* $H_2(M; \mathbb{Z})$ *with* $c^j = \sum_{1 \leq i \leq n} c_i^j R_i$, *the self-intersection form of M can be represented by an integer matrix* $Q(P, \mathrm{gl})$ *whose* (j, l)*th entry is given by* $\sum_{1 \leq i \leq n} \mathrm{gl}(R_i) c_i^j c_i^l$.

(vi) *Suppose that* $H_1(P; \mathbb{Z}) = 0$. *Then if* $\det(Q(P, \mathrm{gl})) \neq 0$, *we have* $\# H_1(\partial M; \mathbb{Z}) = |\det(Q(P, \mathrm{gl}))|$. *Otherwise,* $H_1(\partial M; \mathbb{Z})$ *is infinite.*

Proof: Facts (i) and (iv) are a consequence of the fact that M retracts on P, and P is a CW complex without 3-cells. Facts (ii) and (iii) result from $H_3(M; \mathbb{Z}) = 0$ (P contains no 3-cells), $H_1(M, \partial M; \mathbb{Z}) = 0$ (P has codimension 2 in M), from the isomorphism $H_2(M, \partial M) \cong$

free$(H_2(M)) \oplus$ tors$(H_1(M))$, and from the exact homology sequence

$$0 \to H_3(M, \partial M) \to H_2(\partial M) \to H_2(M) \to H_2(M, \partial M) \to H_1(\partial M) \to H_1(M) \to 0$$

of the pair $(M, \partial M)$. Fact (v) was proved by Turaev [Turaev 94], and the last is a consequence of (ii) and a classical result of Fox [Fox 54]. □

We will also use the following strong result due to Gordon and Luecke for the part regarding S^3 [Gordon and Luecke 89] and to Gabai [Gabai 87] for the part regarding $S^2 \times S^1$.

Theorem 3.13. *No integer Dehn filling on a nontrivial knot in S^3 produces S^3 or $S^2 \times S^1$.*

3.1.1 Hyberbolic Three-Manifolds and Shadows. Let P be a special polyhedron containing at least one vertex and let P' be the regular neighborhood of sing(P) in P. By Theorem 2.3, P' can be thickened (without the need for any gleams!) to a 4-manifold $M_{(P',\varnothing)}$ diffeomorphic to a regular neighborhood of a graph in $\mathbb{R}^4$ so that $\partial P' \subset \partial M$ is a link in $\partial M \cong \#_k S^2 \times S^1$ (for a suitable k). For each component of $\partial P'$ we define its $\mathbb{Z}_2$-gleam to be the $\mathbb{Z}_2$-gleam of the region of P containing the component, and its *valence* to be equal to the number of vertices of P' touching that region. The following was proved by the author and D. P. Thurston [Costantino and Thurston 05]:

Theorem 3.14.

(i) *The 3-manifold $\partial M_{(P',\varnothing)}$ is a connected sum of $1 - \chi(P')$ copies of $S^2 \times S^1$ in which the link $\partial P'$ has hyperbolic complement whose volume is $2|\chi(P')|\mathrm{vol}_{\mathrm{oct}}$, where $\mathrm{vol}_{\mathrm{oct}}$ is the volume of the regular hyperbolic octahedron.*

(ii) *There is a maximal set of sections of the cusps of $\partial M_{(P',\varnothing)} - \partial P'$ such that the torus corresponding to a component of $\partial P'$ whose valence is q is the one depicted in the left drawing of Figure 9 if its $\mathbb{Z}_2$-gleam is zero and is the one depicted in the right drawing otherwise.*

(iii) *The manifold $M_{(P,\mathrm{gl})}$ is obtained by attaching 2-handles to $M_{(P',\varnothing)}$ along $\partial P'$, and hence $\partial M_{(P,\mathrm{gl})}$ is obtained by an integer Dehn filling of $\partial M_{(P',\varnothing)} - \partial P'$.*

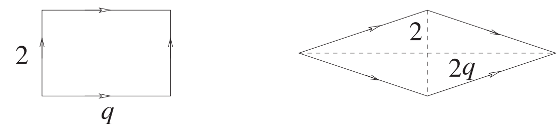

FIGURE 9. The shapes of the section of a cusp of $\partial M_{(P',\varnothing)} - \partial P'$.

Let us recall, suitably adapted to our needs, the following results of I. Agol [Agol 00] and M. Lackenby [Lackenby 00]:

Theorem 3.15. *Let N be a hyperbolic 3-manifold with cusps and let c be a fixed section of a cusp of N. Gluing a solid torus to c through a homeomorphism sending the meridian of the torus to a geodesic in c whose length is greater than 6 produces a 3-manifold N' that is hyperbolic.*

Theorem 3.16. [Agol 00] *Let N be a hyperbolic 3-manifold with cusps; let C_i, $i = 1, \dots, n$, be embedded sections of all the cusps of N, with v_i the volume cut out of N by C_i; and let sl_i, $i = 1, \dots, n$, be the minimal-length geodesics in C_i. Let s be any subset of $\{1, \dots, n\}$ and let N_s be a Dehn filling on N along the cusps C_i, $i \in s$. If for each $i \in s$ the distance between the ith slope of the Dehn filling and sl_i is greater than $18/v_i$, then N_s is hyperbolic.*

The following was proved in [Costantino et al. 06] as a corollary of Theorem 3.14 and Theorem 3.15:

Corollary 3.17. [Costantino et al. 06] *Let (P, gl) be a standard shadow such that for each region R of P we have $|\mathrm{gl}(R)| + v(R) \geq 6$. Then the manifold $\partial M_{(P,\mathrm{gl})}$ is Haken or word hyperbolic, and hence is not S^3 or $\#_k S^2 \times S^1$.*

3.1.2 State-Sum Quantum Invariants. Given a shadow (P, gl) it is fairly easy to compute Reshetikhin–Turaev invariants of $\partial M_{(P,\mathrm{gl})}$ as combinatorial state sums. Instead of plunging into the theoretical aspects of these invariants we limit ourselves to defining these invariants through explicit coefficients in $\mathbb{C}$ (see [Turaev 94] for a complete account). Let $r \geq 3$ be an integer and $t \doteq e^{2\pi i/r} \in \mathbb{C}$; for each $n \in \mathbb{N}$, let

$$[0] = [1] = 1, \quad [n] = \frac{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}, \quad [n]! = \prod_{0 \leq i \leq n} [i].$$

Let us define complex-valued functions on $\mathbb{N}/2$ as follows:

$$w_j = \left(\sqrt{-1}\right)^{2j} \sqrt{[2j+1]}.$$

We say that a triple (i, j, k) of elements of $\mathbb{N}/2$ is *admissible* if the following conditions are satisfied:

$$i + j \geq k, \quad i + k \geq j, \quad j + k \geq i, \quad i + j + k \leq r - 2,$$

for $i + j + k \in \mathbb{N}$. For any triple of elements of $\mathbb{N}/2$ we define

$$\Delta(i, j, k) = \sqrt{\frac{[i + j - k]![i + k - j]![j + k - i]!}{[i + j + k + 1]!}}$$

if the triple is admissible and zero otherwise. Finally, for any 6-tuple of elements of $\mathbb{N}/2$ we define its $6j$-*symbol* as follows:

$$\begin{pmatrix} i & j & k \\ l & m & n \end{pmatrix} = \sum_{z \in \mathbb{Z}} \frac{N(z;\, i, j, k, l, m, n)}{D(z;\, i, j, k, l, m, n)},$$

where

$$\begin{aligned} &N(z;\, i, j, k, l, m, n) \\ &\quad = \left(\sqrt{-1}\right)^{-2(i+j+k+l+m+n)} \Delta(i, j, k)\Delta(i, m, n)\Delta(j, l, n) \\ &\qquad \times \Delta(k, l, m)(-1)^z [z + 1]! \end{aligned}$$

and

$$\begin{aligned} &D(z;\, i, j, k, l, m, n) \\ &\quad = [z - i - j - k]![z - i - m - n]![z - j - l - n]! \\ &\qquad \times [z - k - l - m]![i + j + l + m - z]! \\ &\qquad \times [i + k + l + m - z]![j + k + m + n - z]!, \end{aligned}$$

and where the sum is taken over all z such that all the arguments of the "quantum factorials" in the denominator of the right-hand side are nonnegative. Furthermore, let

$$W \doteq \frac{\sqrt{2r}}{t - t^{-1}},$$

$$S \doteq W^{-1} \sum_{0 \leq i \leq (r-2)/2} (w_i)^4 e^{2\pi\sqrt{-1}(i - i(i+1)/r)}.$$

We define a *coloring* on a special polyhedron (P, gl) as an assignment of an element of $\mathbb{N}/2$ to each region of P. Given a coloring on P, for each region R let

$$w[R] \doteq w_j e^{2\pi\sqrt{-1}\,\mathrm{gl}(R)(i - i(i+1)/r)},$$

where j is the color of R_i. Similarly, we associate with each vertex its $6j$-symbol, where (i, j, k, l, m, n) are the colors of the regions around the vertex, and (i, l), (j, m), and (k, n) are the pairs of colors corresponding to regions that near the vertex, intersect only in the vertex itself. Finally, let $\mathrm{sig}(P, \mathrm{gl})$ be the signature of the self-intersection form of $H_2(M_{(P,\mathrm{gl})}; \mathbb{Z})$, and let $\mathrm{nul}(P, \mathrm{gl})$ be the dimension of the maximal real subspace of $H_2(M_{(P,\mathrm{gl})}; \mathbb{R})$ contained in the annihilator of the form. The *state sum* of (P, gl) is given by

$$\begin{aligned} |(P, \mathrm{gl})|_r &= W^{1-\chi(P)-\mathrm{nul}(P,\mathrm{gl})} S^{-\mathrm{sig}(P,\mathrm{gl})} \\ &\quad \times \sum_{\text{colorings}} \prod_{\text{regions}} w[R] \prod_{\text{vertices}} \begin{pmatrix} i & j & k \\ l & m & n \end{pmatrix}. \end{aligned}$$

Theorem 3.18. [Turaev 94] *Let N be a 3-manifold and let (P, gl) be such that $N = \partial M_{(P,\mathrm{gl})}$. Then $|(P, \mathrm{gl})|_r$ is an invariant of N, denoted by $|N|_r$. If (P', gl') is another polyhedron such that $N = \partial M_{(P',\mathrm{gl}')}$, then $|(P', \mathrm{gl}')|_r = |(P, \mathrm{gl})|_r$. Moreover, if $N = S^3 \#_k S^2 \times S^1$ for some $k \geq 0$, then $|N|_r = 1$, for all $r \geq 3$.*

Remark 3.19.

1. The normalization we have used differs slightly from Turaev's original one to better suit our need of identifying polyhedra with gleams describing "closable" 4-manifolds.

2. The gleam of P is irrelevant for the selection of the admissible colorings, so that the explicit form of the state sum $|(P, \mathrm{gl})|$ does not change if one changes gl. This has allowed us to perform extensive computer-based calculations of Reshetikhin–Turaev invariants of $\partial M_{(P,\mathrm{gl})}$ for a fixed polyhedron with varying gleams.

3.2 Classification of Low-Complexity 4-Manifolds

3.2.1 Zero-Complexity 4-Manifolds. In this subsection we prove the first part of Theorem 3.10 by means of a case-by-case analysis. More precisely, for each polyhedron P of Figure 7 we will list all the possible gleams such that $\partial M_{(P,\mathrm{gl})} = S^3 \#_k S^2 \times S^1$. Then for each of these gleams we identify the 4-manifold obtained by closing $M_{(P,\mathrm{gl})}$.

Case 0.1. In this case $\pi_1(P) = \mathbb{Z}_3$ and $H_2(P) = 0$, so by Proposition 3.12, $\partial M_{(P,\mathrm{gl})}$ cannot have the form $S^3 \#_k S^2 \times S^1$ for any gleam on P.

Case 0.2. In this case P has two regions. Let R_1 be the one passing once over $\mathrm{sing}(P)$ and R_2 the other one. Let P' be a regular neighborhood of $\mathrm{sing}(P)$ in P and let P'_i be the polyhedra obtained by gluing the regions R_i to P'. It can be checked that if R_1 is equipped with gleam gl_1 (necessarily a half-integer), then the pair

$(\partial M_{(P'_1,\mathrm{gl}_1)}, \partial P'_1)$ is $(S^3, T(2\mathrm{gl}_1, 2))$, where $T(p,q)$ is the (p,q)-torus knot. In particular, $\partial P'_1$ is a trivial knot in S^3 only if $\mathrm{gl}_1 = \pm\frac{1}{2}$, and so by Theorem 3.13, if (P, gl) produces a "closable" 4-manifold, then $\mathrm{gl}(R_1) = \pm\frac{1}{2}$. Hence let us now suppose that $\mathrm{gl}(R_1) = \frac{1}{2}$ (up to reversing the orientation of $M_{(P,\mathrm{gl})}$ we can do this). Notice that $H_2(P;\mathbb{Z}) = \mathbb{Z}$, and it is generated by the cycle represented by $2R_1 + R_2$, whose self-intersection is $\mathrm{gl}(R_2) + 4\mathrm{gl}(R_1)$ (see Proposition 3.12). Hence by fact (vi) of Proposition 3.12 we have $2 + \mathrm{gl}(R_2) = \pm 1$ or $2 + \mathrm{gl}(R_2) = 0$, and so $\mathrm{gl}(R_2)$ is in $\{-1,-2,-3\}$. It is not difficult to check that in these cases, using the tricks of Figure 6, P can be simplified to a sphere with gleam respectively $1, 0, -1$. Such spheres are shadows respectively of $\mathbb{CP}^2$, S^4, and $\overline{\mathbb{CP}}^2$. Hence P with gleam $(\frac{1}{2}, -1)$ is a shadow of $\mathbb{CP}^2$, with gleam $(\frac{1}{2}, -2)$ of S^4, and with gleam $(\frac{1}{2}, -3)$ of $\overline{\mathbb{CP}}^2$.

Case 0.3. Let R_1, R_2, and R_3 be the regions of P oriented so that $R_1 + R_3$ and $R_2 + R_3$ are cycles, and let gl_i, $i = 1,2,3$, be their (integer) gleams. It can be checked that $\partial M_{(P,\mathrm{gl})}$ is the Seifert manifold $S^2(\mathrm{gl}_1,1)(\mathrm{gl}_2,1)(\mathrm{gl}_3,1)$, which, according to the classification of Seifert 3-manifolds, can be S^3 or $S^2 \times S^1$ only if $|\mathrm{gl}_i| \le 3$ for all i. Moreover, in the chosen basis of $H_2(M_{(P,\mathrm{gl})})$ the self-intersection matrix of $M_{(P,\mathrm{gl})}$ is (see Proposition 3.12)

$$\begin{pmatrix} \mathrm{gl}_1 + \mathrm{gl}_3 & \mathrm{gl}_3 \\ \mathrm{gl}_3 & \mathrm{gl}_2 + \mathrm{gl}_3 \end{pmatrix}.$$

Hence, by fact (iv) of Proposition 3.12, we must have $(\mathrm{gl}_1 + \mathrm{gl}_3)(\mathrm{gl}_2 + \mathrm{gl}_3) - {\mathrm{gl}_3}^2 = \pm 1, 0$. In particular, it turns out that up to symmetries of P and multiplication by -1 of gl (which changes the orientation of $M_{(P,\mathrm{gl})}$), the only cases are $(k,0,0)$, $(1,\pm 1,k)$, with $k \in \{-3,-2,-1,0,1,2,3\}$. A case-by-case study shows that in the first family, k has to be in $\{-1,0,1\}$, producing respectively $\overline{\mathbb{CP}}^2$, S^4, $\mathbb{CP}^2$. The only interesting cases of the second family turn out to be $(1,-1,1)$ and $(1,-1,3)$, which give $S^2 \times S^2$, $(1,-1,0)$, and $(1,-1,2)$, which give $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, $(1,1,0)$, and $(-1,-1,0)$, which give $\mathbb{CP}^2 \# \mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2 \# \overline{\mathbb{CP}}^2$ respectively.

3.2.2 Complexity-1 4-Manifolds.

Let us first clarify the general strategy we follow for each polyhedron P of Figure 8. Let $R_1, \dots, R_n$ be the regions of P, P' the regular neighborhood of $\mathrm{sing}(P)$ in P, and for each subset s of $\{1,\dots,n\}$, let P_s be the polyhedron obtained by gluing to P' each region R_i with $i \in s$. By Theorem 3.14 the manifold $\partial M_{(P',\varnothing)}$ is hyperbolic with n cusps, and gluing back a region R_i to P' corresponds to performing an integer Dehn filling along the ith cusp. We want to list all the possible gleams on P such that $\partial M_{(P,\mathrm{gl})}$ is $S^3 \#_k S^2 \times S^1$ for some $k \ge 0$. This produces a finite list of nonhyperbolic (possibly partial) Dehn fillings of P' by recursively applying Theorem 3.16. Our main tool is Jeff Weeks's Snappea [Weeks]. So, starting from the hyperbolic manifold $\partial M_{(P',\varnothing)}$ $(s = \varnothing)$, we iterate the following algorithm:

1. Choose sections C_i, $i \notin s$, of the given hyperbolic manifold. Let v_i, $i \notin s$, be the volumes they cut out of the manifold and let g_i, $i \notin s$, be the gleams on the regions R_i corresponding to integer Dehn fillings along shortest geodesics in C_i.
2. For each cusp C_i, $i \notin s$, perform the following steps: Let $s_i = s \cup \{i\}$, and for each integer j such that $-\frac{18}{v_i} \le j \le \frac{18}{v_i}$, let $\mathrm{gl}(R_i) = g_i + j$ and let gl_{s_i} be the set of gleams on the regions with indices in $s \cup \{i\}$:
   - If $\partial M_{(P_{s_i},\mathrm{gl}_{s_i})}$ is not hyperbolic, add $(P_{s_i}, \mathrm{gl}_{s_i})$ to the list of nonhyperbolic fillings of $M_{(P',\varnothing)}$.
   - If it is hyperbolic and has nonempty boundary, apply step 1 to $\partial M_{(P_{s_i},\mathrm{gl}_{s_i})}$. Otherwise, if $j < \frac{18}{v_i}$ increase j. Otherwise, choose another cusp C_k with $k \in \{1,\dots,n\} - i$ and follow step 2.

The result of the above algorithm will be in general a finite list of nonhyperbolic 3-manifolds, possibly with boundary. If all the manifolds in the list are closed, one has a finite number of cases to check. In particular, we did this using Theorem 3.18. In what follows, a clever use of the tools of Section 3.1 allowed us to treat the cases in which some element of the list has nonempty boundary and to show that in fact, if a 4-manifold has a complexity-1 special shadow, then it also has a complexity-0 one. It is worth noting that the above general algorithm was necessary only in few cases, since most of the polyhedra of Figure 8 can be studied "by hand." Let us then start with the easiest cases.

Cases 1.1–1.5. In all these cases $H_1(P)$ is a finite, nontrivial group, and so by Proposition 3.12 there is no gleam on P such that $\partial M_{(P,\mathrm{gl})} = S^3 \#_k S^2 \times S^1$ for some $k \ge 0$.

Cases 1.6–1.7. In these cases P has only one region whose valency is 6. By Corollary 3.17 there is no gleam on P such that $\partial M_{(P,\mathrm{gl})} = S^3 \#_k S^2 \times S^1$.

Case 1.8. Let R_1 be the region whose valency is 5, let R_2 be the other one, and let gl_i, $i = 1,2$, be their gleams. Since the $\mathbb{Z}_2$-gleams of R_1 and R_2 are respectively 1 and 0, then $\mathrm{gl}_1 \in \mathbb{Z}/2$ and $\mathrm{gl}_2 \in \mathbb{Z}$. Following the general algorithm, we obtain a finite list of pairs $(\mathrm{gl}_1, \mathrm{gl}_2)$ such that

$M_{(P,\mathrm{gl})}$ is closed and nonhyperbolic and only one nonclosed case: $(P'_2, \mathrm{gl}_2 = 0)$. Using our state-sum formulation with Reshetikhin–Turaev invariants with $r = 5, 7, 9$ (see Theorem 3.18), we excluded all the closed cases. The nonclosed case corresponds to the infinite family of gleams on P of the form $\mathrm{gl} = (\mathrm{gl}_1, 0)$, $\mathrm{gl}_1 \in \mathbb{Z}/2$, all of which can be simplified by the upper trick of Figure 6, yielding a contractible shadow of S^4.

Case 1.9. This case is similar to the preceding one. Let R_1 and R_2 be respectively the valency-5 and valency-1 regions of P, and let gl_i, $i = 1, 2$, be their gleams (note that in this case, $\mathrm{gl}_i \in \mathbb{Z}$, $i = 1, 2$). The general algorithm gives a list containing only two nonclosed nonhyperbolic manifolds and a finite number of closed ones. As in the preceding case, using Theorem 3.18 we excluded all the closed ones. The first nonclosed nonhyperbolic manifold is $\partial M_{(P'_2,0)}$, whose Dehn fillings correspond to pairs (P, gl) with $\mathrm{gl} = (\mathrm{gl}_1, 0)$, $\mathrm{gl}_1 \in \mathbb{Z}$, which can be simplified to a contractible shadow of S^4 using the upper trick of Figure 6. The second nonhyperbolic manifold is $\partial M_{(P'_1,0)}$. Using S. Matveev's "recognizer" [Matveev 05], we checked that this manifold has JSJ decomposition $D^2(2,1)(3,-2) \cup N^2 \cup D^2(2,1)(3,-2)$ and contains two incompressible tori that can be compressed only if the Dehn filling on the boundary corresponds to the 0 gleam on R_2. But since $S^3 \#_k S^2 \times S^1$ are atoroidal, $\mathrm{gl}_2 = 0$, which falls in the preceding case.

Case 1.10. Let R_1 be the valency-4 region, R_2 and R_3 the remaining two (they are interchangeable through a symmetry of P). It is easy to check that $H_1(P; \mathbb{Z}) = 0$ and $H_2(P; \mathbb{Z}) = \mathbb{Z}$ with generator represented by R_1. By Proposition 3.12, we have $\mathrm{gl}_1 = 0$ or $\mathrm{gl}_1 = \pm 1$. Then, up to multiplying gl by -1, we are reduced to studying two manifolds: $\partial M_{(P'_1,0)}$ and $\partial M_{(P'_1,1)}$. Using Matveev's recognizer, one sees that the first one has JSJ decomposition $N^2 \cup N^2$ (where N^2 is the product of a thrice-punctured sphere with S^1) and contains two incompressible tori, which can be compressed only if at least one of gl_2 and gl_3 is zero, in which case (P, gl) can be simplified using the upper trick of Figure 6, yielding a shadow that is a sphere equipped with a gleam equal to gl_1. In contrast, $\partial M_{(P'_1,1)}$ is hyperbolic and can be treated using the general algorithm. The result is a finite list of closed nonhyperbolic manifolds, which can be excluded using Theorem 3.18, and two nonclosed nonhyperbolic manifolds corresponding respectively to $(P'_{1,2}, (1, 0, \varnothing))$ and $(P'_{1,3}, (1, \varnothing, 0))$, which can be simplified using the tricks of Figure 6.

Case 1.11. Let R_i, $i = 1, 2, 3$, be the region of valency i in P. The 4-manifold $M_{(P,\mathrm{gl})}$ has a handle decomposition induced by P such that the two 1-handles induced by the edges of P are annihilated by the two 2-handles corresponding to R_1 and R_2. Hence $\partial M_{(P'_{1,2},(\mathrm{gl}_1,\mathrm{gl}_2,\varnothing))}$ is the complement of a knot k in S^3, so by Theorem 3.13, we search for the cases in which k is the trivial knot. To do this, we calculated the Alexander polynomial of k using gl_1 and gl_2 as parameters and Turaev's surgery formulas for Reidemeister torsion [Turaev 02]. We then have

$$\Delta(k) = \frac{t + t^2 + t^{3c_1} + t^{(3c_1+2c_2+1)} + t^{(6c_1+2c_2-1)} + t^{(6c_1+2c_2)}}{(1+t)(1+t+t^2)}$$

and

$$c_1 \doteq \mathrm{gl}_1 + \frac{1}{2}, \quad c_2 \doteq \mathrm{gl}_2 - \frac{1}{2}.$$

It is easy to check that the above fraction defines an element of $\mathbb{Z}[t, t^{-1}]$, well defined up to products by $t^{\pm 1}$. Then, to find the cases in which k is an unknot, we study the conditions under which $\Delta(k) = t^r$ for some r. To do this, we associate with $\Delta(k)$ its span, that is, the (well-defined) difference between the highest and the lowest degree in any of its expressions as an element of $\mathbb{Z}[t, t^{-1}]$. It is simple to see that this span depends on gl_1 and gl_2 as a piecewise affine function. A careful analysis of all the possible combinations of $(\mathrm{gl}_1, \mathrm{gl}_2)$ shows that $\mathrm{span}(\Delta(k)) = 0$ in only four cases: $(-\frac{1}{2}, \frac{3}{2})$, $(-\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, -\frac{1}{2})$, $(\frac{1}{2}, -\frac{3}{2})$. But in each of these cases, $|\mathrm{gl}_1| = \frac{1}{2}$. Hence, using the lower-left trick of Figure 6, the polyhedron can be simplified, yielding the polyhedron 0.2 of Figure 7.

Case 1.12. Let R_1 and R_2 be the two valency-2 regions of P, and let R_3 and R_4 be the remaining two. It is easy to see that there are symmetries of P interchanging them in pairs. If either $\mathrm{gl}_3 = \pm\frac{1}{2}$ or $\mathrm{gl}_4 = \pm\frac{1}{2}$, then P can be simplified, and we obtain the special complexity-0 polyhedron. We therefore exclude from now on all the quadruples of gleams satisfying one of the above equalities. The application of the general algorithm produces again a finite list of closed nonhyperbolic Dehn fillings of $\partial M_{(P',\varnothing)}$ and a finite list of nonclosed ones. The former can be shown to be different from $S^3 \#_k S^2 \times S^1$ by means of Theorem 3.18. To exclude the possibility that the latter have integer Dehn fillings of that form, we use the following facts:

1. If S is a Seifert 3-manifold having the homology of S^3 ($S^2 \times S^1$), then $S = S^3$ ($S = S^2 \times S^1$) iff its base orbifold is S^2, its singular fibers are not more than 3, and their degree of singularity is at most 3.

2. The determinant of the self-intersection matrix of $M_{(P,\mathrm{gl})}$ is $(\mathrm{gl}_1+\mathrm{gl}_3+\mathrm{gl}_4)(\mathrm{gl}_2+\mathrm{gl}_3+\mathrm{gl}_4)-(\mathrm{gl}_3-\mathrm{gl}_4)^2$, and by Proposition 3.12, it has to be either 0 or ± 1.

3. $\mathrm{gl}_1, \mathrm{gl}_2 \in \mathbb{Z}$ and $\mathrm{gl}_3, \mathrm{gl}_4 \in \mathbb{Z}/2$.

Up to symmetries of P and multiplication of gl by -1, the list of (partial) nonclosed nonhyperbolic Dehn fillings is encoded by the following quadruples of gleams on P: $(0,\varnothing,\varnothing,\varnothing)$, $(2,-2,\varnothing,\varnothing)$, $(1,\varnothing,\varnothing,\varnothing)$, $(2,\varnothing,-\frac{3}{2},\varnothing)$, $(2,-3,-\frac{5}{2},\varnothing)$, $(2,-3,\frac{3}{2},\varnothing)$, $(3,3,-\frac{3}{2},\varnothing)$, where we have denoted by $\varnothing$ the nonfilled regions. Using Matveev's recognizer, we see that the integer Dehn fillings of the first two quadruples are $S^2(\mathrm{gl}_2,-1)(2\mathrm{gl}_3,-\mathrm{gl}_3+\frac{1}{2})(2\mathrm{gl}_4,-\mathrm{gl}_4+\frac{1}{2})$ and $S^2(2,1)(2\mathrm{gl}_3,2)(2\mathrm{gl}_4,2)$ respectively. Using the above three facts, one can show that these Dehn fillings are "closable" if and only if either $\mathrm{gl}_3=\pm\frac{1}{2}$ or $\mathrm{gl}_4=\pm\frac{1}{2}$, which we have excluded from the beginning. Similarly, a Dehn filling corresponding to $(1,\mathrm{gl}_2,\mathrm{gl}_3,\mathrm{gl}_4)$ gives $D^2(\mathrm{gl}_3+\frac{1}{2},-1)(\mathrm{gl}_4+\frac{1}{2},-1)\cup D^2(2,1)(\mathrm{gl}_2+1,1)$, which contains an incompressible torus unless either gl_3 or gl_4 is $\pm\frac{1}{2}$. The quadruples $(2,\mathrm{gl}_2,-\frac{3}{2},\mathrm{gl}_4)$, $(2,-3,-\frac{5}{2},\mathrm{gl}_4)$, $(2,-3,\frac{3}{2},\mathrm{gl}_4)$ satisfy the equation of fact 2 above only in a finite number of cases, all of which can be excluded by means of Theorem 3.18. The last quadruple, $(3,3,-\frac{3}{2},\mathrm{gl}_4)$, satisfies the determinant equation for all gl_4, but it can be checked that $H_1(\partial M_{(P,\mathrm{gl})};\mathbb{Z})$ always has torsion unless $\mathrm{gl}_4=-\frac{3}{2}$ or $\mathrm{gl}_4=-\frac{5}{2}$. These two cases can then be excluded by means of Theorem 3.18.

3.3 Higher-Complexity Manifolds and Exotic Pairs

Let us provide some examples of 4-manifolds having higher special complexity. Some "trivial" examples can be obtained by applying Theorem 3.9. Each connected sum of a pair of special complexity-0 manifolds has special complexity at most 4; more generally, the special complexity of $k\mathbb{CP}^2\#h\overline{\mathbb{CP}}^2$ is bounded above by $2k+2h$. A first nontrivial example is $\mathbb{RP}^2\tilde{\times}S^2$. Its special shadow with 2-vertices can be constructed by applying Theorem 3.9 to its nonspecial shadow, whose underlying polyhedron is obtained from 0.3 of Figure 7 by gluing two disks with gleams ± 1 and one Möbius strip. More generally, if F is a genus-g surface, the manifold $F\times S^2$ has special complexity bounded above by $4g$ if F is orientable and by $4g+2$ otherwise. We define the complexity of a pair of manifolds as the maximum between the complexities of the two manifolds, and then the following natural question arises:

Question 3.20. (Complexity of exotic pairs.) What pair of homeomophic but nondiffeomorphic closed/nonclosed 4-manifolds have the lowest complexity/special complexity?

We now produce upper estimates to the answer of the above question in the case of nonspecial complexity for closed manifolds and in the case of special complexity for nonclosed ones.

It is not difficult to provide upper estimates for the nonspecial complexity of a class of notable closed 4-manifolds: the elliptic surfaces $E(n)$. Using a Kirby-calculus presentation of these manifolds [Gompf and Stipsicz 99, Theorem 8.3.2], one can check that $c(E(n))\leq 6n+2$. Then an example of an "exotic" 4-manifold with nonspecial complexity less than or equal to 14 is $E(2)\#\overline{\mathbb{CP}}^2$, which is homeomorphic but not diffeomorphic to $3\mathbb{CP}^2\#20\overline{\mathbb{CP}}^2$ (whose complexity is 0). Hence an upper estimate for the answer to the above question in the closed case for nonspecial complexity is 14. We expect such an estimate to be nonoptimal and that lower-complexity examples will be found.

In sharp contrast to the empty-boundary case, examples of homeomorphic but nondiffeomorphic 4-manifolds with boundary are much easier to provide: the 4-thickening of the polyhedron 1.10 of Figure 8 equipped with gleams $(-1,1,2)$ (using the notation of Section 3.2.2) admits a nondiffeomorphic model having a special shadow with 3-vertices (see [Gompf and Stipsicz 99, Theorem 11.4.8]). Hence in particular, an upper estimate for the answer to the above question is 3 even in the case of special complexity.

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