Some Computations Regarding Foulkes′ Conjecture

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CONTENTS

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We describe how certain permutation actions of large symmetric groups can be efficiently implemented on a computer. Using a specially tailored adaptation of a general technique to enumerate huge orbits and substantial distributed computation on a cluster of workstations, we collect further evidence related to the approach to Foulkes' conjecture suggested in [Black and List 89].

1. FOULKES' CONJECTURE

To state Foulkes' conjecture we first introduce some notation. Let $\mathbb{N}$ be the set of positive integers, let $\mathbb{Q}$ be the set of rational numbers, and denote by $M_n := \{1, 2, 3, \ldots, n\}$ for $n \in \mathbb{N}$ the set of natural numbers less than or equal to n. We denote the symmetric group on n points by $S_n := \{\pi : M_n \to M_n \mid \pi \text{ bijective}\}$, with concatenation of maps as product, which we denote as $\pi \circ \varphi$, meaning "first apply φ, then π."

For $m, n \in \mathbb{N}$ let $S_m \wr S_n$ be the wreath product of S_m and S_n, which is a semidirect product of the n-fold direct product $S_m^n := S_m \times \cdots \times S_m$ of copies of S_m and S_n, where the latter acts on the first by permuting the direct factors. Note that S_m^n can be identified with the set of maps $\{f : M_n \to S_m\}$. Hence, $S_m \wr S_n = S_m^n \rtimes S_n$ with product

$$(f, \pi) \cdot (f', \pi') := (f \cdot (f' \circ \pi^{-1}), \pi \circ \pi'),$$

where we multiply maps $f : M_n \to S_m$ pointwise using the product in S_m.

The wreath product $S_m \wr S_n$ has order $|S_m \wr S_n| = (m!)^n \cdot n!$ and embeds into S_{mn} by letting the ith direct factor of S_m^n, for $i = 1, \ldots, n$, permute the points $\{(i-1)m + 1, \ldots, im\}$ and keep all other points in M_{mn} fixed, while S_n acts on M_{mn} by permuting these n blocks; for more details see [James and Kerber 81, Section 4.1]. We denote by $\Omega_{m,n}$ the set $\{(S_m \wr S_n) \circ \pi \mid \pi \in S_{mn}\}$ of right cosets of $S_m \wr S_n$ in S_{mn} and by $\mathbb{Q}\Omega_{m,n}$ the associated permutation right $\mathbb{Q}S_{mn}$-module.

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It is easily seen by an induction argument that for $m \geq n$ we have $|S_n \wr S_m| \leq |S_m \wr S_n|$. Thus we have $|\Omega_{m,n}| \leq |\Omega_{n,m}|$. But in fact much more is conjectured to be true:

Conjecture 1.1. [Foulkes 50] *Let $m, n \in \mathbb{N}$ with $m \geq n$. Then the permutation module $\mathbb{Q}\Omega_{m,n}$ is a $\mathbb{Q}S_{mn}$-submodule of the permutation module $\mathbb{Q}\Omega_{n,m}$.*

In Section 2, we describe how the action of S_{mn} on $\Omega_{m,n}$ can be efficiently implemented on a computer. This implementation will be used for calculations connected to the approach to Foulkes' conjecture suggested in [Black and List 89]. Our description uses the notion of Schur bases, which are introduced in Section 3, while in Section 4, the approach of Black and List is discussed. In Section 5, our particular computational techniques are explained and in Section 6, actual computational results are presented. There we also describe the values of m and n for which the conjecture has been verified computationally so far.

2. IMPLEMENTATION OF THE ACTION OF S_{mn} ON $\Omega_{m,n}$

For this section, let $m, n \in \mathbb{N}$ be fixed. We consider the following set of maps:

$$V_{m,n} := \{v : M_{mn} \to M_n \mid v \text{ takes every value exactly } m \text{ times}\}.$$

One can imagine these maps as tuples of length mn with entries in M_n, each one occuring exactly m times. Hence we will denote such maps as tuples $v = (v_1, v_2, \ldots, v_{mn})$. On the computer they are stored exactly in this way. By way of concatenation of maps, we have two transitive actions on $V_{m,n}$, one on the left and one on the right. The group S_n acts regularly on the left by renaming the entries:

$$S_n \times V_{m,n} \to V_{m,n}, (\pi, v) \mapsto \pi \circ v.$$

The group S_{mn} acts on the right by permuting the entries:

$$V_{m,n} \times S_{mn} \to V_{m,n}, (v, \psi) \mapsto v \circ \psi.$$

These actions commute because of the associativity of concatenation: $(\pi \circ v) \circ \psi = \pi \circ (v \circ \psi)$.

Therefore we obtain an induced action of S_{mn} on the S_n-orbits in $V_{m,n}$. In the following, we omit the "$\circ$" symbol in the notation of S_n orbits, denote the set $\{S_n v \mid v \in V_{m,n}\}$ of S_n-orbits in $V_{m,n}$ by $S_n \backslash V_{m,n}$, and the action of S_{mn} on it by $(S_n v) \circ \psi := S_n(v \circ \psi)$.

From now on let $x \in V_{m,n}$ be the tuple

$$x := (\underbrace{1, \ldots, 1}_{m \text{ times}}, \underbrace{2, \ldots, 2}_{m \text{ times}}, \ \ldots \ , \underbrace{n, \ldots, n}_{m \text{ times}}),$$

i.e., the map that maps $k \in M_{mn}$ to $\lceil k/m \rceil$, the smallest integer greater or equal to k/m. Then the stabilizer $\mathrm{Stab}_{S_{mn}}(x)$ of x in S_{mn} is equal to S_m^n, and the stabilizer $\mathrm{Stab}_{S_{mn}}(S_n x)$ of $S_n x \in S_n \backslash V_{m,n}$ in S_{mn} is equal to $S_m \wr S_n$. Thus the action of S_{mn} on $S_n \backslash V_{m,n}$ is equivalent to the action of S_{mn} on $\Omega_{m,n}$. Hence we identify $\Omega_{m,n}$ and $S_n \backslash V_{m,n}$ from now on.

Passing from S_{mn}-sets to $\mathbb{Q}S_{mn}$-modules, we can consider $\mathbb{Q}V_{m,n}$ as a $\mathbb{Q}S_n$-$\mathbb{Q}S_{mn}$-bimodule, and thus the permutation $\mathbb{Q}S_{mn}$-module $\mathbb{Q}\Omega_{m,n}$ is identified with the $\mathbb{Q}S_{mn}$-submodule $(\mathbb{Q}V_{m,n})^{S_n}$ whose permutation basis consists of the sums $\overline{S_n v} := \sum_{w \in S_n v} w$ over S_n-orbits $S_n v \subseteq V_{m,n}$. Note that $(\mathbb{Q}V_{m,n})^{S_n}$ is the set of elements in $\mathbb{Q}V_{m,n}$ invariant under the left action of S_n.

We introduce the following definition to distinguish one tuple in each S_n-orbit.

Definition 2.1. (S_n-Minimal Tuples.) In the above situation, we call the lexicographically smallest tuple in each S_n-orbit *S_n-minimal.* For each $v \in V_{m,n}$, we call the S_n-minimal tuple in the orbit $S_n v$ the *S_n-minimalization of* v. We denote by $V_{m,n}^{\min}$ the set of S_n-minimal tuples in $V_{m,n}$.

It follows readily from the above, that the action of S_{mn} on $\Omega_{m,n}$ can be implemented on a computer by identifying $\Omega_{m,n}$ with $V_{m,n}^{\min}$ and acting with a map $\psi \in S_{mn}$ on $v \in V_{m,n}^{\min}$ by just S_n-minimalizing $v \circ \psi \in V_{m,n}$. Note the runtime needed to compute an S_n-minimalization, and hence the ψ-image of v, is proportional to the length mn of the tuples.

We note the following characterization of S_n-minimality for later reference.

Proposition 2.2. (Equivalent Characterization of S_n-Minimality.) *A tuple $v \in V_{m,n}$ is S_n-minimal, if and only if it has the following property: for all i, j with $1 \leq i < j \leq n$, the first occurrence of i in v is before the first occurrence of j.*

Proof: Let v be S_n-minimal. If the above property does not hold, we can rename some i and j and get a lexicographically smaller tuple in the same S_n-orbit; a contradiction.

Let v have the above property, and assume v is not S_n-minimal. Then there is a tuple v' in the same S_n-orbit that is lexicographically smaller than v. Let p be the first position where both tuples differ, and let $v_p = j$ and $v'_p = i$ with $i < j$. Because v and v' are in the same S_n-orbit, p is the first position in v with value j and the first position in v' with value i. By the assumed property, the first occurrence of i in v is before p. However, v and v' are equal at positions before p; therefore we have a contradiction. □

3. SCHUR BASES

To describe the approach in [Black and List 89], we recall a few facts about permutation modules and homomorphisms between them. For our purposes, we give a slightly more general description than can be found in [Landrock 83, Chapter II.12].

For this section, let G be a finite group, acting transitively from the right on the sets Ω and Ω'. Let $\omega_1 \in \Omega$ and $\omega'_1 \in \Omega'$, and let $H := \mathrm{Stab}_G(\omega_1)$ and $H' := \mathrm{Stab}_G(\omega'_1)$ be the corresponding stabilizers. As above, let $\mathbb{Q}\Omega$ and $\mathbb{Q}\Omega'$ denote the associated permutation modules. The space $\mathrm{Hom}_{\mathbb{Q}G}(\mathbb{Q}\Omega, \mathbb{Q}\Omega')$ of $\mathbb{Q}G$-homomorphisms from $\mathbb{Q}\Omega$ to $\mathbb{Q}\Omega'$ has a distinguished basis, which can be described as follows:

We decompose Ω' into H-orbits, by choosing $s_1 = 1_G, s_2, \ldots, s_l \in G$ such that

$$\Omega' = \omega'_1 s_1 H \cup \omega'_1 s_2 H \cup \cdots \cup \omega'_1 s_l H$$

is a disjoint union. Note that $\{s_1, s_2, \ldots, s_l\}$ is a set of H'-H-double coset representatives in G.

Using the diagonal action of G on $\Omega' \times \Omega$ and considering the intersection of each G-orbit in $\Omega' \times \Omega$ with $\Omega' \times \{\omega_1\}$, we get the decomposition of $\Omega' \times \Omega$ into G-orbits:

$$\Omega' \times \Omega = (\omega'_1 s_1, \omega_1)G \cup (\omega'_1 s_2, \omega_1)G \cup \cdots \cup (\omega'_1 s_l, \omega_1)G.$$

We describe a homomorphism $\varphi \in \mathrm{Hom}_{\mathbb{Q}G}(\mathbb{Q}\Omega, \mathbb{Q}\Omega')$ by a matrix with respect to the natural bases of $\mathbb{Q}\Omega$, respectively $\mathbb{Q}\Omega'$, where the rows are indexed by Ω' and the columns are indexed by Ω. Denoting the (ω', ω)-entry of the matrix of φ by $\varphi_{\omega',\omega}$, we get $\varphi_{\omega',\omega g} = \varphi_{\omega' g^{-1},\omega}$, or equivalently $\varphi_{\omega' g,\omega g} = \varphi_{\omega',\omega}$, for all $\omega \in \Omega$, $\omega' \in \Omega'$, and $g \in G$, because φ is a $\mathbb{Q}G$-module homomorphism. Thus, the matrix of φ is a unique $\mathbb{Q}$-linear combination of the matrices $A^{(1)}, A^{(2)}, \ldots, A^{(l)}$ defined by

$$A^{(i)}_{\omega',\omega} = \begin{cases} 1 & \text{if } (\omega', \omega) \in (\omega'_1 s_i, \omega_1)G, \\ 0 & \text{if } (\omega', \omega) \notin (\omega'_1 s_i, \omega_1)G. \end{cases}$$

We call $\mathcal{A} := (A^{(1)}, A^{(2)}, \ldots, A^{(l)})$ and the associated $\mathbb{Q}G$-module homomorphisms $(\varphi^{(1)}, \varphi^{(2)}, \ldots, \varphi^{(l)})$ the *Schur basis* of $\mathrm{Hom}_{\mathbb{Q}G}(\mathbb{Q}\Omega, \mathbb{Q}\Omega')$, which hence is in bijection with the G-orbits in $\Omega' \times \Omega$. In particular, for $\omega = \omega_1 g \in \Omega$, where $g \in G$, and thus $H^g = \mathrm{Stab}_G(\omega_1 g)$, we have

$$\varphi^{(i)} : \omega = \omega_1 g \mapsto \sum_{\omega' \in \omega'_1 s_i g H^g} \omega'.$$

We now turn to the concatenation of homomorphisms. For a G-set Ω'', let $H'' := \mathrm{Stab}_G(\omega''_1)$ for some $\omega''_1 \in \Omega''$, and as above we choose a set $\{t_1 = 1_G, t_2, \ldots\}$ of H''-H'-double coset representatives in G and a set $\{u_1 = 1_G, u_2, \ldots\}$ of H''-H-double coset representatives in G. Let $\mathcal{B} := (B^{(1)}, B^{(2)}, \ldots)$ and $\mathcal{C} := (C^{(1)}, C^{(2)}, \ldots)$ denote the Schur bases of $\mathrm{Hom}_{\mathbb{Q}G}(\mathbb{Q}\Omega', \mathbb{Q}\Omega'')$ and $\mathrm{Hom}_{\mathbb{Q}G}(\mathbb{Q}\Omega, \mathbb{Q}\Omega'')$, respectively. We can now write the concatenation $B^{(j)} \circ A^{(i)}$, i.e., the matrix product, in terms of the Schur basis $\mathcal{C}$ of $\mathrm{Hom}_{\mathbb{Q}G}(\mathbb{Q}\Omega, \mathbb{Q}\Omega'')$:

$$\begin{aligned}
(B^{(j)} \circ A^{(i)})_{\omega''_1 u_k,\omega_1} &= \sum_{\omega' \in \Omega'} B^{(j)}_{\omega''_1 u_k,\omega'} \cdot A^{(i)}_{\omega',\omega_1} \\
&= |\{\omega' \in \Omega' \mid (\omega''_1 u_k, \omega') \in (\omega''_1 t_j, \omega'_1)G \\
&\qquad \text{and } (\omega', \omega_1) \in (\omega'_1 s_i, \omega_1)G\}| \\
&= |\{\omega' \in \omega'_1 s_i H \mid (\omega''_1 u_k, \omega') \in (\omega''_1 t_j, \omega'_1)G\}| \\
&= \left|\{\omega' \in \omega'_1 s_i H \mid (\omega''_1, \omega' u_k^{-1}) \in (\omega''_1, \omega'_1 t_j^{-1})G\}\right| \\
&= \left|\omega'_1 s_i H u_k^{-1} \cap \omega'_1 t_j^{-1} H''\right| \\
&= \left|\omega'_1 s_i H \cap \omega'_1 t_j^{-1} H'' u_k\right|.
\end{aligned}$$

4. THE APPROACH OF BLACK AND LIST

In [Black and List 89], the authors describe an approach to prove Foulkes' conjecture that is based on a certain $\mathbb{Q}S_{mn}$-module homomorphism $\varphi^{(m,n)} : \mathbb{Q}\Omega_{m,n} \to \mathbb{Q}\Omega_{n,m}$. Using the language of the previous section, we first introduce a $\mathbb{Q}S_{mn}$-module homomorphism $\widetilde{\varphi}^{(m,n)} : \mathbb{Q}V_{m,n} \to \mathbb{Q}V_{n,m}$, with a view towards efficient implementation.

For a tuple $v \in V_{m,n}$ let $\tilde{v} := (\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_{mn})$, where $\tilde{v}_k := |\{l \in M_k \mid v_l = v_k\}|$. With respect to v, the tuple $\tilde{v}$ has the following property:

> for every $i \in M_n$ and every $j \in M_m$, there is a position $K \in M_{mn}$ with $v_k = i$ and $\tilde{v}_k = j$. $(\star)$

From this property, it follows that $\tilde{v} \in V_{n,m}$. Obviously, the set of all such tuples coincides with $\tilde{v} \circ \mathrm{Stab}_{S_{mn}}(v) \subseteq V_{n,m}$, and $\tilde{v}$ is the lexicographically smallest of them. In particular, we have

$$\tilde{x} = (\underbrace{\underbrace{1,2,\ldots,m}, \underbrace{1,2,\ldots,m}, \ \ldots \ , \underbrace{1,2,\ldots,m}}_{n \text{ times}}).$$

For $v \in V_{m,n}$ and $i = 1,\ldots,n$ let $1 \leq p_{i,1} < p_{i,2} < \cdots < p_{i,m} \leq mn$ be the positions such that $v_{p_{i,j}} = i$, and let $\psi_v \in S_{mn}$ be defined as $\psi_v : p_{i,j} \mapsto (i-1)m+j$, for $i = 1,\ldots,n$ and $j = 1,\ldots,m$. Hence we have $x \circ \psi_v = v$ and $\tilde{x} \circ \psi_v = \tilde{v}$. Thus we conclude that all pairs $(\tilde{v}, v)$, for $v \in V_{m,n}$, belong to one and the same G-orbit in $V_{n,m} \times V_{m,n}$, and hence let $\widetilde{\varphi}^{(m,n)} \in \mathrm{Hom}_{\mathbb{Q}S_{mn}}(\mathbb{Q}V_{m,n}, \mathbb{Q}V_{n,m})$ be the corresponding Schur basis element. As $\mathrm{Stab}_{S_{mn}}(x) = S_m^n$ acts regularly on its orbit $\tilde{x} \circ S_m^n \subseteq V_{n,m}$, for $v \in V_{m,n}$, we have

$$\widetilde{\varphi}^{(m,n)} : v \mapsto \sum_{w \in \tilde{v} \circ \mathrm{Stab}_{S_{mn}}(v)} w = \sum_{\eta \in \mathrm{Stab}_{S_{mn}}(v)} \tilde{v} \circ \eta.$$

Note that, if $\sigma_{m,n} \in S_{mn}$ is defined as

$$\sigma_{m,n} : (i-1)m + j \mapsto (j-1)n + i$$

for $i = 1,\ldots,n$ and $j = 1,\ldots,m$, then $\widetilde{\varphi}^{(m,n)}$ is the Schur basis element corresponding to the S_m^n-S_n^m-double coset $S_m^n \circ \sigma_{m,n} \circ S_n^m$ in S_{mn}.

Next we consider $\mathbb{Q}\Omega_{m,n} = (\mathbb{Q}V_{m,n})^{S_n}$ and $\mathbb{Q}V_{n,m} = (\mathbb{Q}V_{n,m})^{S_m}$. By the description $(\star)$ of the elements of $\tilde{v} \circ \mathrm{Stab}_{S_{mn}}(v) \subseteq V_{n,m}$, for $v \in V_{m,n}$, we conclude that $\tilde{v} \circ \mathrm{Stab}_{S_{mn}}(v)$ is a union of S_m-orbits. Hence by restriction, we obtain a $\mathbb{Q}S_{mn}$-homomorphism

$$\varphi^{(m,n)} := \frac{1}{n!} \cdot \widetilde{\varphi}^{(m,n)}|_{\mathbb{Q}\Omega_{m,n}} : \mathbb{Q}\Omega_{m,n} \to \mathbb{Q}\Omega_{n,m}.$$

Moreover, as for $v' := \pi \circ v$, for $\pi \in S_n$, we have $\widetilde{v'} = \tilde{v}$ and $\mathrm{Stab}_{S_{mn}}(v) = \mathrm{Stab}_{S_{mn}}(v')$; we conclude that $\widetilde{\varphi}^{(m,n)}(v') = \widetilde{\varphi}^{(m,n)}(v)$. In particular, we have $\varphi^{(m,n)}(\overline{S_n x}) = \sum_{\eta \in S_m^n} \tilde{x} \circ \eta$, and hence $\varphi^{(m,n)} \in \mathrm{Hom}_{\mathbb{Q}S_{mn}}(\mathbb{Q}\Omega_{m,n}, \mathbb{Q}\Omega_{n,m})$ is the Schur basis element corresponding to the $(S_m \wr S_n)$-$(S_n \wr S_m)$-double coset $(S_m \wr S_n) \circ \sigma_{m,n} \circ (S_n \wr S_m)$ in S_{mn}.

In other words, if $v \in V_{m,n}$ is an S_n-minimal tuple, then $\varphi^{(m,n)}(\overline{S_n v}) \in \mathbb{Q}\Omega_{n,m}$ is the sum of all $\overline{S_m w}$, for S_m-minimal tuples in $w \in V_{n,m}$ which have the property $(\star)$ with respect to v. This is the original description given in [Black and List 89], where, as the main result, the following proposition is proved:

Proposition 4.1. [Black and List 89] *Let $m \geq n$. If $\varphi^{(m,n)}$ is injective, then $\varphi^{(m,n-1)}$ is also injective. Thus it would be enough for proving Foulkes' conjecture to show that $\varphi^{(m,m)} \in \mathrm{End}_{\mathbb{Q}S_{m^2}}(\mathbb{Q}\Omega_{m,m})$ is injective for all $m \in \mathbb{N}$.*

It has already been observed in [Black and List 89], that $\varphi^{(2,2)}$ and $\varphi^{(3,3)}$ indeed are injective. Moreover, it has been shown in [Jacob 04, Section 4.2] that $\varphi^{(4,4)}$ is injective. In the rest of this paper, we will concentrate on the question of how to decide computationally whether $\varphi^{(5,5)}$ is injective or not. Due to the sheer size of this problem, it can only be tackled using particular techniques, and the answer will be given at the very end.

5. THE COMPUTATIONAL APPROACH

Since $\dim_{\mathbb{Q}}(\mathbb{Q}\Omega_{m,m}) = |\Omega_{m,m}| = \frac{(m^2)!}{(m!)^{m+1}}$, the matrices representing the elements of $\mathrm{End}_{\mathbb{Q}S_{m^2}}(\mathbb{Q}\Omega_{m,m})$ for their natural action on $\mathbb{Q}\Omega_{m,m}$ are extremely big even for small m; for example, for $m = 5$ we have $|\Omega_{m,m}| = 5,194,672,859,376 \sim 5 \cdot 10^{12}$. Hence to examine these endomorphisms, it is necessary to work in a much smaller representation of $\mathrm{End}_{\mathbb{Q}S_{m^2}}(\mathbb{Q}\Omega_{m,m})$. As long as the latter is a faithful representation, the minimum polynomials of the elements of $\mathrm{End}_{\mathbb{Q}S_{m^2}}(\mathbb{Q}\Omega_{m,m})$ are retained, and hence injectivity can be decided using the smaller representation. Motivated by the ideas in [Müller 03], for our computations we use the left regular representation of $\mathrm{End}_{\mathbb{Q}S_{m^2}}(\mathbb{Q}\Omega_{m,m})$, which drastically reduces the size of the representing matrices: $\dim_{\mathbb{Q}}(\mathrm{End}_{\mathbb{Q}S_{m^2}}(\mathbb{Q}\Omega_{m,m}))$ equals the character theoretic scalar product of the permutation character associated to $\Omega_{m,m}$ with itself, which can be evaluated with little effort using the computer algebra system GAP [GAP 02]. For example, for $m = 5$, we find the quite moderate size $\dim_{\mathbb{Q}}(\mathrm{End}_{\mathbb{Q}S_{m^2}}(\mathbb{Q}\Omega_{m,m})) = 1,856$.

According to the description of the concatenation of homomorphisms given in Section 3, we can determine the representing matrix of $\varphi^{(m,m)}$ for its left regular action, with respect to the Schur basis of $\mathrm{End}_{\mathbb{Q}S_{m^2}}(\mathbb{Q}\Omega_{m,m})$, by counting. More precisely, we let $\Omega = \Omega' = \Omega'' = \Omega_{m,m}$ and $\omega_1 = \omega_1' = \omega_1'' = \overline{S_m x}$, as well as $G = S_{m^2}$ and $H = H' = H'' = S_m \wr S_m$, and $s_i = t_i = u_i$ and thus $A^{(i)} = B^{(i)} = C^{(i)}$, for $1 \leq i \leq l = \dim_{\mathbb{Q}}(\mathrm{End}_{\mathbb{Q}G}(\mathbb{Q}\Omega))$. Letting $s_2 := \sigma_{m,m} \in S_{m^2}$, we have $\varphi^{(m,m)} = \varphi^{(2)}$ and thus

$$A^{(2)} \circ A^{(i)} = \sum_{k=1}^{l} \left| \omega_1 \circ s_i \circ H \cap \omega_1 \circ s_2^{-1} \circ H \circ s_k \right| \cdot A^{(k)}.$$

Hence we have reduced the problem of studying $\varphi^{(m,m)}$ to the following tasks:

- Classify the H-orbits of the G-orbit $\Omega_{m,m}$, and thereby find corresponding representatives $\{s_1, s_2, \ldots, s_l\}$ of the H-H-double cosets in G, where

$s_1 = 1_G$ and $s_2 = \sigma_{m,m}$; note that $\sigma_{m,m}$ is an involution.

- Determine

$$p_{2,i,k} := \left|\overline{S_m x} \circ s_i \circ H \cap \overline{S_m x} \circ s_2^{-1} \circ H \circ s_k\right|,$$

by running through the H-orbit

$$\overline{S_m x} \circ s_2^{-1} \circ H = \overline{S_m x} \circ \sigma_{m,m} \circ H = \bigcup_{\eta \in S_m^m} \overline{S_m \tilde{x}} \circ \eta,$$

applying all representatives s_k respectively, and classifying the resulting elements into the H-orbits. Note that in the computer implementation, this is done with S_m-minimal tuples representing S_m-orbits.

- Decide whether the resulting matrix $M := [p_{2,i,j}]_{i,j=1,2,\ldots,l} \in \mathbb{Z}^{l \times l}$ has full $\mathbb{Q}$-rank.

As the numerical data for the case $m = 5$ given below indicate, the subtask of classifying points into H-orbits is still considerable. Its solution deserves a particular technique, which is a specially tailored adaptation of ideas in [Lübeck and Neunhöffer 01] and [Müller 03].

Let $U = S_m^m < S_m \wr S_m = H$ be as in Section 1. Thus, every H-orbit of $\Omega_{m,m}$ or $V_{m,m}$ is comprised of U-orbits. The basic idea now is to define U-minimal points in each U-orbit and store only those. To recognize the H-orbit of a point, we first find its U-minimalization and look that one up. To define the concept of U-minimality we first go back to tuples in $V_{m,m}$:

Definition 5.1. **(U-Minimal Tuple.)** In $V_{m,m}$, we call the lexicographically smallest tuple in each U-orbit *U-minimal*. For any $v \in V_{m,m}$, we call the U-minimal tuple in $v \circ U$ the *U-minimalization of v*.

Lemma 5.2 links the concepts of S_m-minimality and U-minimality in $V_{m,m}$.

Lemma 5.2. *If $v \in V_{m,m}$ is an S_m-minimal tuple, then its U-minimalization is again S_m-minimal.*

Proof: By Proposition 2.2 the tuple v is S_m-minimal, if and only if, for all i, j with $1 \leq i < j \leq m$, the first occurrence of i in v is before the first occurrence of j in v. Since the subgroup U just permutes the entries within the m-blocks, the process of U-minimalization just sorts the entries in each m-block into ascending order.

Let v' be the U-minimalization of v and $1 \leq i < j \leq m$. If the first occurrence of i and that of j in v are in the same m-block, then the same will be true after the sorting within the m-blocks and S_m-minimality is not violated. If they are in different m-blocks, the same holds, because their relative order is not changed at all. □

Definition 5.3. **(U-Minimal S_m-Orbits.)** An S_m-orbit $S_m v \subseteq V_{m,m}$ is called *U-minimal*, if its representing S_m-minimal tuple is a U-minimal tuple.

Since the S_m-orbits in $V_{m,m}$ are identified with $\Omega_{m,m}$, this also defines U-minimal elements of $\Omega_{m,m}$. But note that this does not mean that every U-orbit $S_m v \circ U$ in $V_{m,m}$ contains exactly one U-minimal S_m-orbit; for example, for $m = 5$, there are $2,298,891$ tuples in $V_{5,5}$ which are S_5-minimal and U-minimal at the same time, and therefore represent U-minimal S_5-orbits in $\Omega_{5,5}$, while there are only $190,131$ U-orbits in $\Omega_{5,5}$ altogether. But still, the strategy sketched above works.

6. ACTUAL COMPUTATIONS

From here on, we concentrate on the case $m = 5$, and let $G = S_{25}$ and $U = S_5^5 < S_5 \wr S_5 = H$. It turns out that there are $623,360,743,125,120 \sim 6 \cdot 10^{14}$ tuples in $V_{5,5}$ and $5,194,672,859,376 \sim 5 \cdot 10^{12}$ points in $\Omega_{5,5}$. The H-orbit $\overline{S_5 x} \circ s_2^{-1} \circ H$ has $(5!)^4 = 207,360,000 \sim 2 \cdot 10^8$ points. The number of H-orbits in $\Omega_{5,5}$ is equal to $\dim_{\mathbb{Q}}(\mathrm{End}_{\mathbb{Q}G}(\mathbb{Q}\Omega_{5,5})) = 1,856$. Thus, it is feasible, at least by distributed computing, to run through the H-orbit $\overline{S_5 x} \circ s_2^{-1} \circ H$ and to apply the H-H-double coset representatives $s_1, s_2, \ldots, s_{1856}$, once we have found them. However, as already mentioned above, we have to recognize in which H-orbit a point $\overline{S_5 x} \circ s_2^{-1} \circ h \circ s_k$ lies. Apart from the fact that we can not enumerate $\Omega_{5,5}$ completely, we could not even store an H-orbit number for each such point, as this would need at least $2 \cdot 5,194,672,859,376 \sim 10^{13}$ bytes. If we had to store every single tuple of $\Omega_{5,5}$, the situation would be even worse. To circumvent this, the notion of U-minimality comes into play.

In a precomputation, we classify all $2,298,891$ tuples in $V_{5,5}$ which are S_5-minimal and U-minimal at the same time, into the $1,856$ H-orbits in $\Omega_{5,5}$, build up a database containing these tuples and the associated H-orbit number, and determine suitable group elements $s_1, s_2, \ldots, s_{1856} \in G$.

A note on the classification of the S_5-minimal and U-minimal tuples into the H-orbits in $\Omega_{5,5}$ might be of interest: We first enumerate all these tuples by a standard backtrack method. Then we start putting each of these into a class of its own and begin applying generators of H to tuples, followed by S_5-minimalization and U-minimalization. Whenever we observe that two tuples represent S_5-orbits in the same H-orbit in $\Omega_{5,5}$, we merge their classes. We repeat this until there are only $1,856$ classes left. Hence, this is the distribution of S_5-minimal and U-minimal tuples into the H-orbits in $\Omega_{5,5}$. This approach turns out to work quite efficiently, and from this classification we can read off suitable elements $s_1, \ldots, s_{1856} \in S_{25}$.

The precomputation is implemented in the computer algebra system GAP and takes a few minutes on a modern PC. The resulting database and the elements $s_1, \ldots, s_{1856}$ are written out.

In the main computation, every time an S_5-orbit $S_5 v$, represented by an S_5-minimal tuple v, occurs we compute the S_5-minimal tuple $v' \in S_5 v$ by S_5-minimalization, then we determine the U-minimalization v'' of v', which also is a S_5-minimal tuple by Lemma 5.2. The tuple v'' is in our database, so we can look up the H-orbit number of $S_5 v''$, and, because $S_5 v''$ is in the same U-orbit as $S_5 v$, we have determined the H-orbit number of $S_5 v$ by this method.

The main computation is done in a specially tailored C program. In this part we use distributed computing, because different instances of the program on different machines can apply different elements s_k, each having the precomputed database available. After some 14 hours of computation on about 11 modern PCs, i.e., about 150 hours of CPU time, we get the resulting matrix $M \in \mathbb{Z}^{1856 \times 1856}$, representing $\varphi^{(5,5)}$ in the left regular represention of $\mathrm{End}_{\mathbb{Q}G}(\mathbb{Q}\Omega_{5,5})$.

The source code of the GAP and C programs that were used can be downloaded from the following web pages: http://www.math.rwth-aachen.de/~Max.Neunhoeffer/Mathematics/foulkes.html; http://www.expmath.org/expmath/volumes/14/14.3/Neuenhoeffer/foulkes.zip.

Finally, it remains to decide whether or not M has full $\mathbb{Q}$-rank. Actually, determining the $\mathbb{Q}$-rank or even the kernel of an integer matrix of size $1,856 \times 1,856$ is not a completely trivial task. An approach to find a vector $0 \neq v \in \mathbb{Q}^{1 \times 1856}$ with $v \cdot M = 0$ is to reduce M modulo p, where p is a suitable prime and find p-adic approximations of v inductively, until a rational lift is equal to v. This has been described in [Dixon 82]; a sample implementation is available through the function `RationalSolutionIntMat` in the GAP package EDIM [Lübeck 04].

It turns out that the matrix M does not have full $\mathbb{Q}$-rank. Actually, using the GAP package IntegralMeatAxe [Müller 04], which also employs p-adic techniques, it is possible to compute the kernel of M, which turns out to have $\mathbb{Q}$-dimension 15.

Therefore, $\varphi^{(5,5)}$ is not invertible, and hence the approach in [Black and List 89] in general does not work. Note that this does not imply a counterexample to Foulkes' conjecture. Actually Foulkes' conjecture has already been verified in [Foulkes 50] for all cases $n < m = 5$. In addition, we have used the SYMMETRICA program (see [Kerber and Kohnert 92]) to verify the conjecture for all cases with $m \leq 14$ and $n \leq 4$ and for all cases with $m \leq 12$ and $n + m \leq 17$. For bigger cases, some multiplicities of simple modules in the permutation modules are greater than 2^{31}, such that integer overflows occur on our 32 bit machines.

ADDENDUM

We have recently learned that Proposition 3.9 in [Briand 04] is a counterexample to Howe's conjecture [Howe 88], which is a strengthening of Foulkes' conjecture, and moreover is also a counterexample to Stanley's conjecture [Stanley 00, page 304], which is a generalisation of Foulkes' conjecture. We would like to thank Malek Abdesselam for pointing us in that direction.

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