Semi-Presentations for the Sporadic Simple Groups

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CONTENTS

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A semi-presentation for a group G is a set of relations which characterises a set of generators of G up to automorphism. We discuss some techniques for finding semi-presentations and illustrate them by exhibiting semi-presentations on standard generators for the 26 sporadic simple groups and their automorphism groups. We then show how these semi-presentations were used to check the data in the World Wide Web Atlas of Group Representations [Wilson et al. 04].

1. INTRODUCTION AND MOTIVATION

Given two groups G and H which are known to be isomorphic, it is often useful to have an explicit isomorphism $\theta : G \to H$ between them. For example, if G is a small degree permutation group and H is a group of matrices of very large dimension, we can use θ to turn difficult calculations in H into easy calculations in G. As another example, suppose G and H are groups of matrices. We can distinguish many conjugacy classes in G by looking at element orders and traces in G, but it is often the case that there are nonconjugate elements $t, u \in G$ satisfying $o(t) = o(u)$ and $\mathrm{tr}(t) = \mathrm{tr}(u)$. If, however, $\mathrm{tr}(\theta(t)) \neq \mathrm{tr}(\theta(u))$ then we know t and u cannot be G-conjugate. We can therefore distinguish more conjugacy classes than we initially thought.

For this purpose, [Wilson 96] introduced the concept of *standard generators* for a group G, i.e., an n-tuple $(x_1, \ldots x_n)$ of generators which could be specified up to group automorphisms by properties $R_j(x_1, \ldots x_n)$ $(1 \leq j \leq m)$ independent of the representation. Typically, the properties R_j would include the conjugacy classes of each of the x_i and the orders of certain short words in the x_i.

Example 1.1. Standard generators of the Mathieu group $G = \mathrm{M}_{24}$ are elements x and y where x is in class $2B$, y is in class $3A$, xy has order 23 and $xyxyxy^2xyxy^2xy^2$ has order 4. (We will follow ATLAS [Conway et al. 85] conventions for naming groups and conjugacy classes throughout.) Thus if x', y' are elements of $H \cong \mathrm{M}_{24}$ with these properties, there is an isomorphism $\theta : G \to H$ induced by $x \mapsto x'$, $y \mapsto y'$. If we write elements of G as words in x and y, then the corresponding element in H is the same word with primed letters replacing unprimed ones.

For each group G with standard generators defined, the properties R_j have been chosen so that in any group of the right isomorphism type, it is relatively easy to find an n-tuple of standard generators. In this paper, we will consider the inverse problem of checking whether a given n-tuple of elements of G forms an n-tuple of standard generators of G. Note that this is not a trivial problem in general: the properties R_j specify the conjugacy classes of each of the elements of the n-tuple, and in some representations it may be difficult to tell apart conjugacy classes containing elements of the same order. We can solve this problem by providing characterisations of standard generators of G which only specify the orders of group elements. We will call such characterisations *semi-presentations*, and we will give a formal definition in Section 2.

Our main motivation for considering this problem is so that we can check data integrity in the World Wide Web Atlas of Group Representations [Wilson et al. 04, Wilson 98]. The Web Atlas includes explicit matrices for standard generators of hundreds of different groups in thousands of representations. With a project of this size, it is very likely that mistakes will occur, and we wanted an automated way of checking the representations in the Atlas. We do this by converting each semi-presentation to a black box algorithm which can determine whether or not a pair of elements of G is a pair of standard generators; this is straightforward. This black box algorithm can be applied to each representation of G in the Web Atlas to check whether the correct generators have been supplied. Once the semi-presentation has been found and converted to a black box algorithm, a computer can easily be programmed to apply it to all the representations for a given group and to report any representations that failed.

The main part of this paper is a collection of semi-presentations for the sporadic simple groups (Section 4) and their automorphism groups (Section 5) on standard generators. (These semi-presentations can now be found in the Web Atlas in the form of black box algorithms.) In Section 6 we summarise the results of applying these semi-presentations to the representations given in the Web Atlas and show how some errors were uncovered.

2. SEMI-PRESENTATIONS

Before giving a formal definition of semi-presentations, we will give an example. Consider the simple Tits group $G = {}^2F_4(2)'$ of order 17971200. A pair of elements $x, y \in G$ is a pair of standard generators if x is in class $2A$, y has order 3 and xy has order 13. Call a pair (x, y) *allowable* if $o(x) = 2$, $o(y) = 3$ and $o(xy) = 13$. Not all allowable pairs are pairs of standard generators, because x could be an element of order 2 which is not in class $2A$. To prove that a given allowable pair is actually a pair of standard generators, we use the following facts about G (which can be derived from the character table and power maps of G [Conway et al. 85]):

1. The group G has exactly two conjugacy classes of elements of order 2.
2. If $a, b \in G$ are both in class $2B$, then ab must have order 1, 2, 3, 4, 5, 6, 8, 12 or 13.
3. If c is an element of order 6 or 12 in G, then c powers up to an involution in class $2B$.

Let x and y be standard generators of G. We can check (for example, by using the 26-dimensional representation of G over GF(2)) that the element $z = xy^2xyxy$ has order 12, so by Fact 3, z^6 is in class $2B$. We can also check that xz^6 has order 10, so by Fact 2, x is not in class $2B$, and by Fact 1, it must be in class $2A$.

Thus we have found some conditions on the orders of words in $x, y \in G$ which prove that x and y are standard generators. These conditions can be tested in any representation of G, regardless of whether we can distinguish conjugacy classes. These conditions form a semi-presentation for G on its standard generators, and we denote them with double angled brackets $\langle\langle \quad \rangle\rangle$. In our case, we write the semi-presentation as

$$\langle\langle x, y, (z) \mid o(x) = 2, o(y) = 3, o(xy) = 13,\\ o(z) = 12, o(xz^6) = 10; z := xy^2xyxy\rangle\rangle,$$

which should be thought of as a more readable form of

$$\langle\langle x, y \mid o(x) = 2, o(y) = 3, o(xy) = 13,\\ o(xy^2xyxy) = 12, o(x(xy^2xyxy)^6) = 10\rangle\rangle.$$

The formal definition is as follows:

Definition 2.1. A *semi-presentation for G* on an n-tuple $(g_1, \dots g_n)$ of generators of G is a finite k-tuple of words $w_i(x_1, \dots x_n)$, $1 \le i \le k$, together with a finite k-tuple of positive integers o_i, $1 \le i \le k$ such that:

1. $o(w_i(g_1, \dots g_n)) = o_i$ for $1 \le i \le k$; and
2. for each n-tuple $h = (h_1, \dots h_n)$ of elements of G satisfying
$$o(w_i(h_1, \dots h_n)) = o_i \quad (1 \le i \le k)$$
there is a group automorphism $\theta : G \to G$ such that
$$\theta(h_j) = g_j \quad (1 \le j \le n)$$
(the universality condition).

We will be fairly informal in our descriptions of semi-presentations, but it will always be clear how to convert a description into a formal semi-presentation as defined above. If P is a semi-presentation for G, we write $G \approx P$.

Our choice of the term 'semi-presentation' comes from the fact that a presentation for G on a set of generators gives a semi-presentation very easily, provided G is simple. Suppose the presentation is

$$G \cong \langle x_1, \dots x_n \,|\, p_i(x_1, \dots x_n) = 1, 1 \le i \le k \rangle. \qquad (2\text{–}1)$$

Firstly, we rewrite the presentation in a more helpful format. For $1 \le i \le k$, choose a word $w_i(x_1, \dots x_n)$ as short as possible such that p_i is some power of w_i (possibly $p_i = w_i$). Let o_i be the actual order of $w_i(x_1, \dots x_n)$ in G; if the presentation has been written irredundantly, we will have $p_i = w_i^{o_i}$. Then we certainly have

$$G \cong \langle x_1, \dots x_n \,|\, w_i(x_1, \dots x_n)^{o_i} = 1, 1 \le i \le k \rangle. \qquad (2\text{–}2)$$

Lemma 2.2. *If G is simple and $o_j \neq 1$ for some $1 \le j \le k$, then we have*

$$G \approx \langle\langle x_1, \dots x_n \,|\, o(w_i(x_1, \dots x_n)) = o_i, 1 \le i \le k \rangle\rangle. \qquad (2\text{–}3)$$

Proof: If $Y = (y_1, \dots y_n)$ is an n-tuple of elements of G satisfying the conditions

$$o(w_i(y_1, \dots y_n)) = o_i \quad (1 \le i \le k),$$

then by Equation (2–2), the map

$$\theta : G \to \langle Y \rangle, \qquad x_i \mapsto y_i \quad (1 \le i \le k)$$

is a surjective homomorphism. Since $o(w_j(y_1, \dots y_n)) \neq 1$, $\langle Y \rangle$ is not the trivial group, and since G is simple, θ must be an isomorphism. Thus the relations in Equation (2–3) specify the set of generators up to automorphism. □

Semi-presentations are in general much easier to find than presentations, but they provide less information.

Example 2.3. The Web Atlas [Wilson et al. 04] gives the following presentation for $L_2(7)$ on its standard generators:

$$L_2(7) \cong \langle x, y \,|\, x^2 = y^3 = (xy)^7 = [x, y]^4 = 1 \rangle.$$

This leads to the following semi-presentation on the same generators:

$$L_2(7) \approx \langle\langle x, y \,|\, o(x) = 2, o(y) = 3, o(xy) = 7, o([x, y]) = 4 \rangle\rangle.$$

In fact, we can drop the last relation, so

$$\langle\langle x, y \,|\, o(x) = 2, o(y) = 3, o(xy) = 7 \rangle\rangle. \qquad (2\text{–}4)$$

However, (2–4) is also a valid semi-presentation for $L_2(8)$ on its standard generators, showing that two nonisomorphic groups can have the same semi-presentation (in contrast to the situation with presentations). Observe that if we turn (2–4) into a presentation

$$\langle x, y \,|\, x^2 = y^3 = (xy)^7 = 1 \rangle$$

then we obtain an infinite group.

2.1 The Standard Relations

We will fix some notation. Let G be a group, and let x and y be standard generators for G (for simpler notation, we will assume $n = 2$ from now on). We seek a semi-presentation for G on x and y. The properties R_j which define standard generators for G give the conjugacy classes of x and y; call them $\mathcal{C}_1$ and $\mathcal{C}_2$. Moreover, they give the orders $o_1, \dots o_t$ of various words $w_1(x, y), \dots w_t(x, y)$ in x and y. In all cases, we fix

$$w_1(x, y) = x,$$
$$w_2(x, y) = y,$$
$$w_3(x, y) = xy,$$

although there may be some extra words. These words w_i and orders o_i give us a good start for a semi-presentation for G on x and y. We call them the *standard relations*, and in a semi-presentation, they will be abbreviated to '**std**'.

Example 2.4. From the Web Atlas or Table 2, standard generators of the Fischer group Fi_{22} are x and y where

x is in class $2A$, y has order 13, xy has order 11 and $(xy)^3xy^2xy(xy^2)^2$ has order 12. In our notation we have $\mathcal{C}_1 = 2A$, $\mathcal{C}_2 = 13A$, $o_1 = 2$, $o_2 = 3$, $o_3 = 11$, $o_4 = 12$ and

$$w_4(x, y) = (xy)^3xy^2xy(xy^2)^2.$$

The semi-presentation that we give in Section 4.13 is

$$\mathrm{Fi}_{22} \approx \langle\langle x, y, (z) \,|\, \mathbf{std}, o(z) = 30, \\ o(xz^{15}) = 3; z := xyxy^2xy^2\rangle\rangle.$$

This is an abbreviation for

$$\mathrm{Fi}_{22} \approx \langle\langle x, y, (z) \,|\, o(x) = 2, o(y) = 13, o(xy) = 11, \\ o((xy)^3xy^2xy(xy^2)^2) = 12, o(z) = 30, \\ o(xz^{15}) = 3; z := xyxy^2xy^2\rangle\rangle.$$

Once we have the standard relations, to complete the semi-presentation we only need to check $x \in \mathcal{C}_1$ and $y \in \mathcal{C}_2$. If G has unique conjugacy classes of elements of orders o_1 and o_2, then the standard relations alone give a semi-presentation. Otherwise, we have some work to do; we either need to prove that the standard relations are sufficient to give a semi-presentation, or we need to find some extra relations.

2.2 Some Notation for Pairs

Let a, b, c be positive integers. An (a, b)*-pair* is a pair (x, y) of elements of G such that x has order a and y has order b. An (a, b, c)*-pair* is an (a, b)-pair (x, y) with the additional property that xy has order c. We call an (o_1, o_2, o_3)-pair an *allowable pair*.

We also use this notation to allow specific conjugacy classes as well as element orders. For instance, a $(2A, 13, 11)$-pair in $G = \mathrm{Fi}_{22}$ is a pair (x, y) of elements of G such that x is in conjugacy class $2A$, y has order 13 and xy has order 11.

2.3 Structure Constants

In most of what follows, we will assume that it is practical to perform simple computations with the group G and that the character table and power maps of G are known. In particular, this means that the *(symmetrised) structure constants*

$$\xi(\mathcal{C}_i, \mathcal{C}_j, \mathcal{C}_k) = \\ \frac{|G|}{|C_G(g_i)||C_G(g_j)||C_G(g_k)|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g_i)\chi(g_j)\chi(g_k)}{\chi(1)} \tag{2–5}$$

can be calculated for any triple $(\mathcal{C}_i, \mathcal{C}_j, \mathcal{C}_k)$ of G-conjugacy classes with representatives g_i, g_j and g_k respectively. These numbers are important because of the well-known formula:

$$\xi(\mathcal{C}_i, \mathcal{C}_j, \mathcal{C}_k) = \sum \frac{1}{|C_G(\langle g_i, g_j, g_k\rangle)|}, \tag{2–6}$$

where the sum is over conjugacy classes of triples g_i, g_j, g_k, from classes $\mathcal{C}_i$, $\mathcal{C}_j$, $\mathcal{C}_k$ respectively, satisfying $g_ig_jg_k = 1$. See [James and Liebeck 93, Chapter 28] and [Isaacs 94, Problem 3.9].

3. IDENTIFYING CONJUGACY CLASSES

As we have remarked, the standard relations establish all the properties required for the pair (x, y) to be standard generators except for their conjugacy classes. In this section, we will describe some techniques we can use to prove that x and y are in the correct conjugacy classes.

3.1 Fingerprinting

This is the method of choice if G is fairly small because it leads to very short semi-presentations. For larger groups, it usually requires too much computer time or memory, and we are forced to use other methods. The basic idea is to find an invariant for each automorphism class of pairs (x, y) and show that there is a particular value of this invariant which is only taken by standard generators. This invariant then gives one or more relations which are added to the set of standard relations to give a semi-presentation for G on its standard generators.

In some cases, fingerprinting was carried out when standard generators for G were first being defined; [Wilson 96] illustrates the case $G = \mathrm{J}_1$. The fingerprinting that we carry out will usually involve many more cases, as we may have to consider all the conjugacy classes containing elements of a given order.

Let F_2 be a free group with free generators t and u. Given elements x, y of G, there is a homomorphism $\phi_{x,y} : F_2 \to G$ given by $t \mapsto x$, $u \mapsto y$. Fix an n-tuple $W = (w_1, \ldots w_n)$ of elements of F_2. We define the W-*fingerprint* (or simply *fingerprint*) of the pair (x, y) to be the n-tuple of integers $(f_i)_{i=1}^n$ where $f_i = o(\phi_{x,y}(w_i))$. Clearly, automorphic pairs give rise to the same fingerprint, so we can talk about the fingerprint of an automorphism class of pairs.

If we wish to fingerprint the classes of allowable pairs (an important special case), we proceed as follows. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be conjugacy classes containing elements of orders o_1 and o_2 respectively.

1. Find representatives a, b in conjugacy classes $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively.
2. Choose random elements $g \in G$ until $o(ab^g) = o_3$.
3. Calculate the W-fingerprint of (a, b^g). If it has been seen before, go back to Step 2.
4. Add the W-fingerprint to the list of fingerprints. If we have seen the right number of fingerprints (see below), then stop. Otherwise, go back to Step 2.

We do this for all possible choices of $\mathcal{C}_1$ and $\mathcal{C}_2$. If the procedure does not appear to be terminating, it may mean that the set W is too small. In this case, we need to start again with a larger set.

Example 3.1. Let $G = \mathrm{M}_{22}$. In this case, the allowable pairs are the $(2, 4, 11)$-pairs. Let

$$W = (tu^2, tutu^2, [t, u], tututu^2, tutu^2tu^3).$$

Then the six W-fingerprints for allowable pairs are given in Table 1.

In each case, we need to know K, the number of W-fingerprints for $(\mathcal{C}_1, \mathcal{C}_2, o_3)$-pairs (when W is sufficiently large). This information is partly provided by ξ, the sum of the $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ symmetrised structure constants (where the sum is over conjugacy classes $\mathcal{C}_3$ of elements of order o_3). There is a positive contribution to ξ for each conjugacy class of allowable pairs; if an allowable pair generates a subgroup which has centralizer of order c in G, then this class contributes $1/c$ to the value of ξ (see Equation (2–6)). Note however, that automorphic pairs always have the same W-fingerprint, even though the sum for ξ counts nonconjugate automorphic pairs separately. There are additional discrepancies if there are two allowable pairs which generate subgroups H_1, H_2 which are isomorphic but nonconjugate. In this case, there may be only one fingerprint, even though the two allowable pairs are not automorphic in G. We sometimes have to examine structure constants in subgroups of G so that we can be sure of the correct value of K.

$\mathcal{C}_2$	$o(xy^2)$	$o(xyxy^2)$	$o([x,y])$	$o(xyxyxy^2)$	$o(xyxy^2xy^3)$
4A	5	11	6	11	5
4A	6	8	5	8	5
4B	5	11	5	11	6
4B	6	7	4	7	5
4B	4	8	6	6	6
4B	6	7	4	7	6

TABLE 1. W-fingerprints for allowable pairs in M_{22}.

Once we have a complete set of fingerprints for all allowable pairs, we pick out a subset of W which characterises the fingerprint corresponding to the standard generators. This, together with the standard relations, gives a semi-presentation. Occasionally, one of the standard relations is redundant because of the extra conditions we supply.

We make the obvious changes to the method if we wish to find fingerprints of another type (say, if there are too many fingerprints of allowable pairs to find).

Example 3.2. Let $G = \mathrm{Co}_1$, considered in Section 4.12. We are able to find conditions that establish $x \in 2B$ quite easily, using the method in Section 3.2. To find conditions showing $y \in 3C$, we could try fingerprinting the $(2, 3, 40)$-pairs, but there are a lot of them, and they are hard to analyse because not all of them generate G. Instead, we find a $2A$-element t and a $(2A, 3, 36)$-pair (t, y^g) and find fingerprints for these. This enables us to find conditions which prove that y is a $3C$-element.

3.2 Involutions

The following simple lemma is extremely useful for identifying conjugacy classes of involutions.

Lemma 3.3. *If $a, b \in G$ are involutions such that ab has odd order, then a is G-conjugate to b.*

The typical application of Lemma 3.3 is to establish the conjugacy class of an involution a. We find a *reference involution* b which is known to be in class $\mathcal{C}$ (usually by powering up an element of suitable order) and then find a conjugate b^g of b such that ab^g has odd order. Then by Lemma 3.3, $a \in \mathcal{C}$. This turns out to be quite useful, as we frequently have $o_1 = 2$. Indeed, for the sporadic groups, the only exception is $G = \mathrm{Co}_3$.

Example 3.4. For the Harada-Norton group $G = \mathrm{HN}$, considered in Section 4.22, we can find a condition that checks that x is in class $2A$ as follows: all elements of order 22 power up to class $2A$, and we know by the standard relations that xy has order 22. Thus we can take $(xy)^{11}$ as our reference involution. Now we search for $z \in G$ such that $x[(xy)^{11}]^z$ has odd order; we can take $z = xy^2xyxyxyxy^2$ (which gives order 5).

3.3 Elements of Even Order

If we are fortunate, we may be able to distinguish classes of even order by powering them up to involutions and using Lemma 3.3. This does not work if the classes power up to the same conjugacy class of involutions, but we may be able to use their centralizers to distinguish them.

Suppose a is an element of order $2n$, $\mathcal{C}$ and $\mathcal{C}'$ are conjugacy classes of elements of order $2n$, and p is a prime which divides $|C_G(\mathcal{C})|$ but not $|C_G(\mathcal{C}')|$. Suppose further that we know that a is either in $\mathcal{C}$ or $\mathcal{C}'$, and we wish to prove that it is in $\mathcal{C}$. We will try to find an element b of order divisible by p which commutes with a; because of the centralizer orders, this will show that a is in $\mathcal{C}$.

Our strategy is:

1. Find words in standard generators which generate the involution centralizer $C_G(t)$, $t = a^n$ (or a sufficiently large subgroup thereof).

2. Find words in the generators of $C_G(t)$ which commute with a, generating a sufficiently large subgroup of $C_G(a)$.

3. Find an element in $C_G(a)$ with order divisible by p. By the centralizer orders, we must have that a is in $\mathcal{C}$.

This strategy is feasible because it is easy to find elements in an involution centralizer.

Lemma 3.5. [Bray 00] *Let $t, g \in G$, with t an involution. Let n be the order of the element $t \cdot t^g$. Define*

$$z = \begin{cases} (t \cdot t^g)^{n/2}, & \text{if } n \text{ is even,} \\ g(t \cdot t^g)^{(n-1)/2}, & \text{if } n \text{ is odd.} \end{cases} \tag{3–1}$$

Then z commutes with t.

Usually, a few iterations of this lemma with several elements $g \in G$ gives a set of generators for $C_G(t)$.

3.4 Zero-Valued Structure Constants

Sometimes structure constants can be used to establish the conjugacy class of an element. A typical case is when we have classes $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_2'$ and $\mathcal{C}_3$ where $\mathcal{C}_2$ and $\mathcal{C}_2'$ contain elements of the same order and

$$\xi(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) > 0,$$
$$\xi(\mathcal{C}_1, \mathcal{C}_2', \mathcal{C}_3) = 0.$$

Thus if $x \in \mathcal{C}_1$ and $y \in \mathcal{C}_2 \cup \mathcal{C}_2'$, we can establish that $y \in \mathcal{C}_2$ if we can find a conjugate y^g of y such that $xy^g \in \mathcal{C}_3$.

This can be seen as a special case of the fingerprinting method described in Section 3.1.

Example 3.6. The group $G = \mathrm{Fi}_{23}$ treated in Section 4.14 has standard generators $x \in 2B$ and $y \in 3D$ such that xy has order 28. Suppose we have checked that all the orders are correct, and that we are also able to establish that x is in class $2B$. Then, because $\xi_G(2B, 3A, 28) = 0$, we know that y cannot be in $3A$. For the same reason, y cannot be in $3B$ or $3C$. Thus y must be in $3D$, and no further checking is needed.

4. THE SPORADIC SIMPLE GROUPS

In this section, we will give semi-presentations for the 26 sporadic simple groups on their standard generators. In each case, we will omit the standard relations (abbreviated **std**), but they can easily be extracted from Table 2, which gives the definitions of standard generators [Wilson et al. 04, Wilson 96]. The full semi-presentations can be found in the Web Atlas in the form of black box algorithms.

The computer programs we used to find and check these semi-presentations were written in GAP [GAP 04] except for those involving the Monster group $\mathbb{M}$, which were written in C [Kernighan and Ritchie 88]. These programs can be found on the website listed in Section 7.

4.1 Mathieu Group M_{11}

There are unique conjugacy classes of elements of orders 2 and 4, so the standard relations suffice to give a semi-presentation. We have

$$\mathrm{M}_{11} \approx \langle\langle x, y \mid \mathbf{std} \rangle\rangle.$$

4.2 Mathieu Group M_{12}

There are four automorphism classes of $(2, 3, 11)$-pairs, only one of which is a $(2B, 3B, 11)$-pair. We have

$$\mathrm{M}_{12} \approx \langle\langle x, y \mid \mathbf{std}, o(xyxyxy^2) = 6 \rangle\rangle.$$

4.3 Mathieu Group M_{22}

We need to show that y is in class $4A$ rather than $4B$. There are two automorphism classes of $(2, 4A, 11)$-pairs and four classes of $(2, 4B, 11)$-pairs. We found fingerprints for all of these in Example 3.1. We have

$$\mathrm{M}_{22} \approx \langle\langle x, y \mid \mathbf{std}, o([x, y]) = 6 \rangle\rangle.$$

4.4 Mathieu Group M_{23}

There are unique conjugacy classes of elements of orders 2 and 4, so the standard relations suffice to give a semi-presentation. We have

$$\mathrm{M}_{23} \approx \langle\langle x, y \mid \mathbf{std} \rangle\rangle.$$

G	$\mathcal{C}_1$	$\mathcal{C}_2$	o_3	w_4	o_4
M_{11}	$2A$	$4A$	11	$xyxy^2xy^3$	5
M_{12}	$2B$	$3B$	11		
M_{22}	$2A$	$4A$	11	$xyxy^2$	11
M_{23}	$2A$	$4A$	23	$(xy)^3xy^2xy(xy^2)^2$	8
M_{24}	$2B$	$3A$	23	$(xy)^2xy^2xy(xy^2)^2$	4
J_1	$2A$	$3A$	7	$xyxy^2$	19
J_2	$2B$	$3B$	7	$xyxy^2$	12
J_3	$2A$	$3A$	19	$xyxy^2$	9
J_4	$2A$	$4A$	37	$xyxy^2$	10
Co_3	$3A$	$4A$	14		
Co_2	$2A$	$5A$	28		
Co_1	$2B$	$3C$	40	$xyxy^2$	6
Fi_{22}	$2A$	$13AB$	11	$(xy)^3xy^2xy(xy^2)^2$	12
Fi_{23}	$2B$	$3D$	28		
Fi_{24}'	$2A$	$3E$	29	$xyxyxy^2$	33
HS	$2A$	$5A$	11		
Suz	$2B$	$3B$	13	$xyxy^2$	15
McL	$2A$	$5A$	11	$(xy)^3xy^2xy(xy^2)^2$	7
He	$2A$	$7C$	17		
Ru	$2B$	$4A$	13		
O'N	$2A$	$4A$	11		
HN	$2A$	$3B$	22	$xyxy^2$	5
Th	$2A$	$3A$	19		
Ly	$2A$	$5A$	14	$xyxyxy^2$	67
B	$2C$	$3A$	55	$(xy)^3xy^2xy(xy^2)^2$	23
$\mathbb{M}$	$2A$	$3B$	29		
$M_{12}.2$	$2C$	$3A$	12	$xyxy^2$	11
$M_{22}.2$	$2B$	$4C$	11		
$J_2.2$	$2C$	$5AB$	14		
$J_3.2$	$2B$	$3A$	24	$xyxy^2$	9
$Fi_{22}.2$	$2A$	$18E$	42		
Fi_{24}	$2C$	$8D$	29		
HS.2	$2C$	$5C$	30		
Suz.2	$2C$	$3B$	28		
McL.2	$2B$	$3B$	22	$(xy)^3xy^2xy(xy^2)^2$	24
He.2	$2B$	$6C$	30		
O'N.2	$2B$	$4A$	22		
HN.2	$2C$	$5A$	42		

TABLE 2. Standard generators for the sporadic almost-simple groups.

4.5 Mathieu Group M_{24}

The $(2B, 3B, 23)$ structure constants are entirely accounted for by the subgroup $L_2(23)$, so we only have one fingerprint for this class of pairs. There are two fingerprints for $(2B, 3A, 23)$-pairs (one of which gives standard generators) and two fingerprints for $(2A, 3B, 23)$-pairs. We have

$$M_{24} \approx \langle\langle x, y \,|\, \mathbf{std}, o(xyxyxy^2) = 12\rangle\rangle.$$

4.6 Janko Group J_1

There are unique conjugacy classes of elements of orders 2 and 3, so the standard relations suffice to give a semi-presentation. We have

$$J_1 \approx \langle\langle x, y \,|\, \mathbf{std}\rangle\rangle.$$

4.7 Janko Group J_2

The only nonzero $(2, 3, 7)$-structure constants in J_2 are from $(2A, 3B, 7A)$ (which is accounted for by $L_2(7)$, a group which contains no elements of order 12) and $(2B, 3B, 7A)$. Thus the standard relation $o(xyxy^2) = 12$ is sufficient to ensure that the pair is $(2B, 3B)$. We have

$$J_2 \approx \langle\langle x, y \,|\, \mathbf{std}\rangle\rangle.$$

4.8 Janko Group J_3

The $(2A, 3B, 19)$-structure constant in $J_3.2$ is 5, but there are only four fingerprints because two of the automorphism classes of pairs generate $L_2(19)$, which has an outer automorphism not realised in $J_3.2$. There are two classes of $(2A, 3A, 19)$-pairs, one of which gives standard generators. We have

$$J_3 \approx \langle\langle x, y \,|\, \mathbf{std}, o(xyxyxy^2) = 17\rangle\rangle.$$

In fact, the standard relation $o(xyxy^2) = 9$ is superfluous in the above.

4.9 Janko Group J_4

We use Lemma 3.3 to show that x is a $2A$-element (we find our reference $2A$-element by powering up an element of order 24). To show that y is in $4A$, we follow the method in Section 3.3 and find an element in $C_G(y)$ with order 20 (the centralizers of elements in classes $4B$ and $4C$ have orders not divisible by 5). We have

$$\begin{aligned} J_4 \approx \langle\langle x, y, (z, c, d, e, f, g) \,|\, \mathbf{std}, o(z) = 24,\\ o(x(z^{12})^{xy^3xy^3}) = 11, o(g) = 20, o([g, y]) = 1;\\ z := xyxyxy^2, c := xyxy^3xyxy,\\ d := xy^2xy^3xy^2xy, e := c(y^2(y^2)^c)^5,\\ f := d(y^2)(y^2)^d, g := (efe)^3(fe)^4f\rangle\rangle. \end{aligned}$$

4.10 Conway Group Co_3

The structure constants are quite large for this group, so we try to eliminate as many cases as we can before fingerprinting. Firstly, we check that x is not in $3B$ by checking $o(xy^2) = 24$ (all elements of order 4 have their squares in $2A$, so y^2 is in $2A$, and the $(2A, 3B, 24)$-structure constant is zero). Secondly, we show that y is in $4A$ by finding an element w in its centralizer which has order

5. This leaves the single class of $(3A, 4A, 14)$-pairs and 341 classes of $(3C, 4A, 14)$-pairs, for which we found fingerprints. It turns out that the relations we added so far eliminate all the $(3C, 4A, 14)$-pairs. We have

$$\begin{aligned}\mathrm{Co}_3 \approx \langle\langle x, y, (u, v, w) \,|\, \mathbf{std}, o(xy^2) = 24,\\ o(w) = 5, o([w, y]) = 1; u := (y^2(y^2)^{xy^2})^3,\\ v := xyxy^3x^2(y^2(y^2)^{xyxy^3x^2})^2,\\ w := (uv^2)^3(uv)^6\rangle\rangle.\end{aligned}$$

4.11 Conway Group Co_2

We can show that x is a $2A$-element by using Lemma 3.3 (where the reference $2A$-element is obtained by powering up xy, which is known to have order 28).

Structure constants and fingerprinting show that there is a single automorphism class of pairs of type $(2A, 5A, 28)$ corresponding to standard generators and a single class of pairs of type $(2A, 5B, 28)$ generating a subgroup of $N(2A)$. It turns out that we do not need to add any relations to those we have found already to eliminate this second possibility. We have

$$\mathrm{Co}_2 \approx \langle\langle x, y \,|\, \mathbf{std}, o(x(xy)^{14}) = 3\rangle\rangle.$$

4.12 Conway Group Co_1

We can show that x is a $2B$-element by using Lemma 3.3 (where the reference $2B$-element is obtained by powering up an element of order 42). The structure constant $(2B, 3D, 40)$ is quite large, and to show that y is a $3C$-element, it is easier to look at $(2A, 3, 36)$-pairs (see Example 3.2). We can find a $2A$-element by powering up xy (which is known to be of order 40). There are only two fingerprints to consider here: one for $3C$ (which generates a subgroup of $N(2A)$ and has centralizer 2 in G) and one for $3D$ (which generates $\mathrm{U}_6(2){:}3$, with centralizer 1). We have

$$\begin{aligned}\mathrm{Co}_1 \approx \langle\langle x, y, (z, a, b) \,|\, \mathbf{std}, o(z) = 42, o(x^{y^2}z^{21}) = 11,\\ o(ab) = 36, o(ab^2abab) = 18;\\ z := xy(xyxy^2)^2, a := (xy)^{20},\\ b := y^{xyxyxyxy^2}\rangle\rangle.\end{aligned}$$

4.13 Fischer Group Fi_{22}

All that needs to be checked is that x is a $2A$-element. We can use Lemma 3.3, taking the 15th power of an element of order 30 as the reference involution. We have

$$\begin{aligned}\mathrm{Fi}_{22} \approx \langle\langle x, y, (z) \,|\, \mathbf{std}, o(z) = 30,\\ o(xz^{15}) = 3; z := xyxy^2xy^2\rangle\rangle.\end{aligned}$$

4.14 Fischer Group Fi_{23}

The $(2B, \mathcal{C}, 28)$ structure constant is zero for $\mathcal{C} = 3A, 3B, 3C$, so we only need to show that x is a $2B$-element (see Example 3.6). We can do this by using Lemma 3.3, taking the 14th power of xy (having order 28) as the reference involution. We have

$$\mathrm{Fi}_{23} \approx \langle\langle x, y \,|\, \mathbf{std}, o(x^{y^2}(xy)^{14}) = 5\rangle\rangle.$$

4.15 Fischer Group Fi_{24}'

The only nonzero structure constant $\xi(2A, \mathcal{C}, 29)$ for a conjugacy class $\mathcal{C}$ containing elements of order 3 is for $\mathcal{C} = 3E$. Thus it suffices to check that x is in $2A$. We can do this by using Lemma 3.3, taking the 30th power of an element z of order 60 as the reference involution. We have

$$\begin{aligned}\mathrm{Fi}_{24}' \approx \langle\langle x, y, (z) \,|\, \mathbf{std}, o(z) = 60,\\ o(x(z^{30})^{xyxy}) = 5; z := (xy)^6y\rangle\rangle.\end{aligned}$$

4.16 Higman-Sims Group HS

We found representatives of the classes $2A$, $2B$, $5A$, $5B$ and $5C$ and found fingerprints for all the automorphism classes of $(2, 5, 11)$-pairs. There are 84 in total. We have

$$\mathrm{HS} \approx \langle\langle x, y \,|\, \mathbf{std}, o(xy^2) = 10, o(xyxy^2) = 15\rangle\rangle.$$

4.17 Suzuki Group Suz

We have to consider $(2A, 3C, 13)$-pairs (three fingerprints), $(2B, 3B, 13)$-pairs (five fingerprints) and $(2B, 3C, 13)$-pairs (63 fingerprints). Note that there are six automorphism classes of $(2B, 3C, 13)$-pairs generating $\mathrm{L}_2(25)$, but there are only three fingerprints for these because $\mathrm{L}_2(25)$ has an extra outer automorphism which is not realised in $\mathrm{Aut(Suz)} \cong \mathrm{Suz}.2$. Thus the $(2B, 3C, 13)$ structure constant is 66 rather than 63. We have

$$\mathrm{Suz} \approx \langle\langle x, y \,|\, \mathbf{std}, o(xyxyxy^2) = 12\rangle\rangle.$$

4.18 McLaughlin Group McL

We have to consider $(2A, 5A, 11)$-pairs and $(2A, 5B, 11)$-pairs. There are two fingerprints for the former type, one of which gives standard generators. We found 52 fingerprints for the latter type: 34 from pairs generating McL, 14 from pairs generating M_{22}, two from pairs generating M_{11} and two from pairs generating $\mathrm{L}_2(11)$. We account for the structure constant

$$\xi_{\mathrm{McL}.2}(2A, 5B, 11) = 65 = 34 + 14 \times 2 + \frac{1}{2}(2 + 2 \times 2)$$

by observing that:

- the 14 pairs generating M_{22} are counted twice (because M_{22} has an outer automorphism not realised in McL.2);
- the subgroups M_{11} and $L_2(11)$ have centralizer of order 2 in McL.2; and
- the subgroup $L_2(11)$ has an outer automorphism not realised in McL.2.

We have

$$\mathrm{McL} \approx \langle\langle x, y \,|\, \mathbf{std}, o(xy^2) = 12\rangle\rangle.$$

4.19 Held Group He

We check that x is a $2A$-element by using Lemma 3.3, taking the 5th power of an element of order 10 as the reference $2A$-element. We then consider $(2A, 7A/B, 17)$-pairs (two fingerprints), the single $(2A, 7C, 17)$-pair and the $(2A, 7D/E, 17)$-pairs (28 fingerprints). It turns out that the relations we have already are enough to prove that y is in $7C$. We have

$$\mathrm{He} \approx \langle\langle x, y, (z) \,|\, \mathbf{std}, o(z) = 10, \quad o(xz^5) = 3; z := xy^2xyxy^2xy^2\rangle\rangle.$$

4.20 Rudvalis Group Ru

The structure constants $(2, 4, 13)$ for $G = \mathrm{Ru}$ are fairly complicated, as there are a number of different subgroups of G which can be generated in this way. The only such subgroups which have nontrivial centralizer in G are Sz(8) and $2 \times \mathrm{Sz}(8)$. Each is contained in a maximal subgroup $(2^2 \times \mathrm{Sz}(8)) : 3$, so each is centralized by a subgroup 2^2. Both Sz(8) and $2 \times \mathrm{Sz}(8)$ can be $(2, 4, 13)$-generated in four different ways (up to automorphisms), but because the automorphism of order 3 acts simultaneously on 2^2 and Sz(8), there are 12 automorphism classes of $(2, 4, 13)$-pairs in Ru generating $2 \times \mathrm{Sz}(8)$ and only four such for Sz(8). This information, together with the structure constants for Ru and its subgroups, tells us how many fingerprints there should be. The details are given in Table 3; overall there are 118 fingerprints to find. We have

$$\mathrm{Ru} \approx \langle\langle x, y \,|\, \mathbf{std}, o(xy^2) = 14, o(xyxy^2) = 29\rangle\rangle.$$

$\mathcal{C}_1$	$\mathcal{C}_2$	Number of fingerprints $(\mathcal{C}_1, \mathcal{C}_2, 13)$	Subgroups arising
$2A$	$4A$	5	$L_3(3){:}2$, $L_2(25)$
$2A$	$4B$	4	Sz(8)
$2A$	$4C$	7	Ru, ${}^2F_4(2)'$
$2A$	$4D$	29	Ru, $L_3(3)$, ${}^2F_4(2)'$, $L_2(25).2$
$2B$	$4A$	1	Ru
$2B$	$4B$	10	Ru, $L_2(13).2$
$2B$	$4C$	32	Ru
$2B$	$4D$	30	Ru, $2 \times \mathrm{Sz}(8)$

TABLE 3. Fingerprints for Ru.

4.21 O'Nan Group O'N

Here it is sufficient to show that y is a $4A$-element. We can do this by using the method of Section 3.3. We find an element z of order divisible by 3, 5 or 7 in the centralizer of y (as $|C_G(4B)| = 2^8$). Indeed, it is only necessary for z to be in the normalizer of the $4A$-element (i.e., the involution centralizer $C_G(2A)$) as the odd-order part of z will then centralize y. Hence a suitable z is fairly easy to find. We have

$$\mathrm{O'N} \approx \langle\langle x, y, (z) \,|\, \mathbf{std}, o(z) = 5, \quad [y, z] = 1; z := xyxy(y^2(y^2)^{xyxy})^5\rangle\rangle.$$

4.22 Harada-Norton Group HN

We have $\xi(2A, 3A, 22) = 0$, so it is sufficient to show that x is in $2A$ (see Example 3.4). We have

$$\mathrm{HN} \approx \langle\langle x, y \,|\, \mathbf{std}, o(x[(xy)^{11}]^{xy^2xyxyxyxy^2}) = 5\rangle\rangle.$$

4.23 Thompson Group Th

Here it is sufficient to show that y is a $3A$-element. Observe that

$$A_4 \cong \langle g, h \,|\, g^3 = h^3 = (gh)^2 = 1\rangle \qquad (4\text{–}1)$$

and g is conjugate to h^{-1} in A_4. Thus we can show y is a $3A$-element by taking another $3A$-element v^{-1} (we take the 7th power of an element z of order 21) and then finding an element w such that yv^w has order 2. We have

$$\mathrm{Th} \approx \langle\langle x, y, (z, v, w) \,|\, \mathbf{std}, o(z) = 21, o(yv^w) = 2; \quad z := (xy)^3y, v := z^7, \quad w := xy^2(xy)^4(xy^2)^2(xy)^2(xy^2)^5(xy)^3\rangle\rangle.$$

4.24 Lyons Group Ly

We must show that y is a $5A$-element. To reduce the number of fingerprints to search, we instead look for $(3A, 5, 14)$-pairs (r, s). We can find a $3A$-element r by powering up an element of order 42.

We have

$$\xi_{\mathrm{Ly}}(3A, 5A, 14) = 1/3, \qquad (4\text{–}2)$$

$$\xi_{\mathrm{Ly}}(3A, 5B, 14) = 38/3. \qquad (4\text{–}3)$$

These structure constants are entirely accounted for by the maximal subgroup 3·McL:2. All the $(3A, 5, 14)$-pairs in McL:2 generate McL, so all the $(3A, 5, 14)$-pairs in Ly generate 3·McL, which is centralized by a group of order 3 in Ly. The structure constants (4–2) and (4–3) then show that there is just one fingerprint for $5A$ and 38 for $5B$. We have

$$\mathrm{Ly} \approx \langle\langle x, y, (z, r, s) \,|\, \mathbf{std}, o(z) = 42, o(rs) = 14, o(rsrs^2) = 30; z := (xy)^5(xy^2)^2, r := z^{14}, s := y^{xyxy^2xyxyxy^2} \rangle\rangle.$$

4.25 Baby Monster Group B

To show that x is in $2C$, we use Lemma 3.3, taking the 26th power of an element of order 52 as our reference $2C$-element.

To show that y is in $3A$, we observe that all $(2A, 3, 8)$-pairs in B are $(2A, 3A, 8)$-pairs (as can be seen from the structure constants). We found an element $z \in 2A$ by taking the 19th power of an element of order 38 and then found a conjugate y^g of y such that zy^g has order 8. (Orders 2, 4 or 14 would also have worked.) Hence (z, y^g) is a $(2A, 3, 8)$-pair, so y must be in $3A$. We have

$$\mathrm{B} \approx \langle\langle x, y, (u, v) \,|\, \mathbf{std}, o(u) = 52, o(xu^{26}) = 35, o(v) = 38, o(v^{19}y^x) = 8; u := (xyxy^2)^2(xy)^2(xyxy^2)^2; v := (xy)^3(xy^2xy)^2xy(xyxy^2)^2xy^2 \rangle\rangle.$$

4.26 Monster Group $\mathbb{M}$

The smallest nontrivial representation of $\mathbb{M}$ over any field has dimension 196,882, and while standard generators of $\mathbb{M}$ in the 196,882-dimensional representation over GF(2) have been computed, it is prohibitively slow and expensive in terms of storage to perform calculations with such enormous matrices. Instead, we use the computer construction in [Linton et al. 98] with an implementation by Parker and the second author in C [Kernighan and Ritchie 88].

Before performing any calculations in the computer construction, we came up with the following strategy for finding a semi-presentation:

1. Find standard generators x, y of $\mathbb{M}$.

2. Find a word c in x and y whose order is in

   $$S = \{34, 38, 50, 54, 62, 68, 94, 104, 110\},$$

   and power it up to a give an element $d \in 2A$.

3. Find a word e such that $o(xd^e)$ is in

   $$U = \{1, 3, 5\}.$$

   This would prove that x is a $2A$-element, and also that y is not a $3A$-element (because $o(xy) = 29$).

4. Find a word f such that $o(xy^f)$ is in

   $$V = \{19, 25, 31, 34, 50, 55, 68, 94\}.$$

   The structure constants for $\mathbb{M}$ then imply that y is not a $3C$-element, so it must be a $3B$-element.

To find standard generators, we powered up a representative of class $4B$ from [Barraclough and Wilson 05] to give an involution x, and then looked for conjugates y of a $3B$ class representative b from [Wilson 01] such that $o(xy) = 29$. (We will be able to prove retrospectively that x and y are in the correct classes.) We used

$$x = (DC^3D^2CD^2CD^2CD^2CDCDCDC)^2,$$
$$b = (ABABAB^2AB)^7,$$
$$y = ((ABABAB^2AB)^7)^{TBC^3BT},$$

where the elements $A, B, C, D, T \in \mathbb{M}$ are as described in [Linton et al. 98].

We then followed the strategy described above. We have

$$\mathbb{M} \approx \langle\langle x, y, (c, d, f) \,|\, \mathbf{std}, o(c) = 50, o(xd) = 5, o(xy^f) = 34; c := (xy)^4(xy^2)^2, d := c^{25}, f := xyxyxyxyxy^2 \rangle\rangle.$$

Notice that Step 3 turned out to be unnecessary, as we can take $e = 1$.

5. THE SPORADIC AUTOMORPHISM GROUPS

In this section, we will give semi-presentations for the 12 almost-simple groups G which are not simple themselves but have a sporadic simple group as their derived subgroup. As before, we abbreviate the standard relations to '**std**'.

5.1 Mathieu Group $\mathrm{M}_{12}.2$

Here we know that xy is an outer element (it has order 12) and y is inner (it has order 3), and so x must be outer, and hence must be a $2C$-element. There are three fingerprints for $(2C, 3A, 12)$-pairs (corresponding to the three conjugacy classes of elements of order 12) and seven fingerprints for $(2C, 3B, 12)$-pairs. We have

$$\mathrm{M}_{12}.2 \approx \langle\langle x, y \mid \mathbf{std}, o((xy)^3xy^2) = 6\rangle\rangle.$$

The relation $o(xyxy^2) = 11$ becomes redundant when this extra condition is added.

5.2 Mathieu Group $\mathrm{M}_{22}.2$

There are 13 automorphism classes of $(2, 4, 11)$-pairs to consider, arising from the different combinations of classes of elements of orders 2 and 4. The subgroups generated in this way have a trivial centralizer. We have

$$\mathrm{M}_{22}.2 \approx \langle\langle x, y \mid \mathbf{std}, o(xyxy^2) = 10\rangle\rangle.$$

5.3 Higman-Sims Group HS.2

We found 29 fingerprints for $(2, 5, 30)$-pairs; the two subgroups generated with a $(2C, 5B, 30)$-pair are $5 \times \mathrm{S}_5$ and $2 \times \mathrm{A}_8$; they have centralizers of orders 5 and 2 (respectively) in G. All other subgroups thus generated have trivial centralizers in G. The structure constants then show that there are exactly 29 automorphism classes of $(2, 5, 30)$-pairs. We have

$$\mathrm{HS}.2 \approx \langle\langle x, y \mid \mathbf{std}, o([x, y]) = 3\rangle\rangle.$$

5.4 Janko Group $\mathrm{J}_2.2$

There are five automorphism classes of $(2, 5, 14)$-pairs to find: a unique class of $(2C, 5AB, 14)$-pairs (the standard generators) and four classes of $(2C, 5CD, 14)$-pairs. All these pairs generate $\mathrm{J}_2.2$. We have

$$\mathrm{J}_2.2 \approx \langle\langle x, y \mid \mathbf{std}, o(xy^2) = 24\rangle\rangle.$$

5.5 Janko Group $\mathrm{J}_3.2$

By considering element orders in J_3, x must be in class $2B$. There are eight automorphism classes of $(2B, 3, 24)$-pairs, two of which correspond to class $3A$. All the pairs generate $\mathrm{J}_3.2$. We have

$$\mathrm{J}_3.2 \approx \langle\langle x, y \mid \mathbf{std}, o(xyxyxyxy^2) = 9\rangle\rangle.$$

The standard relation $o(xyxy^2) = 9$ is redundant.

5.6 McLaughlin Group McL.2

The only nonzero $(2, 3, 22)$ structure constants come from $(2B, 3B)$-pairs, so the elements must be in the correct conjugacy classes. We have

$$\mathrm{McL}.2 \approx \langle\langle x, y \mid \mathbf{std}\rangle\rangle.$$

5.7 Suzuki Group Suz.2

There are 32 automorphism classes of $(2, 3, 28)$-pairs, 31 of which generate Suz.2, and one of which (the unique class of $(2C, 3C, 28)$-pairs) generates the subgroup $\mathrm{S}_4 \times \mathrm{L}_3(2)$. We have

$$\mathrm{Suz}.2 \approx \langle\langle x, y \mid \mathbf{std}, o(xyxyxy^2xy^2) = 7\rangle\rangle.$$

5.8 Held Group He.2

We can show that x is a $2B$-element by using Lemma 3.3, taking the 12th power of an element of order 24 as our reference involution. This then implies that y must be an outer element of order 6.

To show that y is in $6C$, we will use the method of Section 3.3 and find an element of order 15 which commutes with y (the classes $6D$ and $6E$ have centralizers whose orders are not divisible by 5). We have

$$\begin{aligned}\mathrm{He}.2 \approx \langle\langle x, y, (z, t) \mid \mathbf{std}, o(z) = 24, o(xz^{12}) = 17,\\ o(t) = 15, o([t, y]) = 1; z := xy^2xy^2xy,\\ t := (y^3(y^3)^x)^4((y^3(y^3)^{xy^2xy})^2\rangle\rangle.\end{aligned}$$

5.9 O'Nan Group O'N.2

By the orders of y and xy, we know that x is an outer element, so it must be in class $2B$. There are two classes containing elements of order 4, and we want to show that y is in $4A$. Because the $(2B, 4B, 22)$ structure constant is rather large, we chose not to find fingerprints. Instead, we use the method of Section 3.3, and find an element z of order 5 which commutes with y. Since $|C(4B)| = 512$ is not divisible by 5, y must be a $4A$-element. We have

$$\begin{aligned}\mathrm{O'N}.2 \approx \langle\langle x, y, (t, z) \mid \mathbf{std}, o(z) = 5, o([y, z]) = 1;\\ z := (t(y^2(y^2)^t)^7)^2, t := xy^2xyx\rangle\rangle.\end{aligned}$$

5.10 Fischer Group $\mathrm{Fi}_{22}.2$

We show that x is in class $2A$ by using Lemma 3.3, taking our reference $2A$-element as the 11th power of an element of order 22.

Because x is an inner element and xy has order 42, y must be an outer element, so it is in one of the classes $18E$, $18F$, $18G$ or $18H$. We can show that it is in either $18E$ or $18F$ by considering the 9th power map. The element xy has order 42, so $(xy)^{21}$ is a $2D$-element. Lemma 3.3 then allows us to show that y^9 is a $2D$-element, so y is in class $18E$ or $18F$. This leaves five automorphism classes of $(2A, 18E/F, 42)$-pairs to test, two of which generate the subgroup $3 \times U_4(3).2^2$ with a centralizer of order 3 in G. Each fingerprint gives a different value of $o(xy^8)$, but it turns out that the relations added so far already eliminate the possibility that y is in class $18F$. We have

$$\mathrm{Fi}_{22}.2 \approx \langle\langle x, y, (z) \,|\, \mathbf{std}, o(z) = 22, o(xz^{11}) = 3, o((y^9)^{xy^3}(xy)^{21}) = 3; z := xyxy^5xy^4\rangle\rangle.$$

5.11 Fischer Group Fi_{24}

We check that x is a $2C$-element by using Lemma 3.3, taking the 27th power of an element of order 54 as the reference involution.

The $(2C, 8, 29)$ structure constants are zero except for $8D$ ($\xi = 1$) and $8F$ ($\xi = 10$). The only maximal subgroup of Fi_{24} containing an element of order 29 is $29{:}28$, so all the $(2C, 8, 29)$-pairs generate Fi_{24}. Thus we need to find 11 different fingerprints. We have

$$\mathrm{Fi}_{24} \approx \langle\langle x, y, (z) \,|\, \mathbf{std}, o(z) = 54, o(xz^{27}) = 3, o(xy^2) = 20; z := xyxy^6\rangle\rangle.$$

5.12 Harada-Norton Group HN.2

By considering element orders, we know that y is an inner element and xy is an outer element. Hence x is an outer element of order 2, so it must be in class $2C$.

To show that y is in $5A$, we consider $(2A, 5, 22)$-pairs. We have

$$\xi_{\mathrm{HN}.2}(2A, 5A, 22) = 1/4,$$
$$\xi_{\mathrm{HN}.2}(2A, 5B, 22) = 0,$$
$$\xi_{\mathrm{HN}.2}(2A, 5CD, 22) = 1,$$
$$\xi_{\mathrm{HN}.2}(2A, 5E, 22) = 25/4 = 4 + 9/4.$$

We claim that there are (respectively) 1, 0, 1 and 13 classes of $(2, 5, 22)$-pairs for the classes $5A$, $5B$, $5CD$ and $5E$. We can easily find this many fingerprints; we will show that there cannot be any more.

Certainly any $(2A, 5, 22)$-pair must be contained in HN. The only maximal subgroup of HN to contain elements of order 22 is 2.HS.2, and in fact these elements are contained in 2.HS. Thus any $(2, 5, 22)$-subgroup of 2.HS.2 has centralizer of order 4 in HN.2, because we have

$$2.\mathrm{HS} < 4.\mathrm{HS} < \mathrm{HN}.2,$$

and any subgroup of HS containing elements of orders 11 and 5 must have a trivial centralizer in HS. Thus each class of pairs either contributes $1/4$ (if it is contained in 2.HS.2) or 1 (if it generates HN).

We consider each class in turn:

- The structure constant shows that there is exactly one class of $(2A, 5A, 22)$-pairs.
- There are no $(2A, 5B, 22)$-pairs, because the structure constant is zero.
- By considering the fusion between 2.HS.2 and HN and the structure constants in 2.HS.2, we know that all $(2A, 5CD, 22)$-pairs generate HN. So there is only one class of $(2A, 5CD, 22)$-pairs.
- We observe that four of the fingerprints for the $(2A, 5E, 22)$-pairs have element orders in the set $\{9, 19, 21, 25, 35\}$, showing that they cannot be contained in 2.HS.2 and must therefore generate HN. This leaves a contribution of $25/4 - 4 = 9/4$, and there are $13 - 4 = 9$ fingerprints unaccounted for, so there must be exactly nine classes generating subgroups of 2.HS.2.

After fingerprinting, it turns out that if (a, b) is a $(2A, 5, 22)$-pair then

$$b \in 5A \Leftrightarrow o(ab^2(ab)^3) = 22. \tag{5–1}$$

We can find a $2A$-element t by powering up an element z of order 60. Then we can look for $g \in G$ such that (t, y^g) is a $(2A, 5, 22)$-pair. We then use the criterion in Equation (5–1) to check that y^g (and hence y) is a $5A$ element. We have

$$\mathrm{HN}.2 \approx \langle\langle x, y, (t, z) \,|\, \mathbf{std}, o(z) = 60, o(ty) = 22, o(ty^2(ty)^3) = 22; z := xy^3(xy)^4, t := z^{30}\rangle\rangle.$$

6. RESULTS OF TESTING THE REPRESENTATIONS IN THE WEB ATLAS

We used our semi-presentations to test the representations of sporadic simple and almost-simple groups given in the Web Atlas. As we expected, the vast majority of the representations satisfied the relevant semi-presentations, but a few mistakes were discovered.

- Matrices purporting to generate a 483-dimensional representation of M_{23} over GF(7) were included, but they failed to satisfy the semi-presentation. In fact no such representation of M_{23} exists [Jansen et al. 95].
- One of the 896-dimensional representations of HS over GF(4) was incorrect, as the product of the two generators had order exceeding 100.
- Matrices purporting to generate a 104-dimensional representation of He.2 over GF(5) in fact generated a group of order 30, 240.

- The 924-dimensional representation of $Fi_{22}.2$ over GF(3) had nonstandard generators; the second generator given was xy rather than y.

7. SUPPLEMENTARY INFORMATION

The main GAP programs that were used to prepare this paper can be found at: http://www.expmath.org/expmath/volumes/14/14.3/Nickerson/supp.zip. We also include human-readable and computer-readable tables giving representatives for the pairs involved in fingerprinting (as words in standard generators). These tables are intended for researchers who wish to reproduce the results in this paper.

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