Drawing Bers Embeddings of the Teichmüller Space of Once-Punctured Tori

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CONTENTS

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We present a computer-oriented method of producing pictures of Bers embeddings of the Teichmüller space of once-punctured tori. The coordinate plane is chosen in such a way that the accessory parameter is hidden in the relative position of the origin. Our algorithm consists of two steps. For each point in the coordinate plane, we first compute the corresponding monodromy representation by numerical integration along certain loops. Then we decide whether the representation is discrete by applying Jørgensen's theory on the quasi-Fuchsian space of once-punctured tori.

1. INTRODUCTION

Let Γ be a Fuchsian group acting on the unit disk $\mathbb{D}$ uniformizing a Riemann surface, and $B_2(\mathbb{D}, \Gamma)$ the complex Banach space of holomorphic quadratic differentials for Γ on $\mathbb{D}$ with finite norm. It is well known that the Teichmüller space $T(\Gamma)$ of Γ can be realized as a bounded contractible open set in $B_2(\mathbb{D}, \Gamma)$ through the Bers embedding. Throughout the paper, the space $T(\Gamma)$ is understood as the image of the Bers embedding.

In 1972, Bers wrote [Bers 72, page 278], with the notation changed to conform with ours, "Unfortunately, there is no known method to decide whether a given $\phi \in B_2(\mathbb{D}, \Gamma)$ belongs to $T(\Gamma)$. This is so even if $d = \dim B_2(\mathbb{D}, \Gamma) < \infty$. Even the case $d = 1$ is intractable."

In what follows, we will assume that the quotient Riemann surface $\mathbb{D}/\Gamma$ is a once-punctured torus T so that the Teichmüller space $T(\Gamma)$ has complex dimension one. In this case, two elements $\alpha, \beta \in \Gamma$ are called *standard generators* if the oriented intersection number $i(\alpha, \beta)$ in D/Γ with respect to the orientation coming from the complex structure of D is equal to $+1$.

In this paper, we provide an algorithm for producing the picture of $T(\Gamma)$, or more generally, the "discreteness locus" concerning the holonomy representations in $B_2(\mathbb{D}, \Gamma)$, present the pictures of $T(\Gamma)$ in $B_2(\mathbb{D}, \Gamma)$ for

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several Γ's, and explain our algorithm for producing such pictures. We then describe our experiments concerning an open problem posed by C. McMullen [McMullen 96] on the self-similarity of Bers slices.

To describe the idea of the algorithm, let us recall some basic facts in Teichmüller theory. For every ϕ in $B_2(\mathbb{D}, \Gamma)$, there exists a locally univalent meromorphic function f_ϕ on $\mathbb{D}$ with $\{f_\phi, z\} = \phi(z)$, where $\{f, \cdot\}$ is the Schwarzian derivative of f. The function f_ϕ is called a developing map of ϕ and is unique up to postcomposition by Möbius transformations. The homomorphism $\theta_\phi : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$ defined by

$$f_\phi \circ \gamma = \theta_\phi(\gamma) \circ f_\phi, \quad \gamma \in \Gamma,$$

is called the *holonomy representation* of Γ associated with $\phi \in B_2(\mathbb{D}, \Gamma)$ and is unique up to Möbius conjugacy. Note that this homomorphism θ_ϕ is *type-preserving* in the sense that $\mathrm{tr}[\theta_\phi(\alpha), \theta_\phi(\beta)] = -2$ for any standard generators α, β of Γ. We consider the set $K(\Gamma)$, which is defined as the set of all $\phi \in B_2(\mathbb{D}, \Gamma)$ for which $\theta_\phi(\Gamma)$ is discrete in $\mathrm{PSL}(2, \mathbb{C})$, i.e., $\theta_\phi(\Gamma)$ is a Kleinian group. Then $T(\Gamma)$ is equal to the component of $\mathrm{Int}\, K(\Gamma)$ containing the origin [Shiga 87]. The other components of $\mathrm{Int}\, K(\Gamma)$ other than $T(\Gamma)$ are called *exotic*. Already in 1969, Maskit [Maskit 69] pointed out the existence of exotic components, and in recent years, many authors have been studying the structure of the set $K(\Gamma)$ (see, for instance, [Shiga and Tanigawa 99], [Tanigawa 99], [Ito 00], and [Miyachi 03]). Though Goldman [Goldman 87] succeeded in enumerating all the components of $\mathrm{Int}\, K(\Gamma)$ in terms of integral measured foliations, the shape and configuration of these components are still unclear.

We actually draw the picture of $K(\Gamma)$ in $B_2(\mathbb{D}, \Gamma)$ for the given group Γ. The algorithm involves the following two steps: For each element ϕ in $B_2(\mathbb{D}, \Gamma) \cong \mathbb{C}$:

Step 1. Compute the holonomy representation θ_ϕ.

Step 2. Decide whether the image $\theta_\phi(\Gamma)$ is discrete in $\mathrm{PSL}(2, \mathbb{C})$.

These steps will be described in the following sections.

Remark 1.1. The first and second named authors proposed a different approach for drawing pictures of the Bers embedding in [Komori and Sugawa 04]. One can find a numerical method that enables us to present:

(i) the image of the holonomy representation corresponding to a given cusp boundary point,

(ii) the generators of a Fuchsian group uniformizing a given once-punctured torus,

(iii) values of the accessory parameter (see Section 2.2), and

(iv) pictures of pleating loci.

On the other hand, the present approach has the following merits:

1. We do not have to calculate the accessory parameter to get the picture.

2. We can draw the pictures of exotic components besides the Bers slice.

Remark 1.2. Our definition of the (Bers embedded) Teichmüller space is different from the standard one. In the standard definition, our space $T(\Gamma)$ is the Teichmüller space of the surface $\mathbb{D}^*/\Gamma$, the mirror image of $\mathbb{D}/\Gamma$, where $\mathbb{D}^*$ is the exterior of the unit disk $\mathbb{D}$.

2. HOLONOMY REPRESENTATION

In this section we will describe an algorithm that takes an element ϕ of $B_2(\mathbb{D}, \Gamma)$ as the input and returns a holonomy representation θ_ϕ as the output. To make our calculation easier, we will work with a four-times-punctured sphere. For a detailed exposition, see [Komori and Sugawa 04].

2.1 Commensurability Relations

Fix a pair of standard generators (α, β) of Γ. Then the once-punctured torus T admits an intermediate covering space, the complex plane $\mathbb{C}$ minus lattice points $L_\tau = \{m + n\tau; m, n \in \mathbb{Z}\}$, so that α and β correspond to the generators

$$z \to z + 1, \quad z \to z + \tau$$

for L_τ, where τ is a complex number with $\mathrm{Im}\, \tau > 0$.

We observe that the mapping $z + L_\tau \mapsto 2z + L_\tau$ induces an unbranched covering of the four-times-punctured torus $\widetilde{T} = (\mathbb{C} - \frac{1}{2}L_\tau)/L_\tau$ onto T. We now choose a four-times-punctured sphere $S = \widehat{\mathbb{C}} - \{0, 1, \infty, \lambda\}$ so that T and S have the common covering space $\widetilde{T}$. Set $e_1 = \wp(1/2)$, $e_2 = \wp(\tau/2)$, $e_3 = \wp((1 + \tau)/2)$, and

$$\lambda = \frac{e_3 - e_2}{e_1 - e_2},$$

where $\wp$ is the Weierstrass $\wp$-function with period lattice L_τ. Then a covering projection π of $\widetilde{T}$ onto S is given by

$$\pi(z + L_\tau) = \frac{\wp(z) - e_2}{e_1 - e_2}.$$

Note that $\lambda = \lambda(\tau)$ is known to be an elliptic modular function.

The canonical projection $\mathbb{D} \to \mathbb{D}/\Gamma = T$ induces the universal cover $\tilde{q} : \mathbb{D} \to \widetilde{T}$. Let Γ_S be the covering group of the universal covering projection $p = \pi \circ \tilde{q}$ of $\mathbb{D}$ onto S. Note that we have $B_2(\mathbb{D}, \Gamma_S) = B_2(\mathbb{D}, \Gamma)$ (see [Komori and Sugawa 04]).

Let $B_2(S)$ be the Banach space of (hyperbolically) bounded holomorphic quadratic differentials on S. By definition, the spaces $B_2(\mathbb{D}, \Gamma_S)$ and $B_2(S)$ are isomorphic via the pullback $p^* : B_2(S) \to B_2(\mathbb{D}, \Gamma_S)$ defined by $p^*\psi = \psi \circ p \cdot (p')^2$. The rational function

$$\psi_0(z) = \frac{1}{z(z-1)(z-\lambda)} \tag{2–1}$$

gives a nontrivial bounded quadratic differential $\psi_0(z)dz^2$, which forms a basis of the Banach space $B_2(S)$ since $\dim B_2(S) = 1$. Therefore each element $\phi \in B_2(\mathbb{D}, \Gamma) = B_2(\mathbb{D}, \Gamma_S)$ can be written as $\phi = t\phi_0$, where t is a complex number and $\phi_0 = p^*(\psi_0)$.

2.2 The Monodromy of a Four-Times-Punctured Sphere

Now, for each $\phi = t\phi_0$, consider the developing map $f_\phi : \mathbb{D} \to \widehat{\mathbb{C}}$. Our idea is to compute f_ϕ on S instead of $\mathbb{D}$.

For this purpose, we take the branch P of p^{-1} around $p(0)$ so that $P(p(0)) = 0$ and put $g(z) := f_\phi(P(z))$. Then we have

$$\{g, z\} = \{f_\phi, P(z)\}(P'(z))^2 + \{P, z\} = t\psi_0(z) + \{P, z\}. \tag{2–2}$$

For $\{P, z\}$ in (2–2) we use the next lemma:

Lemma 2.1. [Forsyth 02, Ch. X, p. 492] *The Schwarzian derivative of P is of the form*

$$\{P, z\} = \frac{1}{2z^2} + \frac{(1-\lambda)^2}{2(z-1)^2(z-\lambda)^2} + \frac{c(\lambda)}{z(z-1)(z-\lambda)} \tag{2–3}$$

on S, where $c(\lambda)$ is a constant determined by λ and called the accessory parameter.

By the above lemma and (2–2), $\{g, z\}$ is globally defined on $\widehat{\mathbb{C}} - \{0, 1, \infty, \lambda\}$. Combining (2–1), (2–2), and (2–3), the equation to solve is

$$2y'' + \left(\frac{1}{2z^2} + \frac{(1-\lambda)^2}{2(z-1)^2(z-\lambda)^2} + \frac{t'}{z(z-1)(z-\lambda)}\right) y = 0, \tag{2–4}$$

where we have set $t' = t + c(\lambda)$. As is well known, $\{y_1/y_0, z\} = \{g, z\}$. Hence, $f_\varphi = M \circ (y_1/y_0) \circ p$ around the origin for some Möbius transformation M.

We now describe how to compute the monodromy. Let γ_S be an element of Γ_S. We start with the pair (y_0, y_1) of fundamental solutions of (2–4) determined by the initial conditions

$$y_0(z_0) = 1, \qquad y_0'(z_0) = 0,$$

and

$$y_1(z_0) = 0, \qquad y_1'(z_0) = 1,$$

at a fixed point z_0 in S. Then we continue them analytically along a closed path in S corresponding to γ_S. Returning to the starting point, we arrive at a new pair of solutions (Y_0, Y_1). However, these new solutions must be linear combinations of the original solutions. Thus we have

$$Y_0 = Dy_0 + Cy_1, \quad Y_1 = By_0 + Ay_1,$$

for some complex numbers A, B, C, and D. We define

$$\tilde{\theta}_\psi(\gamma_S) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$$

for each $\gamma_S \in \Gamma_S$. We note that by the monodromy theorem, the matrix is independent of the particular choice of the path corresponding to γ_S.

Since $f_\phi \circ \gamma_S$ corresponds to

$$\frac{Y_1}{Y_0} = \frac{A(y_1/y_0) + B}{C(y_1/y_0) + D},$$

we obtain the following lemma.

Lemma 2.2. *The monodromies θ_ϕ and $\tilde{\theta}_\psi$ are essentially the same. More precisely, on $\Gamma \cap \Gamma_S$, θ_ϕ is equal to the* $\mathrm{PSL}(2, \mathbb{C})$ *representation induced by $\tilde{\theta}_\psi$ up to Möbius conjugacy.*

So we can calculate θ_ϕ on S by means of (2–4). The reader can find a reason why the holonomy representation of Γ_S takes the values in $\mathrm{SL}(2, \mathbb{C})$ in [Komori and Sugawa 04, Remark 4.1].

2.3 Markov Triples

Though α and β are in Γ, they are not in Γ_S, on which $\tilde{\theta}_\psi$ is defined. In other words, α and β do not correspond to the closed paths in S. So we need a little more calculation to end this section.

Let A and B be the matrices in $\mathrm{SL}(2, \mathbb{C})$ such that $\pm A = \theta_\phi(\alpha)$ and $\pm B = \theta_\phi(\beta)$ in $\mathrm{PSL}(2, \mathbb{C})$. Set $x = \operatorname{tr} A$, $y = \operatorname{tr} B$, and $z = \operatorname{tr} AB$. The triple (x, y, z) is well defined up to changing the signs of any two entries. It determines θ_ϕ up to conjugacy in $\mathrm{PSL}(2, \mathbb{C})$. In the next section, this holonomy is represented using Jørgensen's normalization and denoted by $\theta_{x,y,z}$. Since our homomorphism is type-preserving, the well-known trace identity $2 + \operatorname{tr}[X, Y] = (\operatorname{tr} X)^2 + (\operatorname{tr} Y)^2 + (\operatorname{tr} XY)^2 - \operatorname{tr} X \operatorname{tr} Y \operatorname{tr} XY$ implies the relation

$$x^2 + y^2 + z^2 = xyz. \tag{2–5}$$

Conversely, given any triple (x, y, z) satisfying (2–5), we can reconstruct the image of the group Γ up to conjugacy. We call such a triple of complex numbers a *Markov triple*.

Thus it suffices to compute x and y. Again by the trace identity $\operatorname{tr} X \operatorname{tr} Y = \operatorname{tr} XY + \operatorname{tr} XY^{-1}$, we have

$$x = \sqrt{\operatorname{tr} \tilde{\theta}_\psi(\alpha^2) + 2}, \qquad y = \sqrt{\operatorname{tr} \tilde{\theta}_\psi(\beta^2) + 2}. \tag{2–6}$$

Now we can calculate $\tilde{\theta}_\psi(\alpha^2)$ and $\tilde{\theta}_\psi(\beta^2)$ by solving equation (2–4) because α^2 and β^2 are in Γ_S.

Let us summarize the algorithm in this section. The inputs are $\lambda \in \mathbb{C}$ to specify Γ and $t \in \mathbb{C}$ to specify $\phi \in B_2(\mathbb{D}, \Gamma)$. We solve equation (2–4) numerically to get $\tilde{\theta}_\psi(\alpha^2)$ and $\tilde{\theta}_\psi(\beta^2)$. Using equation (2–6) and equation (2–5), we calculate and return the Markov triple.

2.4 Technical Remarks

The simple loops in S separating $\{0, 1\}$ from $\{\infty, \lambda\}$ and separating $\{0, \lambda\}$ from $\{1, \infty\}$ have two intersection points and correspond to α^2 and β^2, respectively, with suitably chosen orientations. Practically, we choose polygonal curves with a common endpoint as such loops. For each oriented line segment of such curves, we solve the differential equation (2–4) numerically and find the transition matrix of it along the segment. Then the ordered products of the transition matrices corresponding to the polygonal curves are representatives of α^2 and β^2 in $\mathrm{SL}(2, \mathbb{C})$ (see [Komori and Sugawa 04] for details). Here, we may think of a value of the parameter t' as being given in (2–4) instead of t, so that we do not care about the value of the accessory parameter $c(\lambda)$.

3. JØRGENSEN'S THEORY TO DECIDE DISCRETENESS

The input of the algorithm of this section is a Markov triple and the output is the answer "discrete," "indiscrete," or "undecided."

The general idea is to try to construct a Ford fundamental region of the given Markov triple. If the image of the corresponding holonomy representation is indiscrete, the term "Ford fundamental region" does not make sense and our process of constructing it will fail. Then we will search for evidence of its indiscreteness.

This algorithm is based on Jørgensen's theory of once-punctured tori [Jørgensen 03]. An exposition of this theory with proofs is in preparation [Akiyoshi et al., to appear]. This algorithm may not halt in finite time for some inputs. For example, if $\mathbb{H}^3/\theta_{x,y,z}(\Gamma)$ is geometrically infinite or a $\mathbb{Z}$-covering space of a punctured torus bundle over the circle, our algorithm will not stop in finite time. In practice, we will stop our calculation at a certain time and give the answer "undecided."

3.1 Notation

Let T be a once-punctured torus. We fix standard generators α, β of the fundamental group of T. Let θ be a type-preserving $\mathrm{PSL}(2, \mathbb{C})$ homomorphism of $\pi_1(T)$. By taking a lift to $\mathrm{SL}(2, \mathbb{C})$, θ can be specified by the Markov triple $x = \operatorname{tr} \theta(\alpha)$, $y = \operatorname{tr} \theta(\alpha\beta)$, and $z = \operatorname{tr} \theta(\beta)$ up to conjugation in $\mathrm{PSL}(2, \mathbb{C})$. The Markov triple is then well defined up to simultaneous change of signs in a pair of elements in the triple. We denote this representation by $\theta_{x,y,z}$.

Recall that a *slope* in T is the isotopy class of an essential simple closed curve on T. By choosing a basis of $H_1(T; \mathbb{Z})$, a slope is represented by a number in $\mathbb{Q} \cup \{1/0 = \infty\}$. To fix our notation, we choose α and β as the basis so that the slope of α and β are $1/0$ and $0/1$ respectively. For a slope $q \in \mathbb{Q} \cup \{1/0\}$, set $S_q = \{g \in \pi_1(T) \mid \text{slope of } g = q\}$. Note that $\alpha \in S_{1/0}$, $\beta \in S_{0/1}$, and $\alpha\beta \in S_{1/1}$. We identify the set of slopes as a subset of $\partial\mathbb{H}^2$. Two rational numbers p/q and r/s are *Farey neighbors* if $|ps - qr| = 1$. By joining all pairs of Farey neighbors by geodesics, we get the *Farey tessellation* of $\mathbb{H}^2$ by ideal triangles. Note that the slopes of α, β, and $\alpha\beta$ form an ideal triangle of the above tessellation. By taking the dual graph of this triangulation, we have a trivalent graph Σ properly embedded in $\mathbb{H}^2$. With each vertex v in Σ we can associate a subset S_v of $\pi_1(T)$ by

$$S_v = S_{q_1} \cup S_{q_2} \cup S_{q_3},$$

where slopes $q_1, q_2, q_3 \in \mathbb{Q} \cup \{\infty\}$ are the ideal vertices of the triangle in the Farey tessellation that is dual to v. Set $I_v = \{\text{isometric hemisphere of } g \mid g \in S_v\}$.

Jørgensen's theory of punctured tori claims that if the image of the holonomy representation $\theta_{x,y,z}$ is discrete, then there is a path P in Σ that depends on (x, y, z) such that the boundary of the Ford region is given by $\bigcup_{v \in P} I_v$. After the Jørgensen normalization, which will be introduced in Section 3.2, we can define an "upward" and a "downward" direction in P. We will say that some vertex $v' \in P$ is the upper (lower) neighbor of $v \in P$

if v' is adjacent to v and the direction from v to v' is upward (downward). We will also use terms like "upper endpoint" and "lower endpoint" of P for the endpoints of P.

In the next subsection, we recall Jørgensen's description. It describes the Ford region for a given discrete representation $\theta_{x,y,z}(\Gamma)$ with $v_0 \in P$, where $v_0 \in \Sigma$ is the dual of 1/0, 0/1, and 1/1. After this subsection, we will describe our algorithm.

3.2 Jørgensen's Description of the Ford Region

The Ford region of $\theta_{x,y,z}$ is defined (if the image of $\theta_{x,y,z}$ is discrete) to be the set of points lying above the isometric hemispheres of all elements in $\theta_{x,y,z}(\Gamma)$ not fixing ∞. Recall that the isometric hemisphere $I(A)$ for $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{C})$ with $A(\infty) \neq \infty$ is the hemisphere in $\mathbb{H}^3$ with radius $1/|c|$ centered at $-d/c \in \mathbb{C} = \partial\mathbb{H}^3 - \{\infty\}$. In order to obtain a fundamental region for $\theta_{x,y,z}(\Gamma)$, we have to take the intersection of this Ford region with some fundamental region for the stabilizer of ∞.

Now let (x, y, z) be a Markov triple. We can reconstruct $\theta_{x,y,z}$ using Jørgensen's normalization [Jørgensen 03]:

$$\theta_{x,y,z}(\alpha) = \frac{1}{x}\begin{pmatrix} xy - z & y/x \\ xy & z \end{pmatrix}, \tag{3–1}$$

$$\theta_{x,y,z}(\beta) = \frac{1}{x}\begin{pmatrix} xz - y & -z/x \\ -xz & y \end{pmatrix}.$$

Then we can check that

$$\theta_{x,y,z}(\alpha\beta) = \begin{pmatrix} x & -1/x \\ x & 0 \end{pmatrix}, \tag{3–2}$$

$$\theta_{x,y,z}(K) = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix},$$

where $K = [\alpha, \beta]$. The isometric hemispheres of α, $\alpha\beta$, and β are centered at $-z/xy$, 0, and y/zx with radii $1/y$, $1/x$, and $1/z$ respectively. It is easy to see that the isometric hemispheres of α^{-1}, $(\alpha\beta)^{-1}$, and $\beta^{-1}K^{-1}$ are the translated images of the above three hemispheres by $z \mapsto z + 1$. Since $\theta_{x,y,z}(\Gamma)$ contains the action $\theta_{x,y,z}(K)$ of translation $z \mapsto z + 2$, we have a bi-infinite sequence of translated images of the above three isometric hemispheres with symmetry of translation by one. Thus, we have a sequence of isometric hemispheres

$$\begin{aligned} \ldots,\ & I_{-4} = I(\alpha^{-1}K),\ I_{-3} = I((\alpha\beta)^{-1}K), \\ & I_{-2} = I(\beta^{-1}),\ I_{-1} = I(\alpha),\ I_0 = I(\alpha\beta), \\ & I_1 = I(\beta),\ I_2 = I(\alpha^{-1}),\ I_3 = I((\alpha\beta)^{-1}), \\ & I_4 = I(\beta^{-1}K^{-1}),\ I_5 = I(\alpha K^{-1}),\ \ldots . \end{aligned}$$

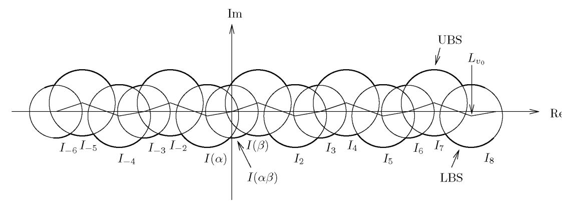

FIGURE 1. Isometric hemispheres.

See Figure 1. Note that $I_{n+3} = \sqrt{\theta_{x,y,z}(K)}(I_n)$ for any $n \in \mathbb{Z}$, where $\sqrt{\theta_{x,y,z}(K)}$ is the translation $z \mapsto z+1$. The group elements that correspond to I_{3n}, I_{3n+1}, and I_{3n+2} belong to $S_{\alpha\beta}$, S_β, and S_α respectively. Set $I_{1/1} := \{I_{3n}\}_{n\in\mathbb{Z}}$, $I_{0/1} := \{I_{3n+1}\}_{n\in\mathbb{Z}}$, $I_{1/0} := \{I_{3n+2}\}_{n\in\mathbb{Z}}$. As a set, $\{I_n\}_{n\in\mathbb{Z}}$ is equal to I_{v_0}. We denote by L_{v_0} the polyline of infinite length given by connecting the centers of I_n and I_{n+1} for each $n \in \mathbb{Z}$.

Since we made the assumption $v_0 \in P$ at the end of Section 3.1, the following assertions follow from Jørgensen's theory:

(C1) Consecutive isometric hemispheres intersect with each other.

(C2) The polyline L_{v_0} has no self-intersection.

So we have two sequences of subarcs of $\partial I_n \subset \mathbb{C}$: the upper boundary sequence UBS and the lower boundary sequence LBS. See Figure 1.

For UBS and LBS, the set of subarcs can be divided into three groups: those that come from $I_{1/0}$, from $I_{0/1}$, and from $I_{1/1}$. Let us consider UBS. We have three cases:

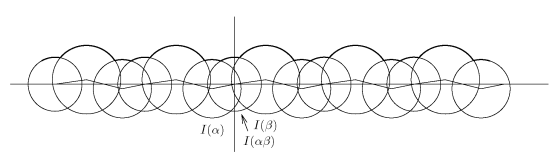

FIGURE 2. Case (S1).

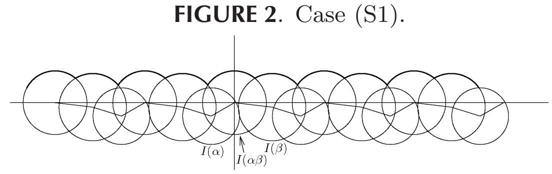

FIGURE 3. Case (S2).

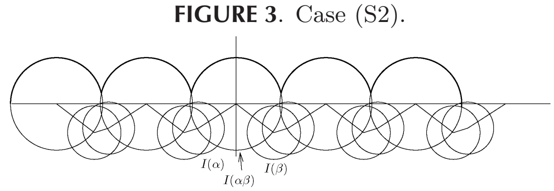

FIGURE 4. Case (S3).

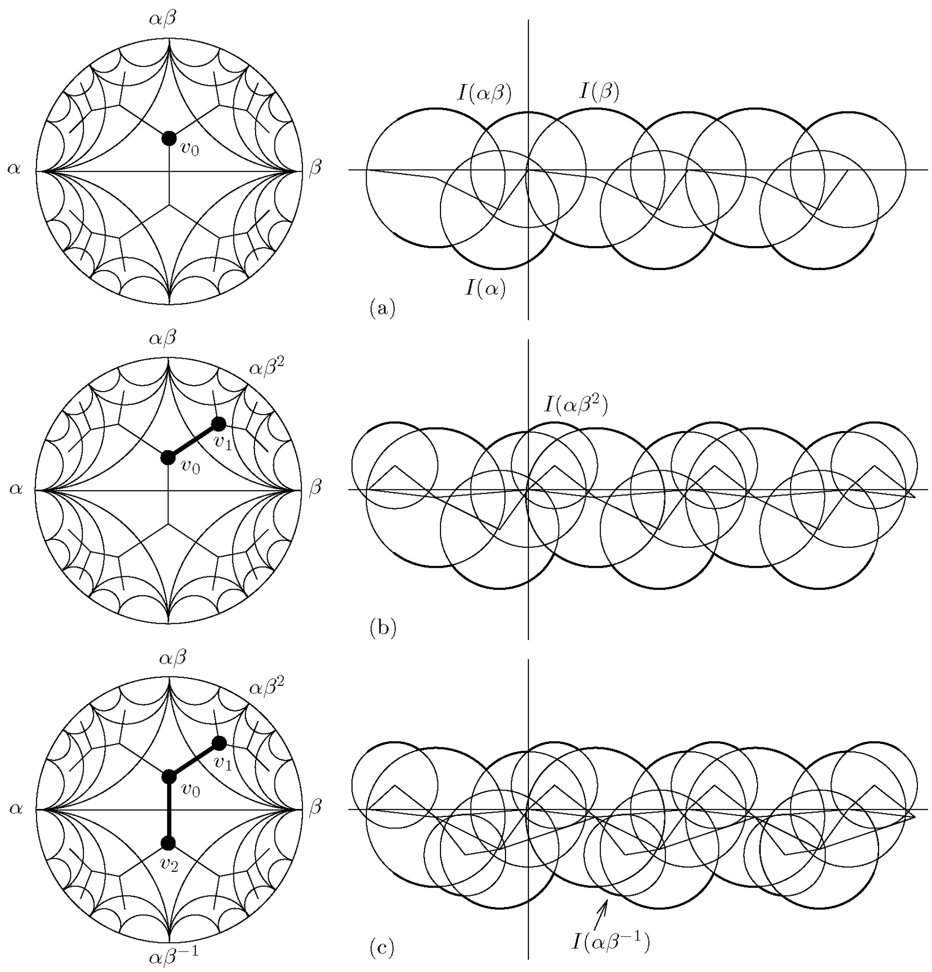

FIGURE 5. Markov triple $(x, y, z) = (2.536 - 1.115i, 2.616 - 0.645i, 2.203 + 0.660i)$. Left: Farey diagram and its dual graph Σ. Right: isometric hemispheres in upper-half-space model. (a) v_0: starting point (dual of $\triangle 01\infty$) (b) v_1: the upper neighbor of v_0, top endpoint since it is of (S1) for UBS. (c) v_2: lower neighbor of v_0, lower endpoint since it is of (S1) for LBS. $v_2v_0v_1$: Jørgensen's path. We conclude that $\theta_{(x,y,z)}$ is discrete.

(S1) All the groups of subarcs $I_{1/0}, I_{0/1}$, and $I_{1/1}$ appear in the sequence (Figure 2).

(S2) Only two groups of subarcs appear in the sequence and one group, say $I_{1/0}$, does not (Figure 3).

(S3) Only one group, say $I_{0/1}$, appears in the sequence (Figure 4).

The method to find the upper neighbor and decide whether it is an upper endpoint is as follows: in case (S1), v_0 is the upper endpoint and there is no upper neighbor vertex for v_0. Next, suppose that UBS is of case (S2), and for the Farey triangle $\triangle q_1q_2q_3$ that is dual to v_0, only the slope q_1 does not appear in UBS. There is a unique Farey triangle $\triangle q_2q_3q_4$ that is adjacent to $\triangle q_1q_2q_3$ along the geodesic connecting q_2 and q_3, and let v' be the dual vertex of it. Then v' is the upper neighbor of v_0. In the case of (S3), there are two possible choices for the upper neighbor; the choice is given in [Wada].

For LBS and lower neighbor, the rule is the same.

For example, Figure 5 (a) depicts the case in which both UBS and LBS of v_0 are of case (S2). The left-hand figure is the Farey diagram, and its dual graph Σ. The right-hand figure is a picture of isometric hemispheres I_{v_0}. Note that $I(\alpha)$ does not belong to UBS, and the slope of α is $1/0$. In this case, v_1, which is the dual to the Farey triangle $0/1$, $1/1$, and $1/2$, is the upper neighbor of v_0. See Figure 5 (b).

Since UBS of v_1 is of case (S1), it is the upper endpoint.

If we carry out the same process for the downward direction, we reach the vertex v_2 in Figure 5 (c), which turns out to be the lower endpoint. In this case, we conclude that the Jørgensen path P is $v_1v_0v_2$.

3.3 The Algorithm

In this subsection we discuss the algorithm. Here we do not assume that $v_0 \in P$. We also consider a condition for indiscreteness not mentioned in the previous subsection.

Starting from v_0, we search Σ for Jørgensen's path. If we arrive at a new vertex in Σ, we get a new slope $q \in \mathbb{Q} \cup \{1/0\}$. Then we check the Shimizu–Leutbecher lemma below for the elements of S_q, and say "indiscrete" and stop the calculation if the condition is satisfied.

Lemma 3.1. (Shimizu-Leutbecher.) *Suppose that a subgroup* Γ *of* $\mathrm{SL}(2,\mathbb{C})$ *contains* $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$*. If there exists an element* $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in \Gamma$ *with* $0 < |c| < 1$*, then* Γ *is indiscrete.*

Since the radius of the isometric hemisphere for $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ is $1/|c|$, it follows that, in our setting, if there exists an isometric hemisphere of radius greater than 1, then the group is indiscrete.

After starting from v_0, our first task is to search for a vertex that satisfies the condition (C1). For $v \in \Sigma$, where $v =$ dual of $\triangle q_1q_2q_3$, a simple calculation shows that (C1) is equivalent to the triangle inequality for $|\tau(q_1)|$, $|\tau(q_2)|$, and $|\tau(q_3)|$, where $\tau(q) := \operatorname{tr} g$ with $g \in S_q$. So if v fails to satisfy (C1), one real number, say $|\tau(q_1)|$, is too large. Then we move to the adjacent vertex v', which is the dual to the Farey triangle of q_2, q_3 and the new slope; i.e., we discard the slope q_1. We repeat this process to find a vertex that satisfies (C1).

We remark that we don't know whether this process always terminates in finite time. In our implementation of the algorithm, we fix a large number, and if the number of iterations exceeds this limit, we give up our calculation trying to construct the Ford region and search for evidence of its indiscreteness.

Now suppose that we have found a vertex with (C1) satisfied. Next, we keep moving to the upper neighbor defined by the rule in the previous subsection until we must stop at some vertex v. Here we have the same remark as in the previous paragraph. We don't know whether this process always terminates in finite time. We fix a large number, and when the number of iterations exceeds this limit, we stop the process and search for evidence of its indiscreteness. We call this vertex v an *uppermost vertex*, and we have two cases for v:

(U1) We stop because of case (S1).

(U2) We stop because v fails to satisfy (C2). (In this case, UBS and LBS are not well defined because we have used the condition (C2) to define UBS and LBS.)

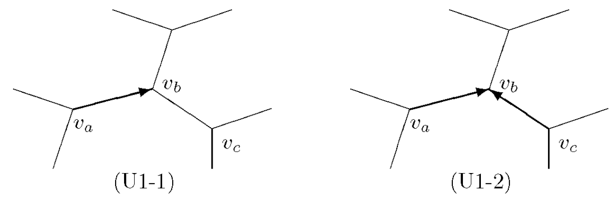

FIGURE 6. Case (U1). Suppose that during the process, we have moved upward in Σ from v_a to v_b, which turns out to be an uppermost vertex. Case (U1-1) For v_b, v_a is the lower neighbor. Case (U1-2) For v_b, v_c is the lower neighbor. (The direction of the arrows is from lower vertex to upper vertex.)

For later purposes, we define two subcases in case (U1). See Figure 6:

(U1-1) The lower neighbor vertex of v is where we come from.

(U1-2) The lower neighbor vertex of v is not where we come from.

We define the notions (D1), (D2), (D1-1), and (D1-2) for LBS in the same way. In the case (U1), we change our direction and start moving to the lower neighbor. In the case (U2), we move to a neighbor by the rule we established heuristically and consider this direction as "lower" and start moving to the lower neighbor.

The rule is as follows: Suppose that the uppermost vertex v that violates the condition (C2) is the dual of $\triangle q_1q_2q_3$, and a part of the polyline L_v is given by connecting the centers of the isometric hemispheres $I(q_1)$, $I(q_2)$, $I(q_3)$, $K'I(q_1)$, where K' is the translation $z \mapsto z+1$, in this order. Suppose also that the segment connecting $I(q_1)$ and $I(q_2)$ intersects the segment connecting $I(q_3)$, $K'I(q_1)$. There is a unique Farey triangle $\triangle q_1q_2q_4$ that is adjacent to $\triangle q_1q_2q_3$ along the geodesic connecting q_1 and q_2. Also, there is a unique Farey triangle $\triangle q_1q_3q_5$

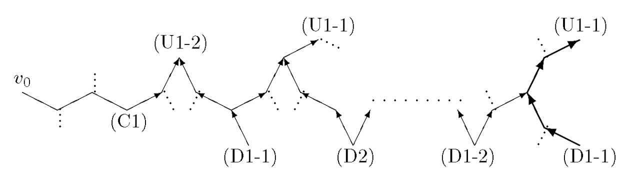

FIGURE 7. An example of the whole process. In Σ, starting from v_0, we search for a vertex with (C1) satisfied. Then we go upward until we reach a vertex at (U1) or (U2), say (U1-2). Then we go downward until (D1) or (D2), say (D1-1). We continue this alternating process of visiting vertices until we can find the Jørgensen path as illustrated with heavy arrows or until some isometric hemisphere corresponding to the vertex that we visit violates the Shimizu–Leutbecher condition.

which is adjacent to $\triangle q_1q_2q_3$ along the geodesic connecting q_1 and q_3. These two triangles are our candidates for the lower neighbor. If the vertex dual of $\triangle q_2q_3q_4$ is the vertex we come from in this upward/downward process, we choose the vertex dual of $\triangle q_1q_3q_5$ as the lower neighbor. If the vertex dual of $\triangle q_1q_3q_5$ is the vertex we come from, we choose the vertex dual of $\triangle q_2q_3q_4$ as the lower neighbor.

In both these cases we keep moving in the direction of the lower neighbor vertex in Σ. For the lowermost vertex, we have the same cases (D1-1), (D1-2), and (D2) as above and again change our direction to move upward. We continue this process for upper and lower directions alternately.

If we can find a path P in Σ such that one end v_U is of case (U1-1) and the other end v_L is of (D1-1) and we can go from v_U to v_L by going downward and from v_L to v_U by going upward, then this is the Jørgensen path P. (See Figure 7.)

In this case, the conditions for the Poincaré fundamental polygon theorem are satisfied, and the output of our algorithm is "discrete." For a detailed discussion of Jørgensen's theory, see [Akiyoshi et al., to appear].

4. PICTURES

In the following pages we present several pictures produced by our method.

In Figure 8, $\lambda = \frac{1}{2}$, and the corresponding once-punctured torus T is the square torus with one point removed. It is known that the accessory parameter $c\left(\frac{1}{2}\right)$ is equal to $\frac{1}{2}$, and we take the center and the range to be $\frac{1}{2}$ and $\pm\frac{1}{4}$ respectively. In the discreteness locus, a color is given according to the length of Jørgensen's path P mentioned in the previous section.

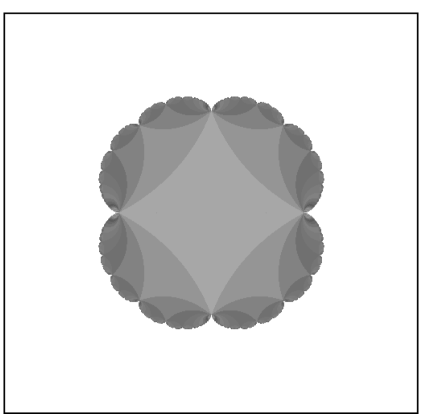

FIGURE 8. $\lambda = \frac{1}{2}$, center$= \frac{1}{2}$, range$= \pm\frac{1}{4}$.

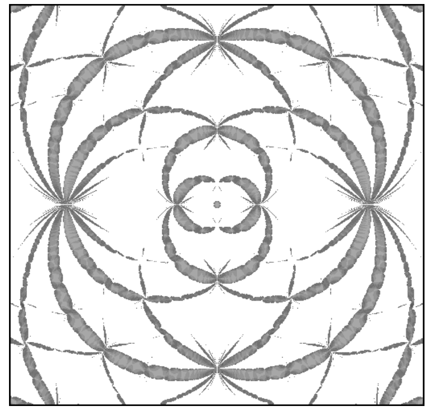

FIGURE 9. $\lambda = \frac{1}{2}$, center$= \frac{1}{2}$, range$= \pm 8$.

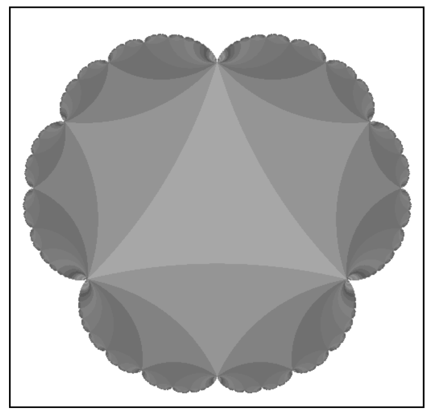

FIGURE 10. $\lambda = \frac{1}{2} + \frac{\sqrt{3}}{2}\,i$, center$= \frac{1}{2} + \frac{1}{2\sqrt{3}}\,i$, range$= \pm\frac{1}{4}$.

Figure 9 is a blowup of Figure 8. Many exotic components appear in this picture.

For Figure 10, $\lambda = \frac{1}{2} + \frac{\sqrt{3}}{2}\,i$ and T is a once-punctured torus with hexagonal symmetry. For the range of the parameter $t + c(\lambda)$, the center is $\frac{1}{2} + \frac{1}{2\sqrt{3}}\,i$ and the range

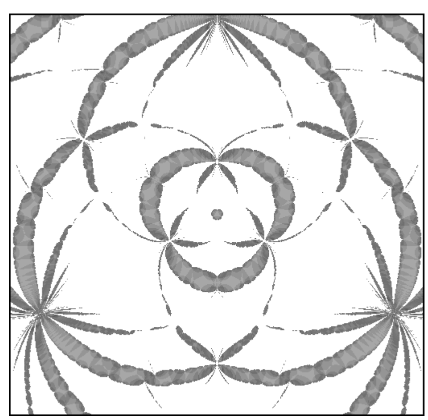

FIGURE 11. $\lambda = \frac{1}{2} + \frac{\sqrt{3}}{2}\,i$, center$= \frac{1}{2} + \frac{1}{2\sqrt{3}}\,i$, range$= \pm 8$.

is $\pm\frac{1}{4}$. Note that to get the picture, we do not have to compute the exact value of the accessory parameter $c(\lambda)$ because it is hidden in the relative position of the origin. Figure 11 is a blowup of Figure 10.

5. AN EXPERIMENT: SELF-SIMILARITY OF A BERS SLICE

In [McMullen 96, p. 178], McMullen asked, "Is the boundary of a Bers slice self-similar?" and carried out a computer experiment for a Maskit slice instead of a Bers slice. His pictures of a part of a Maskit boundary and its blowups suggest an affirmative answer for a Maskit slice. Motivated by his work, we have produced Figures 12, 13, and 14.

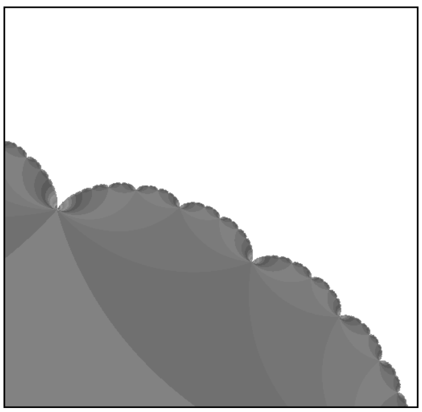

FIGURE 12. center $= 0.569645 + 0.136675i$, range$=\pm 0.0192$.

Figure 12 depicts a part of the Bers slice of a square torus ($\lambda = \frac{1}{2}$). Figures 13 and 14 are the blowups around the limit point $0.569645\ldots + 0.136675\ldots\, i$. Our conclusion is that this part of the boundary appears to have self-similarity around that point with scale factor about

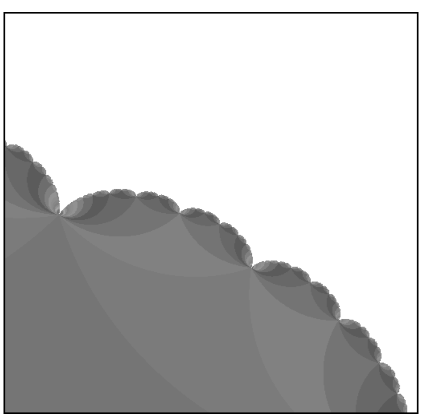

FIGURE 13. center $= 0.569645 + 0.136675\, i$, range$=\pm 0.004$.

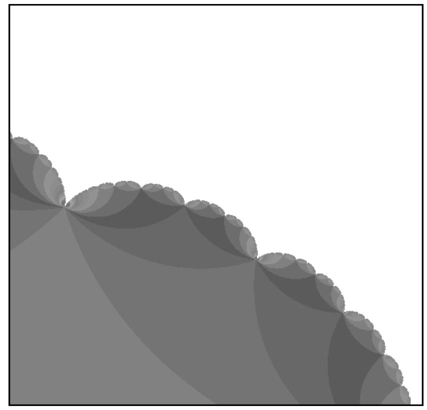

FIGURE 14. center $= 0.569645 + 0.136675\ i$, range$=\pm 0.000833$.

4.8. That point also appears in [Sugawa 02] as the farthest boundary point of the Teichmüller space of the once-punctured square torus from the origin and an observation was made there about the scale factor.

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