Noncollision Singularities: Do Four Bodies Suffice?

Joseph L. Gerver

CONTENTS

2000 AMS Subject Classification: Primary 70F10; Secondary 70F15, 70F16

Keywords: Noncollision singularities, four-body problem, n-body problem

A heuristic model is presented for a solution of the planar Newtonian four-body problem which has a noncollision singularity.

1. INTRODUCTION

Consider a system of n point bodies, $Q_1, \ldots, Q_n$, in $\mathbf{R}^2$ or $\mathbf{R}^3$ with Newtonian gravitational potential. Let m_i, $\mathbf{r}_i(t)$, and $\mathbf{v}_i(t)$ be the mass, position, and velocity, respectively, of body i $(1 \leq i \leq n)$ at time t, and let G be the gravitational constant. The potential energy of the system is $-U$, where

$$U = \sum_{i<j} \frac{Gm_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|}, \tag{1–1}$$

and the equation of motion, for each i, is

$$m_i \mathbf{r}_i''(t) = \nabla_i U, \tag{1–2}$$

where $\nabla_i U$ is the gradient of U considered as a function of $\mathbf{r}_i$ alone, with the positions of the other bodies fixed.

If for some time t^*, $\lim_{t\to t*} \mathbf{r}_i(t)$ and $\lim_{t\to t*} \mathbf{v}_i(t)$ exist, and $\lim_{t\to t*} |\mathbf{r}_i(t) - \mathbf{r}_j(t)| \neq 0$, for all i and j, then a solution can be extended to an interval around t^*. If not, then t^* is a *singularity*. If

$$\lim_{t\to t*} \mathbf{r}_i(t) = \lim_{t\to t*} \mathbf{r}_j(t), \tag{1–3}$$

then we say that there is a *collision* between bodies i and j at $t = t^*$. Can there be a singularity without a collision? For example, Poincaré suggested, $\mathbf{r}_i(t)$ might tend to infinity, or oscillate wildly (like $\sin \frac{1}{t}$) as $t \to t^*$.

Although Poincaré never wrote anything about noncollision singularities, Painlèvé gave him credit for being the first to ask this question. Painlèvé himself was able to prove in 1897 that in a three-body system, every singularity is a collision [Painlevé 97]. Whether noncollision singularities exist for larger systems remained an open question for almost one hundred years. Von Zeipel showed in 1908 that the diameter of any system having such a singularity would have to grow infinitely large [Von

1058-6458/2003 $0.50 per page
Experimental Mathematics **12**:2, page 187

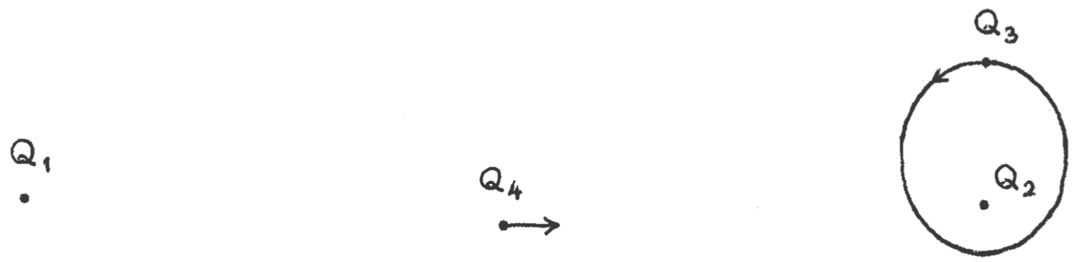

FIGURE 1.

Zeipel 08, McGehee 84], while Saari showed in 1972 that the bodies would also have to oscillate wildly [Saari 72]. A few years later, Saari proved that in a four-body system, noncollision singularities are unlikely, in the sense that the set of initial conditions leading to these singularities has measure zero [Saari 77]. It is still an open question whether this set has measure zero when there are five or more bodies. Meanwhile, in 1974, Mather and McGehee showed that if the solution is allowed to be continued through an infinite number of binary collisions, then there exist noncollision singularities with four bodies on the line [Mather and McGhee 75]. Finally, in 1988, Xia found an example of a true noncollision singularity, with no binary collisions, involving five bodies in three-dimensional space [Xia 92], and soon afterwards, Gerver found an example in the plane, involving a large, but finite, number of bodies [Gerver 91].

It is still not known whether there exist true noncollision singularities with four bodies, even in three-dimensional space, nor is it known how many bodies are required in the plane. In this paper, we suggest an answer to both questions by presenting a model for a noncollision singularity with four bodies in the plane. There are, of course, many gaps that must be filled in before this model becomes a proof of the existence of such singularities.

2. THE MODEL

As above, we let Q_1, Q_2, Q_3, and Q_4 be point masses in the plane, with Newtonian potential. We take the gravitational constant to be $\mu << 1$, and we let $m_1 = m_2 = \mu^{-1}$ and $m_3 = m_4 = 1$.

Initially, Q_3 is in an elliptical orbit about Q_2. The distance between Q_1 and Q_2 is much greater than the semimajor axis of the orbit of Q_3. Q_1 and Q_2 are moving away from each other much more slowly (by a factor on the order of μ) than the orbital velocity of Q_3, while Q_4 moves back and forth between Q_1 and the orbiting pair (Figure 1). The total energy and angular momentum of the system are both zero.

Each time Q_4 encounters the orbiting pair, it extracts either energy or angular momentum. It alternates between extracting energy and angular momentum at successive encounters, and whenever it extracts one, it leaves the other essentially unchanged. When it extracts energy, at every second encounter, Q_4 increases its own velocity by a sizable fraction. The distance between Q_1 and Q_2 also increases from one energy extracting encounter to the next, but only by a factor of $1 + O(\mu)$. Thus, the time between successive energy extracting encounters decreases in geometric progression. After a finite time, Q_4 will have traveled back and forth between Q_1 and Q_2 an infinite number of times, and Q_1 and Q_2 will have moved an infinite distance apart. At the same time, the orbit of Q_3 will have shrunk to zero, but there is no collision between Q_2 and Q_3, because both bodies escape to infinity.

Because $m_3 = m_4$, it is also possible to arrange things so that at each encounter, Q_3 and Q_4 switch places. The energy and angular momentum of the body orbiting Q_2 still decrease in the same manner as before, but the identity of that body keeps alternating between Q_3 and Q_4. Thus, the lim sup of $|\mathbf{r}_i - \mathbf{r}_j|$ is infinity for every $i \neq j$, although the lim inf is zero, unless $\{i, j\} = \{1, 2\}$. In all previous examples of noncollision singularities (McGehee and Mather, Xia, and Gerver), $\lim |\mathbf{r}_i(t) - \mathbf{r}_j(t)| = 0$ for some $\{i, j\}$ as t approaches the singularity. In what follows, we shall assume that the body orbiting Q_2 is always Q_3, and the body moving back and forth between Q_1 and Q_2 is always Q_4. But almost everything we say also applies when Q_3 and Q_4 keep switching places.

Because Q_3 is in an elliptical orbit, its energy is negative. The kinetic energy of Q_1 and Q_2 are neglible (on

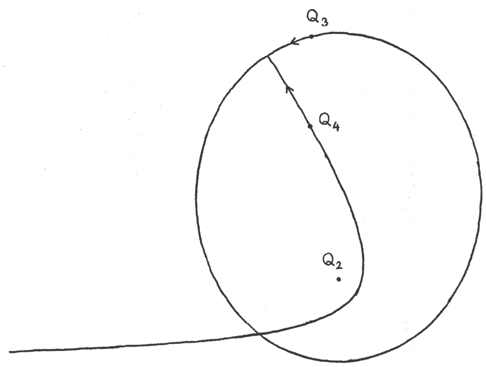

FIGURE 2.

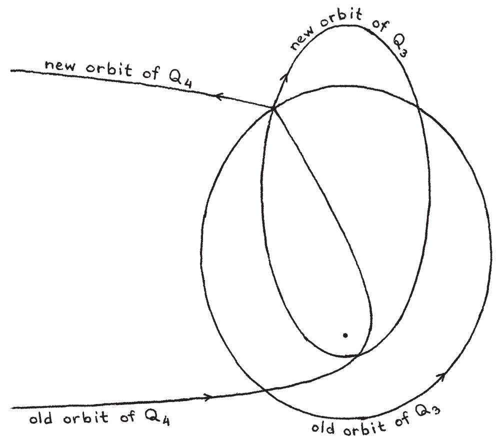

FIGURE 3.

the order of μ) compared to the energy of Q_3, but since the total energy is zero, Q_4 must have a positive energy which nearly cancels the negative energy of Q_3. Thus, Q_4 approaches Q_2 along a hyperbolic orbit. The semimajor axis of the hyperbola is negative, but it must have nearly the same absolute value as the semimajor axis of the ellipse. Because the asymptotes of the hyperbola do not cross at a particularly small angle, Q_4 never gets much closer than Q_3 does to Q_2. Even when Q_3 and Q_4 are at comparable distances from Q_2, their mutual attraction is neglible, and they continue to adhere closely to their respective conic sections. Only when Q_3 and Q_4 approach each other much more closely than Q_2 do their orbits change significantly. Around the time of a near collision between Q_3 and Q_4, their paths can be approximated by hyperbolic orbits around their common center of gravity. But when we compute the new orbits of Q_3 and Q_4 around Q_2 after their near collision, we can approximate the near collision by an actual elastic collision between Q_3 and Q_4.

We model the encounter between Q_4 and the orbiting pair as follows: Q_2 remains fixed at the origin, while Q_3 travels around it in an elliptical orbit. Q_4 approaches from the left along a hyperbolic orbit, with the incoming asymptote parallel to the x-axis. (We take Q_1 to lie on the x-axis at $-\infty$.) We ignore the attractive force between Q_3 and Q_4. The positive energy of Q_4 exactly cancels the negative energy of Q_3. An elastic collision occurs between Q_3 and Q_4 (Figure 2). The velocities of these bodies after the collision are uniquely determined by the fact that the collision is elastic (i.e., momentum and kinetic energy are conserved), and by the fact that Q_4 must end up in a hyperbolic orbit approaching an asymptote parallel to the x-axis, moving in the negative x direction (Figure 3).

In our approximate model, once we are given the initial orbit of Q_3, we are free to choose any point on that orbit for the elastic collision. That means we must be free to choose a hyperbolic orbit for Q_4 which intersects the orbit of Q_3 at the chosen point of collision, and we must be free to choose a relative phase for Q_3 and Q_4 so that both bodies arrive at the collision point at the same time. This freedom is a reasonable feature of our model, because a small change in the position of Q_4 at the time of its encounter with Q_1 will cause a small change in its direction of motion afterwards, and in the limit as the x-coordinate of Q_1 goes to $-\infty$, this small change in direction becomes a large change in the y-intercept of the incoming asymptote of the orbit of Q_4 around Q_2, without affecting the direction of the asymptote, which remains parallel to the x-axis. Likewise, a small change in the position of Q_3 in its orbit at the time of one close encounter with Q_4 will result in a small change in the semimajor axis of the new orbit of Q_3 after the encounter. This in turn affects the number of revolutions of Q_3 until its next encounter with Q_4, and thus results in a large change in the position of Q_3 at the next encounter. So by fine-tuning the initial conditions, it ought to be possible to adjust the y-coordinate of the incoming asymptote of Q_4 and the phase of Q_3 at every future encounter between these two bodies.

3. TRANSFER OF ANGULAR MOMENTUM

Returning to our approximate model, we suppose that initially the orbit of Q_3 has eccentricity ε_0, where $0 < \varepsilon_0 < \frac{1}{2}\sqrt{2}$, that the angular momentum of Q_3 is positive, so that Q_3 travels counterclockwise around Q_2, that the major axis of the orbit coincides with the y-axis, and the periapsis lies on the negative y-axis. We also assume, without loss of generality, that the semimajor axis of the orbit is 1.

Let $\varepsilon_1 = \sqrt{1-\varepsilon_0^2}$. Note that $\frac{1}{2}\sqrt{2} < \varepsilon_1 < 1$ so that $\varepsilon_0 < \varepsilon_1$. We will show that for a suitable choice of the point of collision between Q_3 and Q_4, the orbit of Q_3 after the collision will have eccentricity ε_1, with negative (clockwise) angular momentum. The major axis of the new orbit will still coincide with the y-axis, with the periapsis at $y < 0$, and the semimajor axis will still be 1. At this encounter, Q_4 extracts angular momentum, but no energy, from Q_3.

We let the elastic collision occur at (X, Y), where

$$X = -\varepsilon_0\varepsilon_1 \tag{3–1}$$

and

$$Y = \varepsilon_0 + \varepsilon_1. \tag{3–2}$$

This point is on both the old orbit of Q_3,

$$\frac{x^2}{\varepsilon_1^2} + (y-\varepsilon_0)^2 = 1, \tag{3–3}$$

and the new orbit,

$$\frac{x^2}{\varepsilon_0^2} + (y-\varepsilon_1)^2 = 1. \tag{3–4}$$

Note that the old orbit has semiminor axis ε_1, and ε_1 is also the angular momentum of Q_3 before the collision. The semiminor axis of the new orbit is ε_0 and the angular momentum of Q_3 after the collision is $-\varepsilon_0$. The energy of Q_3, both before and after the collision, is $-\frac{1}{2}$.

Let (v_x, v_y) and (u_x, u_y) be the velocity of Q_3 immediately before and after the collision, respectively. We know, from the angular momentum and energy of the old and new orbits, that

$$Xv_y - Yv_x = \varepsilon_1, \tag{3–5}$$

$$\frac{1}{2}(v_x^2 + v_y^2) - (X^2+Y^2)^{-1/2} = -\frac{1}{2}, \tag{3–6}$$

$$Xu_y - Yu_x = -\varepsilon_0, \tag{3–7}$$

and

$$\frac{1}{2}(u_x^2 + u_y^2) - (X^2+Y^2)^{-1/2} = -\frac{1}{2}. \tag{3–8}$$

The above equations, along with the fact that the major axes of both orbits coincide with the y-axis (which tells us that $Xv_x + Yv_y < 0$ and $Xu_x + Yu_y > 0$), uniquely determine v_x, v_y, u_x, and u_y, viz.

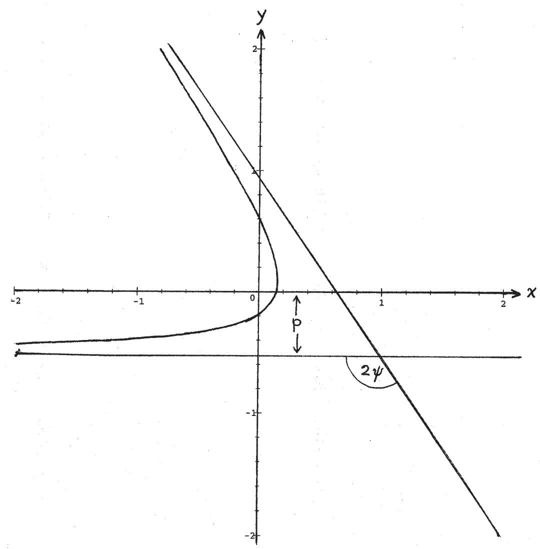

FIGURE 4.

$$v_x = \frac{-\varepsilon_1^2}{\varepsilon_0\varepsilon_1 + 1}, \tag{3–9}$$

$$v_y = \frac{-\varepsilon_0}{\varepsilon_0\varepsilon_1 + 1}, \tag{3–10}$$

$$u_x = \frac{\varepsilon_0^2}{\varepsilon_0\varepsilon_1 + 1}, \tag{3–11}$$

and

$$u_y = \frac{\varepsilon_1}{\varepsilon_0\varepsilon_1 + 1}. \tag{3–12}$$

We must find old and new hyperbolic orbits for Q_4, both with energy $+\frac{1}{2}$, with the incoming asymptote of the old orbit and the outgoing asymptote of the new orbit both parallel to the negative x-axis, such that both orbits intersect the point (X, Y) and the total momentum of Q_3 and Q_4 is the same before and after the collision.

Both hyperbolas will have semimajor axis -1, with one focus at the origin, and one asymptote $y = -p$ for some real number p (Figure 4). Suppose the asymptotes intersect at an angle of 2ψ. Let

$$\tilde{x} = x\cos\psi + y\sin\psi \tag{3–13}$$

and

$$\tilde{y} = y\cos\psi - x\sin\psi + \csc\psi. \tag{3–14}$$

Then the equation of the hyperbola is

$$\tilde{y}^2 - (\tilde{x}\tan\psi)^2 = 1 \tag{3–15}$$

and $p = \cot\psi$ (Figure 5).

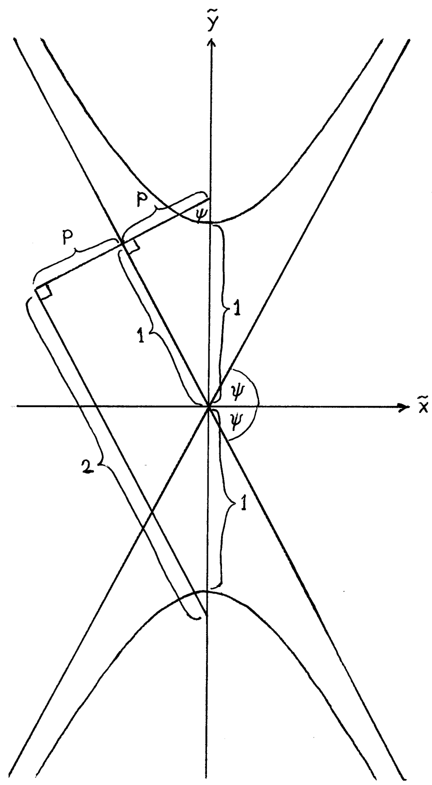

FIGURE 5.

Translating back to x and y coordinates, we have

$$(y\cos\psi - x\sin\psi + \csc\psi)^2 - (x\sin\psi + y\sin\psi\tan\psi)^2 = 1, \tag{3–16}$$

or

$$y^2(\cos^2\psi - \sin^2\psi\tan^2\psi) - 2xy(\sin\psi\cos\psi + \sin^2\psi\tan\psi) + 2y\cot\psi - 2x + \csc^2\psi = 1. \tag{3–17}$$

A few trig identities yield

$$y^2(1-\tan^2\psi) - 2xy\tan\psi + 2y\cot\psi - 2x + 1 + \cot^2\psi = 1, \tag{3–18}$$

or

$$y^2\left(1-\frac{1}{p^2}\right) - \frac{2xy}{p} + 2yp - 2x + p^2 = 0. \tag{3–19}$$

Multiplying by p^2 and collecting like powers of p, we get

$$p^4 + 2yp^3 + (y^2-2x)p^2 - 2xyp - y^2 = 0 \tag{3–20}$$

or

$$(p^2 + yp - x - r)(p^2 + yp - x + r) = 0, \tag{3–21}$$

where $r = \sqrt{x^2+y^2}$.

Each factor on the left-hand side of (3–21) represents one branch of the hyperbola. The first factor is the branch closest to the focus at the origin, occupied by Q_2. This branch is the projection onto the xy-plane of the intersection of the plane $z = p^2 - yp - x$ with the half-cone $z = r$. The second factor is the branch closest to the empty focus, the projection of the intersection of the same plane with the half-cone $z = -r$. Q_4 follows the first branch, so if the orbit of Q_4 is to intersect the point (X, Y), we must have

$$p^2 + Yp - X - R = 0, \tag{3–22}$$

where $R = \sqrt{X^2+Y^2}$. Thus, $p = p_1$ or p_2, where

$$p_1 = \frac{-Y + \sqrt{Y^2 + 4(X+R)}}{2} \tag{3–23}$$

and

$$p_2 = \frac{-Y - \sqrt{Y^2 + 4(X+R)}}{2}. \tag{3–24}$$

We choose $y = -p_1$ to be the incoming asymptote of the old orbit of Q_4, and $y = -p_2$ to be the outgoing asymptote of the new orbit.

Let $\hat{v}_x$ and $\hat{v}_y$ be the x and y components of the velocity of Q_4 going into the collision, and let $\hat{u}_x$ and $\hat{u}_y$ be the components of its velocity coming out. The angular momentum of Q_4 is p_1 before the collision and $-p_2$ afterwards, while its energy is $\frac{1}{2}$ both before and after. Thus,

$$X\hat{v}_y - Y\hat{v}_x = p_1, \tag{3–25}$$

$$\frac{1}{2}(\hat{v}_x^2 + \hat{v}_y^2) - R^{-1} = \frac{1}{2}, \tag{3–26}$$

$$X\hat{u}_y - Y\hat{u}_x = -p_2, \tag{3–27}$$

and

$$\frac{1}{2}(\hat{u}_x^2 + \hat{u}_y^2) - R^{-1} = \frac{1}{2}. \tag{3–28}$$

These constraints determine $\hat{v}_x$, $\hat{v}_y$, $\hat{u}_x$, and $\hat{u}_y$, once we know the signs of $X\hat{v}_x + Y\hat{v}_y$ and $X\hat{u}_x + Y\hat{u}_y$. The fact that the incoming asymptote of the old orbit and the outgoing asymptote of the new orbit are parallel to the x-axis constrains both signs to be positive. We conclude that

$$\hat{v}_x = 1 - \frac{Y}{Rp_1}, \tag{3–29}$$

$$\hat{v}_y = \frac{1}{Rp_1}, \tag{3–30}$$

$$\hat{u}_x = -1 + \frac{Y}{Rp_2}, \tag{3–31}$$

and

$$\hat{u}_y = \frac{-1}{Rp_2}. \tag{3–32}$$

Indeed,

$$R = \sqrt{(\varepsilon_0\varepsilon_1)^2 + (\varepsilon_0+\varepsilon_1)^2} = \sqrt{\varepsilon_0^2\varepsilon_1^2 + \varepsilon_0^2 + 2\varepsilon_0\varepsilon_1 + \varepsilon_1^2}$$
$$= \sqrt{\varepsilon_0^2\varepsilon_1^2 + 2\varepsilon_0\varepsilon_1 + 1}$$
$$= \varepsilon_0\varepsilon_1 + 1 = 1 - X, \qquad (3\text{–}33)$$

so $X + R = 1$, $p_1^2 + Yp_1 - 1 = 0$, $p_1 = 1/p_1 - Y$, and

$$X\big(\frac{1}{Rp_1}\big) - Y\big(1 - \frac{Y}{Rp_1}\big) = \frac{X+Y^2}{Rp_1} - Y$$
$$= \frac{-\varepsilon_0\varepsilon_1 + (\varepsilon_0+\varepsilon_1)^2}{Rp_1} - Y$$
$$= \frac{\varepsilon_0^2 + \varepsilon_0\varepsilon_1 + \varepsilon_1^2}{Rp_1} - Y$$
$$= \frac{1+\varepsilon_0\varepsilon_1}{Rp_1} - Y = \frac{1}{p_1} - Y = p_1. \qquad (3\text{–}34)$$

Likewise,

$$Y^2 + 1 = \varepsilon_0^2 + 2\varepsilon_0\varepsilon_1 + \varepsilon_1^2 + 1 = 2\varepsilon_0\varepsilon_1 + 2 = 2R, \qquad (3\text{–}35)$$

so

$$\big(1 - \frac{Y}{Rp_1}\big)^2 + \big(\frac{1}{Rp_1}\big)^2 - \frac{2}{R} = 1 - \frac{2Y}{Rp_1} + \frac{Y^2+1}{R^2p_1^2} - \frac{2}{R}$$
$$= 1 - \frac{2Y}{Rp_1} + \frac{2R}{R^2p_1^2} - \frac{2}{R}$$
$$= 1 - \frac{2Y}{Rp_1} + \frac{2}{Rp_1^2} - \frac{2}{R}$$
$$= 1 - \frac{2}{Rp_1^2}(p_1^2 + Yp_1 - 1)$$
$$= 1, \qquad (3\text{–}36)$$

and in a similar manner, we can show that the constraints involving $\hat{u}_x$ and $\hat{u}_y$ are satisfied.

We now need only show that momentum is conserved during the collision between Q_3 and Q_4. That is, we must show that $v_x + \hat{v}_x = u_x + \hat{u}_x$ and $v_y + \hat{v}_y = u_y + \hat{u}_y$. Indeed,

$$u_x - v_x = \frac{\varepsilon_0^2 + \varepsilon_1^2}{\varepsilon_0\varepsilon_1 + 1} = \frac{1}{R}, \qquad (3\text{–}37)$$

and since

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{p_1+p_2}{p_1p_2} = \frac{-Y}{\frac{1}{4}[Y^2 - (Y^2+4)]} = Y, \qquad (3\text{–}38)$$

we have

$$\hat{v}_x - \hat{u}_x = 2 - \frac{Y}{Rp_1} - \frac{Y}{Rp_2} = 2 - \frac{Y^2}{R} = \frac{2R - Y^2}{R} = \frac{1}{R}, \qquad (3\text{–}39)$$

while

$$u_y - v_y = \frac{\varepsilon_0 + \varepsilon_1}{\varepsilon_0\varepsilon_1 + 1} = \frac{Y}{R} \qquad (3\text{–}40)$$

and

$$\hat{v}_y - \hat{u}_y = \frac{1}{Rp_1} + \frac{1}{Rp_2} = \frac{Y}{R}. \qquad (3\text{–}41)$$

4. TRANSFER OF ENERGY

We next examine the second collision, in which Q_4 extracts energy from Q_3, but no angular momentum is exchanged. This time, the orbit of Q_3 before the collision has eccentricity ε_1 and semimajor axis 1, and the orbit afterwards has eccentricity ε_0 and semimajor axis $\varepsilon_0^2/\varepsilon_1^2$. Both orbits have negative angular momentum and major axes coinciding with the y-axis, but before the collision, the periapsis is on the negative y-axis, and afterwards, it is on the positive side (Figure 6). The equation of the old orbit is

$$\frac{x^2}{\varepsilon_0^2} + (y - \varepsilon_1)^2 = 1 \qquad (4\text{–}1)$$

and that of the new orbit is

$$\frac{\varepsilon_1^2}{\varepsilon_0^4}x^2 + \big(\frac{\varepsilon_1^2}{\varepsilon_0^2}y + \varepsilon_0\big)^2 = 1. \qquad (4\text{–}2)$$

Note that Q_3 has angular momentum $-\varepsilon_0$ both before and after the collision, but its energy decreases from $-\frac{1}{2}$ to $-\varepsilon_1^2/2\varepsilon_0^2$.

This time, the collision occurs at $(\tilde{X}, \tilde{Y})$, where

$$\tilde{X} = \varepsilon_0^2 \qquad (4\text{–}3)$$

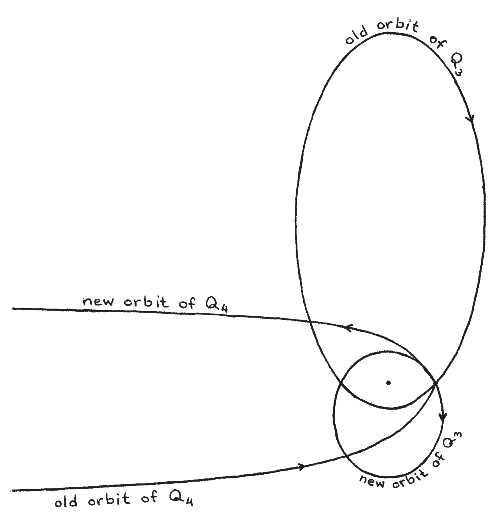

FIGURE 6.

and

$$\tilde{Y} = 0. \tag{4–4}$$

Again we let v_x and v_y be the x and y components of the velocity of Q_3 before the collision, and let u_x and u_y be the components of the velocity afterwards. We have

$$\tilde{X} v_y - \tilde{Y} v_x = -\varepsilon_0, \tag{4–5}$$

$$\frac{1}{2}(v_x^2 + v_y^2) - (\tilde{X}^2 + \tilde{Y}^2)^{-1/2} = -\frac{1}{2}, \tag{4–6}$$

$$\tilde{X} u_y - \tilde{Y} u_x = -\varepsilon_0, \tag{4–7}$$

and

$$\frac{1}{2}(u_x^2 + u_y^2) - (\tilde{X}^2 + \tilde{Y}^2)^{-1/2} = -\frac{\varepsilon_1^2}{2\varepsilon_2^2}, \tag{4–8}$$

while $\tilde{X} v_x + \tilde{Y} v_y < 0$ and $\tilde{X} u_x + \tilde{Y} u_y > 0$. These constraints uniquely determine the velocity of Q_3 before and after the collision:

$$v_x = -\frac{\varepsilon_1}{\varepsilon_0}, \tag{4–9}$$

$$v_y = -\frac{1}{\varepsilon_0}, \tag{4–10}$$

$$u_x = 1, \tag{4–11}$$

and

$$u_y = -\frac{1}{\varepsilon_0}. \tag{4–12}$$

The orbit of Q_4 before and after the collision is easily determined. Beforehand, we can still approximate the energy of Q_4 as $\frac{1}{2}$, because Q_4 transfers a negligible amount of energy (on the order of μ) to Q_1 between its two encounters with Q_3. Thus, the incoming asymptote of Q_4 should be $y = -p$, where p satisfies

$$p^2 + \tilde{Y} p - \tilde{X} - \tilde{R} = 0, \tag{4–13}$$

with $\tilde{R} = \sqrt{\tilde{X}^2 + \tilde{Y}^2} = \varepsilon_0^2$. Hence, $p = \pm\sqrt{2}\varepsilon_0$. We let $p = \sqrt{2}\varepsilon_0$, so that the incoming asymptote is $y = -\sqrt{2}\varepsilon_0$, and the angular momentum of Q_4 before the collision is $\sqrt{2}\varepsilon_0$. Letting $\hat{v}_x$ and $\hat{v}_y$ be the components of the velocity of Q_4 going into the collision, we have

$$\tilde{X}\hat{v}_y - \tilde{Y}\hat{v}_x = p \tag{4–14}$$

and

$$\frac{1}{2}(\hat{v}_x^2 + \hat{v}_y^2) - \tilde{R}^{-1} = \frac{1}{2}, \tag{4–15}$$

or

$$\varepsilon_0^2 \hat{v}_y = \sqrt{2}\varepsilon_0 \tag{4–16}$$

and

$$\frac{1}{2}(\hat{v}_x^2 + \hat{v}_y^2) - \varepsilon_0^{-2} = \frac{1}{2}. \tag{4–17}$$

Thus

$$\hat{v}_y = \frac{\sqrt{2}}{\varepsilon_0} \tag{4–18}$$

and

$$\hat{v}_x = 1. \tag{4–19}$$

Note that the solution $\hat{v}_x = -1$ is ruled out by the condition that the incoming asymptote of the orbit be parallel to the x-axis.

After the collision, the energy of Q_4 is $-\varepsilon_1^2/2\varepsilon_0^2$, so the semimajor axis of the hyperbolic orbit of Q_4 is $-\varepsilon_0^2/\varepsilon_1^2$ and the speed of Q_4 at infinity is $\varepsilon_1/\varepsilon_0$. We can normalize the semimajor axis to -1 if we replace x and y by $x\varepsilon_1^2/\varepsilon_0^2$ and $y\varepsilon_1^2/\varepsilon_0^2$, respectively. If the outgoing asymptote is $y\varepsilon_1^2/\varepsilon_0^2 = -p$, then p must satisfy

$$p^2 + \tilde{Y} p \varepsilon_1^2/\varepsilon_0^2 - \tilde{X}\varepsilon_1^2/\varepsilon_0^2 - \tilde{R}\varepsilon_1^2/\varepsilon_0^2 = 0, \tag{4–20}$$

where again $\tilde{Y} = 0$ and $\tilde{X} = \tilde{R} = \varepsilon_0^2$. Thus, $p = \pm\sqrt{2}\varepsilon_1$. This time, we choose $p = -\sqrt{2}\varepsilon_1$, so the outgoing asymptote is $y\varepsilon_1^2/\varepsilon_0^2 = \sqrt{2}\varepsilon_1$, or $y = \sqrt{2}\varepsilon_0^2/\varepsilon_1$, and the angular momentum of Q_4 is still $(\sqrt{2}\varepsilon_0^2/\varepsilon_1)(\varepsilon_1/\varepsilon_0) = \sqrt{2}\varepsilon_0$. Therefore, if $\hat{u}_x$ and $\hat{u}_y$ are the components of the velocity of Q_4 immediately after the collision, we have

$$\varepsilon_0^2 \hat{u}_y = \sqrt{2}\varepsilon_0 \tag{4–21}$$

and

$$\frac{1}{2}(\hat{u}_x^2 + \hat{u}_y^2) - \frac{1}{\varepsilon_0^2} = \frac{\varepsilon_1^2}{2\varepsilon_0^2}. \tag{4–22}$$

Thus,

$$\hat{u}_y = \frac{\sqrt{2}}{\varepsilon_0} \tag{4–23}$$

and

$$\hat{u}_x = -\frac{\varepsilon_1}{\varepsilon_0}. \tag{4–24}$$

(The outgoing asymptote of the new orbit of Q_4 is parallel to the x-axis, so $\hat{u}_x$ must be negative.)

Because the total energy of Q_3 and Q_4 is zero both before and after the collision, we need only show that momentum is conserved to prove that the collision is elastic. We have

$$u_x - v_x = 1 + \frac{\varepsilon_1}{\varepsilon_0} = \hat{v}_x - \hat{u}_x \tag{4–25}$$

and

$$u_y - v_y = 0 = \hat{v}_y - \hat{u}_y. \tag{4–26}$$

After the second encounter of Q_4 with Q_3, the orbit of Q_3 has the same eccentricity ε_0 that it had before the first encounter, but the orbit is smaller by a factor of $\varepsilon_1^2/\varepsilon_0^2$ and has been reflected around the x axis. We can therefore arrange for Q_4 to have another encounter with

Q_3 after a roundtrip to Q_1, in which the phase of Q_3 is the same as at the first encounter (but reflected about the x axis), so that Q_4 again extracts angular momentum, but no energy, from Q_3, and this can be followed by a fourth encounter in which Q_4 extracts energy from Q_3, but no angular momentum. After this fourth encounter, the orbit of Q_3 is once again reflected about the x-axis, back to the same side where it started, but smaller by a factor of $\varepsilon_1^4/\varepsilon_0^4$. The process can be continued indefinitely, with the orbit of Q_3 shrinking by the same factor of $\varepsilon_1^2/\varepsilon_0^2$ at every second encounter with Q_4. The energy of Q_4 increases by a factor of $\varepsilon_1^2/\varepsilon_0^2$ at each such encounter, and it only loses $O(\mu)$ when it swings around Q_1, so its velocity during the trip between Q_1 and Q_2 (except when it is close to Q_1 or Q_2, and its potential energy is comparable to its kinetic energy) increases by a factor of nearly $\varepsilon_1/\varepsilon_0$, which is greater than 1. Because the distance between Q_1 and Q_2 increases only slightly, by a factor of $1 + O(\mu)$, during each roundtrip of Q_4, the time required for each double roundtrip decreases in geometric progression, by a factor only slightly greater than $\varepsilon_0/\varepsilon_1$; this factor is strictly less than 1, provided we choose ε_0 not too close to $\frac{1}{2}\sqrt{2}$. That means an infinite number of roundtrips occur in a finite time. During each roundtrip, a small fraction of the energy of Q_4 is transferred to the outward motion of Q_1 and Q_2 away from each other (with Q_3 dragged along by Q_2), so after an infinite number of roundtrips, Q_1 and Q_2 will have moved an infinite distance apart along the x-axis. Thus, a noncollision singularity occurs after a finite time.

5. WHAT CAN GO WRONG

Several things could go wrong with this scenario, and we must check that none of them happen.

Before an angular momentum extracting encounter between Q_3 and Q_4, the former travels along an ellipse and the latter along a hyperbola. The encounter occurs when the two bodies pass close to the point of intersection of the ellipse and hyperbola at the same time. But the ellipse and hyperbola intersect at two points, and both bodies pass the wrong point before approaching the correct one (Figure 7). We must check that if the relative phases of Q_3 and Q_4 are such that they approach the right point of intersection at the same time, then they do not also approach the wrong point at the same time. In other words, we need to prove that Q_3 and Q_4 do not both require the same time to move between the two points of intersection. It is straightforward to demostrate this numerically for any given ε_0. For example, when $\varepsilon_0 = \frac{1}{2}$, the right point of intersection is $(X, Y) = (-\frac{1}{4}\sqrt{3}, \frac{1}{2} + \frac{1}{2}\sqrt{3}) = (-.433, 1.366)$ and the wrong point is $(-.480, -.332)$. Q_3 requires 4.226 units of time to move from the wrong point to the right point, and Q_4 requires only .995 units of time. It seems to be true in general that Q_3 requires more time than Q_4, although it is not clear how to prove this for arbitrary ε_0.

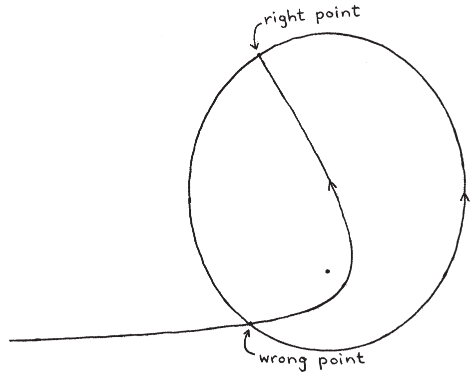

FIGURE 7.

After an angular momentum extracting encounter between Q_3 and Q_4, the new orbits of the two bodies do not cross again. Likewise, the orbits do not cross before an energy extracting encounter, because $\hat{v}_y/\hat{v}_x > v_y/v_x$ (Figure 8). The orbits do cross after an energy extracting encounter, provided $\varepsilon_0 < \frac{1}{3}\sqrt{3}$ (that is, when $\hat{u}_y/\hat{u}_x > u_y/u_x$), but once again, Q_3 takes much longer than Q_4 to traverse its arc (Figure 9). So in both cases, the two bodies do not interfere with each other before

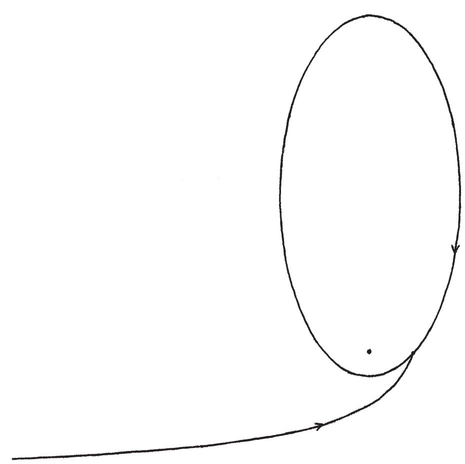

FIGURE 8.

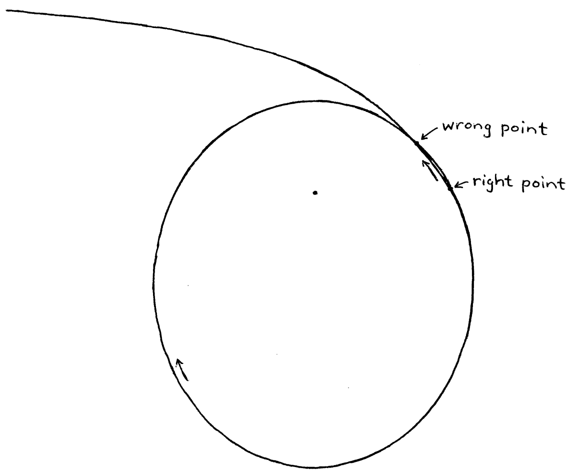

FIGURE 9.

their encounter, and they make a clean getaway afterwards.

Next, we must check that none of the bodies have an actual binary collision before the noncollision singularity. Any such collision would have to involve Q_4, because Q_1 never gets close to Q_2 or Q_3, and these last two cannot collide with each other because Q_3 is always in an elliptical orbit around Q_2 with eccentricity ε_0 or ε_1.

It is easy to check that Q_4 never collides with Q_2. We need only check the four hyperbolic orbits of Q_4 around Q_2, namely before and after an angular momentum extracting encounter with Q_3, and before and after an energy extracting encounter. Specifically, we must calculate the asymptote parallel to the x-axis for each of the four hyperbolas. These are, respectively, $y = \frac{1}{2}(Y - \sqrt{Y^2+4})$, where $Y = \varepsilon_0 + \varepsilon_1$, $y = \frac{1}{2}(Y + \sqrt{Y^2+4})$, $y = -\sqrt{2}\varepsilon_0$, and $y = \sqrt{2}\varepsilon_0^2/\varepsilon_1$. In no case is the asymptote $y = 0$, so there is no collision with Q_2.

To see that Q_4 does not collide with Q_3, we must look at the velocities of both bodies going into and coming out of both the angular momentum and energy extracting encounters. Although we approximated each such encounter by an elastic collision, Q_3 and Q_4 actually travel in close hyperbolic orbits around their center of gravity during these encounters. A true binary collision between these bodies can occur only if the hyperbolas are degenerate. That is, the velocity of each body after the encounter, in the center of mass coordinate system of the two bodies, must be in the direction opposite the velocity of that body before the encounter. In other words, it would require that

$$u_x - \frac{u_x + \hat{u}_x}{2} = -\big(v_x - \frac{v_x + \hat{v}_x}{2}\big) \tag{5–1}$$

and

$$u_y - \frac{u_y + \hat{u}_y}{2} = -\big(v_y - \frac{v_y + \hat{v}_y}{2}\big) \tag{5–2}$$

or, equivalently, $v_x + u_x = \hat{v}_x + \hat{u}_x$ and $v_y + u_y = \hat{v}_y + \hat{u}_y$. At the angular momentum extracting encounter, we have

$$v_x + u_x = \frac{\varepsilon_0^2 - \varepsilon_1^2}{\varepsilon_0\varepsilon_1 + 1} = \frac{-Y}{R}(\varepsilon_1 - \varepsilon_0), \tag{5–3}$$

$$v_y + u_y = \frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_0\varepsilon_1 + 1} = \frac{1}{R}(\varepsilon_1 - \varepsilon_0), \tag{5–4}$$

$$\hat{v}_x + \hat{u}_x = \frac{Y}{Rp_2} - \frac{Y}{Rp_1} = \frac{Y}{R}\big(\frac{p_1 - p_2}{p_1 p_2}\big) = \frac{-Y}{R}\sqrt{Y^2+4}, \tag{5–5}$$

and

$$\hat{v}_y + \hat{u}_y = \frac{1}{Rp_1} - \frac{1}{Rp_2} = \frac{1}{R}\sqrt{Y^2+4}. \tag{5–6}$$

Thus, a collision can occur only if $\varepsilon_1 - \varepsilon_0 = \sqrt{Y^2+4} = \sqrt{2\varepsilon_0\varepsilon_1 + 5}$. Squaring both sides, we see that this implies $1 - 2\varepsilon_0\varepsilon_1 = 2\varepsilon_0\varepsilon_1 + 5$, or $\varepsilon_0\varepsilon_1 + 1 = R = 0$, so no collision occurs during the angular momentum extracting encounter. At the energy extracting encounter, we find that indeed

$$v_x + u_x = 1 - \frac{\varepsilon_1}{\varepsilon_0} = \hat{v}_x + \hat{u}_x, \tag{5–7}$$

but

$$v_y + u_y = -\frac{2}{\varepsilon_0} \tag{5–8}$$

and

$$\hat{v}_y + \hat{u}_y = \frac{2\sqrt{2}}{\varepsilon_0}, \tag{5–9}$$

so again there is no collision between Q_3 and Q_4.

The one remaining possibility is a collision between Q_4 and Q_1. Each time Q_4 passes close to Q_1, it gets sent back toward Q_2 and Q_3. Thus, the velocity of Q_4 is rotated by very nearly 180 degrees during its encounter with Q_1, and the two bodies come close to a binary collision. We must show that the rotation is never exactly 180 degrees in the center of mass coordinate system of the two bodies.

We first look at the path of Q_4 in a Q_2-centered system. We must consider two cases: In the first case, Q_4 encounters Q_1 after extracting angular momentum from Q_3 and before extracting energy. In the second case, Q_4 encounters Q_1 after extracting energy from Q_3 and before extracting angular momentum. After extracting angular momentum, Q_4 moves toward the left along a path asymptotic to the line $y = \frac{1}{2}(\varepsilon_0 + \varepsilon_1 + \sqrt{2\varepsilon_0\varepsilon_1 + 5})$. When it heads back toward Q_2 to extract energy, Q_4

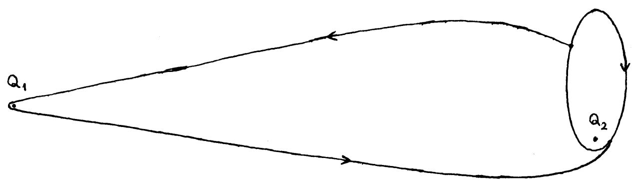

FIGURE 10.

moves along a path asymptotic to the line $y = -\sqrt{2}\varepsilon_0$. Of course, the two lines cannot really be parallel. They must converge slightly and intersect near Q_1 (Figure 10). Because Q_4 has different y-coordinates right after extracting angular momentum and right before extracting energy, its velocity coming out of its encounter with Q_1 cannot be exactly in the opposite direction from its velocity going into the encounter. Likewise, right after extracting energy from Q_3, Q_4 is on a path asymptotic to $y = \sqrt{2}\varepsilon_1$, if we normalize the x and y coordinates so that the semimajor axis of the orbit of Q_3 is 1 (Figure 11). Right before the following angular momentum extracting encounter, the path of Q_4 is asymptotic to $y = \frac{1}{2}(-\varepsilon_0 - \varepsilon_1 + \sqrt{2\varepsilon_0\varepsilon_1 + 5}) < \sqrt{2}\varepsilon_1$. (Recall that this angular momentum extracting encounter, after normalization, is the reflection about the x-axis of the previous angular momentum extracting encounter.) Again, the y-coordinates are different, so the direction of Q_4 is not exactly reversed during its encounter with Q_1.

However, we have been considering the velocity of Q_4 relative to Q_2. What counts is the velocity of Q_4 relative to Q_1. If Q_1 should have a small y-component to its velocity, on the order of the y-component of the velocity of Q_4, then it could still happen that the velocity of Q_4 relative to Q_1 does exactly reverse its direction during the encounter between Q_4 and Q_1, in which case the encounter would be a binary collision.

We can rule out this possibility because the total angular momentum of the system is zero. Recall that Q_3 and Q_4 have mass 1, while Q_1 and Q_2 have mass $\mu^{-1} >> 1$. We normalize the time and distance scales so that the semimajor axis of the orbit of Q_3 is 1, and the mean velocity of Q_3 in its orbit is 1. Then the energy of Q_3 is approximately $-\frac{1}{2}$. Since the kinetic energy of Q_1 and Q_2 are much less than 1 (on the order of μ), and the total energy of the system is zero, the energy of Q_4 is approximately $\frac{1}{2}$, and the velocity of Q_4 (except when it is close to Q_1 or Q_2) is approximately 1. The energy of Q_1 and Q_2 remains on the order of μ times the energy of Q_4, because each time Q_4 encounters Q_1 or the orbiting pair, it transfers to them approximately 2 units momentum. Therefore, the velocity of Q_1 and the orbiting pair away from each other both increase by about 2μ times the velocity of Q_4 at each encounter. But since the velocity of Q_4 increases by a factor of $\varepsilon_1/\varepsilon_0 > 1$ at every second encounter with the orbiting pair, the velocity of Q_1 and the orbiting pair away from each other is of the same order as the change in this velocity. That is, the velocity of Q_1 and the orbiting pair is on the order of μ times the velocity of Q_4, and the kinetic energy of Q_1 and the orbiting pair is on the order of $\mu^{-1}\mu^2 = \mu$ times the energy of Q_4.

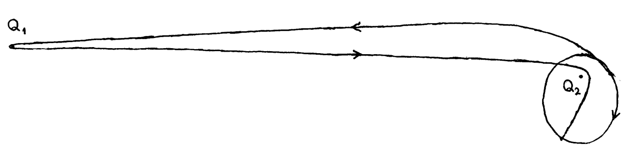

FIGURE 11.

Let χ be the distance between Q_1 and Q_2 (in units of the semimajor axis of the orbit of Q_3). Now the angular momentum of Q_2 and Q_3 about their center of mass is on the order of 1 (it goes from ε_1 to $-\varepsilon_0$ to $-\varepsilon_1$ to ε_0 and back to ε_1 in our normalized units). Likewise, when Q_4 is midway between Q_1 and Q_2, the angular momentum of Q_4 about the center of mass of all four bodies is on the order of 1 (it is $\pm\frac{1}{4}(\varepsilon_0 + \varepsilon_1) \pm \frac{1}{4}\sqrt{2\varepsilon_0\varepsilon_1 + 5}$ or $\pm\sqrt{2}\varepsilon_0$ in normalized units). That means the angular momentum of Q_1 and the orbiting pair about the center of mass of all four bodies must be on the order of 1, so that the total angular momentum of the system is zero. Since Q_1 and the orbiting pair are both about $\frac{1}{2}\chi$ distance units, in the negative and positive x directions, respectively, from their center of mass, both must have momentum with a y-component on the order of χ^{-1}, and velocity with a y-component on the order of $\mu\chi^{-1}$. The y-component of the velocity of Q_4, however, must be on the order of χ^{-1}, because in the time it takes Q_4 to make a roundtrip from Q_2 to Q_1 and back, namely 2χ, the y-coordinate of the position of Q_4 must change by something on the order of 1 (indeed, by either $\sqrt{2}\varepsilon_0 + \frac{1}{2}(\varepsilon_0 + \varepsilon_1 + \sqrt{2\varepsilon_0\varepsilon_1 + 5})$ or $\sqrt{2}\varepsilon_1 + \frac{1}{2}(\varepsilon_0 + \varepsilon_1 - \sqrt{2\varepsilon_0\varepsilon_1 + 5})$ times the semimajor axis). Since $\mu\chi^{-1} << \chi^{-1}$, the y-component of the velocity of Q_1 is much smaller than the y-component of the velocity of Q_4 (specifically, the average of the y-components of the velocity of Q_4 before and after its encounter with Q_1). Thus, our conclusion still holds, even if we measure the velocity of Q_4 relative to Q_1. The angle between the velocity of Q_4 before and after its encounter with Q_1 differs from π by something on the order of χ^{-1}, so the orbit of Q_4 near Q_1 is a true hyperbola, not a degenerate one, with asymptotes intersecting at an angle on the order of

χ^{-1}, close to but certainly not equal to zero, and there is no collision between Q_4 and Q_1.

Next, we must check that by varying the phase of Q_3 at its encounter with Q_4, we can vary the amount of energy transferred to Q_4. Increasing the energy transferred has the effect of increasing the velocity of Q_4, and also increasing, by the same factor, the mean velocity of Q_3 over its entire orbit. But the orbit shrinks, so that Q_3 makes more revolutions in the time that Q_4 travels to Q_1 and back. A small change, on the order of χ^{-1}, in the phase of Q_3 at one encounter, would therefore result in a large change, on the order of 1, in the phase of Q_3 at its next encounter with Q_4. It is this mechanism that enables us to simultaneously select the approximate phase of Q_3 at an infinite number of future encounters.

As we did earlier, we let μ tend to zero so that we can model the encounters as elastic collisions between Q_3 and Q_4, and we assume that the incoming asymptote of the orbit of Q_4 before the encounter, and the outgoing asymptote after the encounter are in the negative x direction. But we do not assume that the encounter occurs at (X, Y), where X and Y are functions of the initial eccentricity ε_0 as defined in (3–1) and (3–2). Instead, we allow Q_3 to be anywhere in its orbit at the time of the encounter. That means ε_1, the eccentricity of the orbit after the encounter, is not necessarily equal to $\sqrt{1-\varepsilon_0^2}$, and the length and direction of the major axis of the orbit are not necessarily unchanged by the encounter. Rather, all three orbital parameters are functions of the phase of Q_3 in its orbit at the time of the encounter (and also functions of ε_0).

Let ϕ_0 be the phase of Q_3 at an angular momentum extracting encounter with Q_4, let ϕ_1 be the phase of Q_3 at the following energy extracting encounter, let s_1 be the semimajor axis of the orbit of Q_3 between the two encounters, and let s_2 be the semimajor axis after the second encounter. We take s_0, the semimajor axis before the first encounter, to be 1. If we hold ε_0 fixed at $\frac{1}{2}$ and vary ϕ_0, then numerical calculations reveal that $ds_1/d\phi_0 = -1.85$ when $\phi_0 = \pi - \arctan(2 + \frac{2}{3}\sqrt{3})$, this being the phase of Q_3 at (X, Y). If, on the other hand, we hold ε_1 fixed at $\frac{1}{2}\sqrt{3}$ and s_1 fixed at 1 while we vary ϕ_1, then $ds_2/d\phi_1 = .38$ when $\phi_1 = 0$. Since neither derivative is zero, we can indeed control the phase of Q_3 at each encounter, at least when $\varepsilon_0 = \frac{1}{2}$.

Finally, we need a method for making small adjustments to the eccentricity and angle of periapsis of the orbit of Q_3. In our approximate model for the encounters between Q_3 and Q_4, we assumed an elastic collision between these bodies, rather than a close approach along hyperbolic orbits. We also neglected the small gravitational attraction between Q_3 and Q_4 when they are not close to each other, and we neglected the gravitational field of Q_1. Finally, we assumed that the encounter between Q_3 and Q_4 occurred at precisely the correct phase of Q_3, but in fact this phase can only be controlled approximately at each encounter, because small changes must be made in the phase in order to control the phase at the next encounter. As a result of all these factors, the eccentricity of the orbit of Q_3 will not precisely alternate between ε_0 and ε_1, and the major axis of the orbit will not always be exactly perpendicular to the line through Q_1 and Q_2. Indeed, after a large number of encounters, the orbit might drift so far from its ideal parameters that we cannot continue.

To avoid this possibility, we must show that whenever the eccentricity and angle of periapsis drift too far, they can be nudged back by slightly changing the phase of Q_3 at the time of its encounter with Q_4. Once again, we fix the eccentricity at $\varepsilon_0 = \frac{1}{2}$, and the angle of periapsis at $\theta_0 = -\frac{\pi}{2}$ (that is, the major axis of the ellipse is perpendicular to the line through Q_1 and Q_2, while Q_3, which has positive angular momentum, is moving toward periapsis when it crosses the line between Q_1 and Q_2) shortly before an angular momentum extracting encounter. As usual, we approximate the encounter by an elastic collision. By varying the phase ϕ_0 of Q_3 at the time of the encounter, we vary both ε_1 and θ_1, the eccentricity and angle of periapsis of the orbit, respectively, after the encounter. The orbit doesn't change again until the following energy extracting encounter. Suppose we independently vary the phase ϕ_1 of Q_3 at the time of the energy extracting encounter, and we let ε_2 and θ_2 be the eccentricity and angle of periapsis, respectively, of the orbit of Q_3 afterwards. Then we can think of ε_2 and θ_2 as functions of two variables ϕ_0 and ϕ_1. (Note that ε_1 is equal to $\sqrt{1-\varepsilon_0^2}$ if $\phi_0 = \pi - \arctan[\varepsilon_0^{-1} + (1-\varepsilon_0^2)^{-1/2}]$, in which case $\varepsilon_2 = \varepsilon_0$ if $\phi_1 = 0$, but in general ε_1 and ε_2 have other values.) Numerical calculations reveal that

$$\frac{\partial \varepsilon_2}{\partial \phi_0} = 1.81, \tag{5–10}$$

$$\frac{\partial \theta_2}{\partial \phi_0} = -2.92, \tag{5–11}$$

$$\frac{\partial \varepsilon_2}{\partial \phi_1} = -.16, \tag{5–12}$$

and

$$\frac{\partial \theta_2}{\partial \phi_1} = 0, \tag{5–13}$$

when $\phi_0 = \pi - \arctan(2 + \frac{2}{3}\sqrt{3})$ and $\phi_1 = 0$. Since the matrix

$$\begin{bmatrix} 1.81 & -2.92 \\ -.16 & 0. \end{bmatrix} \tag{5–14}$$

is nonsingular, we can always bring ε and θ back into line by making small adjustments to ϕ at two successive encounters of Q_3 and Q_4. For example, suppose we want to ensure that ε is always within δ of $\frac{1}{2}$ or $\frac{1}{2}\sqrt{3}$ (depending on whether the last encounter extracted energy or angular momentum) and θ is always within δ of $\frac{\pi}{2}$. For fixed δ, we can choose μ small enough that ε and θ stay within these intervals for an arbitarily large number of encounters. When they come close to drifting out of their intervals, we can recenter them by making adjustments to ϕ at two successive encounters. These adjustments will be on the order of δ, so they will not interfere with our ability to control future values of ϕ, for which we need to make much smaller adjustments, on the order of χ^{-1}.

We remark that s, the semimajor axis of the orbit of Q_3, will also tend to drift away from its nominal value of $\varepsilon_0^{2n}/(1-\varepsilon_0^2)^n$ after $2n$ encounters. But there is no need to adjust s. Because the total energy of the system is zero, the energies of all bodies remain in proportion. As long as s shrinks in geometric progression with every energy extracting encounter (that is, as long as s_{2n+2}/s_{2n} is bounded from above for all n by a constant strictly lessthan 1), there will be an infinite number of encounters in a finite time, and hence a noncollision singularity.

REFERENCES

[Gerver 91] J. L. Gerver. "The Existence of Pseudocollisions in the Plane." *J. Differential Equations* 89 (1991), 1–68.

[Mather and McGhee 75] J. Mather and R. McGehee. "Solutions of the Collinear Four Body Problem which Become Unbounded in Finite Time." In *Lecture Notes in Physics* 38, edited by J. Moser, pp. 575–597. Berlin-New York: Springer-Verlag, 1975.

[McGehee 84] R. McGehee. "A Note on a Theorem of Von Zeipel." Report No. 19 (1984), Institut Mittag-Leffler.

[Painlevé 97] P. Painlevé. *Lecons sur la Théorie Analytique des Équations Différentielles.* Paris: Hermann, 1897.

[Saari 72] D. G. Saari. "Singularities and Collisions of Newtonian Gravitational Systems." *Arch. Rational Mech. Anal.* 49 (1972), 311–320.

[Saari 77] D. G. Saari. "A Global Existence Theorem for the Four Body Problem of Newtonian Mechanics." *J. Differential Equations* 26 (1977), 80–111.

[Von Zeipel 08] H. Von Zeipel. "Sur les singularités du problème des n corps." *Ark. Mat. Astro. Fys.* 4:32 (1908), 1–4.

[Xia 92] Z. Xia. "The Existence of Noncollision Singularities in Newtonian Systems." *Ann. of Math.* 135:2 (1992), 411–468.

Joseph L. Gerver, Department of Mathematics, Rutgers University, Camden, NJ 08102 (gerver@camden.rutgers.edu)

Received March 25, 2003; accepted May 6, 2003.