The Totally Real A_6 Extension of Degree 6 with Minimum Discriminant

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The totally real algebraic number field F of degree 6 with Galois group A_6 and minimum discriminant is determined. It is unique up to isomorphy, and is generated by a root of the polynomial $t^6 - 24t^4 + 21t^2 + 9t + 1$ over the rationals. We also give an integral basis and list the fundamental units and class number of F.

In [Ford and Pohst 1992] we gave details of a computation to determine the (unique) totally real algebraic number field of degree 6 having Galois group A_5 and minimum discriminant. There we indicated how the same computation could be extended to give the totally real sextic field with Galois group A_6 of minimum discriminant. This computation has now been completed.

Theorem. *The smallest possible discriminant for a totally real A_6 extension of degree 6 is $13041^2 = 170\,067\,681$. There is, up to isomorphy, exactly one field F of that discriminant. It is generated by a root ρ of the polynomial*

$$t^6 - 24t^4 + 21t^2 + 9t + 1.$$

The class number of F is 1. An integral basis for F is given by

$$1, \quad \rho, \quad \rho^2, \quad \rho^3, \quad \omega = -\tfrac{2}{3} - \tfrac{1}{3}\rho^2 + \tfrac{1}{3}\rho^4, \quad \rho\omega.$$

A system of fundamental units for F is

$$\varepsilon_1 = \rho$$
$$\varepsilon_2 = 2 + 7\rho - 7\rho^2 - 15\rho^3 + \omega + 2\rho\omega$$
$$\varepsilon_3 = 55 + 427\rho + 145\rho^2 - 466\rho^3 - 19\omega + 61\rho\omega$$
$$\varepsilon_4 = 258 + 1217\rho + 245\rho^2 - 1263\rho^3 - 32\omega + 165\rho\omega$$
$$\varepsilon_5 = 320 + 2467\rho + 817\rho^2 - 2711\rho^3 - 107\omega + 355\rho\omega.$$

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As in the previous paper we searched for a generating element ρ of the field, governed by the inequality $\mathrm{Tr}(\rho^2) \le \frac{3}{2} + (\frac{4}{3}B)^{1/5} = \tilde{B}$, where B is an upper bound on the field discriminant.

The computations were performed on a Digital VaxSystem 4000-90 computer in the Department of Computer Science at Concordia University. The calculations took about 25.28 CPU-hours to reach the bound $\tilde{B} = 34$—the limit used in [Ford and Pohst 1992]—and about 529.13 CPU-hours to reach the bound $\tilde{B} = 48$, which suffices to prove the theorem.

The table opposite lists defining polynomials and field discriminants for all A_6 extension fields discovered in the course of our search. For each discriminant, there is only one field up to isomorphy.

REFERENCES

[Ford and Pohst 1992] D. Ford and M. Pohst, "The Totally Real A_5 Extension of Degree 6 with Minimum Discriminant", *Experimental Math.* **1** (1992), 231–235.

$t^6 - 24t^4 + 21t^2 + 9t + 1$	13041^2
$t^6 + t^5 - 12t^4 - 7t^3 + 21t^2 + 7t - 7$	13867^2
$t^6 - 21t^4 + 4t^3 + 70t^2 - 24t - 4$	14032^2
$t^6 - 10t^4 + t^3 + 28t^2 - 3t - 20$	17267^2
$t^6 + 3t^5 - 5t^4 - 12t^3 + 3t^2 + 7t + 1$	18680^2
$t^6 - 22t^4 + 14t^3 + 86t^2 - 28t - 88$	19580^2
$t^6 - 14t^4 + 6t^3 + 46t^2 - 28t - 8$	20548^2
$t^6 + 3t^5 - 7t^4 - 25t^3 - 6t^2 + 13t + 2$	20795^2
$t^6 + 2t^5 - 20t^4 + 12t^3 + 28t^2 - 24t + 4$	22988^2
$t^6 + 2t^5 - 21t^4 + 4t^3 + 21t^2 + 2t - 2$	23704^2
$t^6 - 18t^4 + 3t^3 + 85t^2 - 13t - 115$	24851^2
$t^6 + t^5 - 15t^4 - 25t^3 + 15t^2 + 20t + 4$	25979^2
$t^6 - 19t^4 + 36t^3 - 7t^2 - 12t - 1$	26272^2
$t^6 + 2t^5 - 19t^4 - 48t^3 - 10t^2 + 15t - 2$	26353^2
$t^6 - 21t^4 + 23t^3 + 32t^2 - 35t + 8$	27014^2
$t^6 - 19t^4 + 4t^3 + 83t^2 - 52t - 33$	28196^2
$t^6 - 19t^4 + 8t^3 + 83t^2 - 88t + 7$	29272^2
$t^6 - 20t^4 + 10t^3 + 75t^2 - 25t - 75$	29525^2
$t^6 + 2t^5 - 21t^4 - 40t^3 + 70t^2 + 115t - 32$	30119^2
$t^6 + 3t^5 - 19t^4 - 25t^3 + 46t^2 + 5t - 2$	30423^2
$t^6 + t^5 - 21t^4 - 36t^3 + 61t^2 + 81t - 64$	30519^2
$t^6 - 13t^4 + 12t^3 + 14t^2 - 12t + 2$	30704^2

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