Recognition of $\mathcal{K}$-Singularities of Functions

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CONTENTS

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We describe a computer program, based on Maple, that decides whether or not a polynomial function has a simple or unimodal singularity at the origin, and determines the $\mathcal{K}$-class of this singularity. The program applies the splitting lemma to the function, in an attempt to reduce the number of variables. Then, in the more interesting cases, linear coordinate changes reduce the 3-jet of the function (or the 4-jet if necessary) to a standard form, and auxiliary procedures complete the classification by looking at higher-order terms. In particular, the reduction procedure classifies cubic curves in $\mathbf{P}^2$.

1. INTRODUCTION

Let $\mathcal{C}_n$ denote the set of germs of smooth functions $(\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$. The group $\mathcal{R}$ of germs of diffeomorphisms of $(\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ acts on $f \in \mathcal{C}_n$ by $hf(x) = f(h^{-1}(x))$. The *contact group* $\mathcal{K}$ is the group of germs of diffeomorphisms $(\mathbf{C}^{n+1}, 0) \to (\mathbf{C}^{n+1}, 0)$ of the form

$$H(x, y) = (h(x), g(x, y)),$$

with $h \in \mathcal{R}$ and $g(x, 0) = 0$. The germ H acts on f by $(h(x), Hf(x)) = H(x, f(x))$, that is,

$$\operatorname{graph}(H.f) = H(\operatorname{graph}(f)).$$

An $\mathcal{R}$- or $\mathcal{K}$-classification of functions is an enumeration of the orbits of the action of $\mathcal{R}$ or $\mathcal{K}$ on $\mathcal{C}_n$. For germs where the origin is an isolated singularity, these orbits have polynomial representatives (normal forms), and one can define invariants such as the *multiplicity* and the *modality* [Arnold 1976]. The $\mathcal{R}$-classification of singularities of modality up to 2 or multiplicity up to 16 is given in [Arnold 1976]. The $\mathcal{K}$-classification of simple singularities (those of modality zero) is given in [Giusti 1983], and that of unimodal singularities is given in [Wall 1983].

Given a function f, it is hopeless, in general, to look for an explicit diffeomorphism $H \in \mathcal{K}$ that

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converts f to a normal form. Any attempt to find the $\mathcal{K}$-class of f, then, has to proceed along different lines. Invariants such as the multiplicity are often difficult to compute and do not always distinguish between different singularities.

The computer program K-type, described in this paper, provides a partial solution to the problem of recognizing the $\mathcal{K}$-class of a function germ. Given a polynomial f in any number of variables, it seeks to determine whether or not f has a simple or unimodal $\mathcal{K}$-singularity at the origin, and, if it does, to find the normal form in [Wall 1983] that f is equivalent to.

K-type is based on Maple [Char et al. 1985]. Its core algorithm follows [Arnold 1976], but additional procedures are used in order to reduce the input variables (essentially implementing the splitting lemma) and to identify, by explicit changes of coordinates, the normal form of a binary quartic or of a cubic curve in $\mathbf{P}^2$. Details are given in the next two sections.

A result in [du Plessis et al.] shows that a large class of map germs are determined by their discriminants. In the case where the target dimension is at most equal to the source dimension, the discriminant is the zero set of the determinant of a square matrix, called the discriminant matrix. A recipe for computing the discriminant matrices of maps lying in an unfolding of a $\mathcal{K}$-class is also given in [du Plessis et al.]. When the target dimension is two, the discriminant function (that is, the determinant of the discriminant matrix) is $\mathcal{K}$-finite if and only if the map is $\mathcal{A}$-finite, and the $\mathcal{K}$-class of the discriminant function is close to determining the $\mathcal{A}$-class of the map. Some of the $\mathcal{A}$-classification in [du Plessis and Tari] was carried out this way.

The need for recognition of the $\mathcal{K}$-classes of functions is discussed also in [du Plessis and Wall], where an instability locus of a map is computed and a search for the different $\mathcal{K}$-classes at points in this locus is carried out. Many of the calculations in [du Plessis and Wall] have been checked using K-type.

2. OVERALL DESCRIPTION OF THE PROGRAM

The input to K-type is a polynomial in one or more variables. The actual call should be of the form

$$\texttt{Kclass}(f,[x_1,\dots,x_n]);$$

where f is the polynomial and $x_1,\dots,x_n$ are the variables. The output is the type of the singularity that f has at the origin, if that type can be found. The notation for $\mathcal{K}$-classes follows [Wall 1983], with the following substitutions: A_k is printed `A[k]`, and likewise for other subscripted capitals; while $W^{\#}_{k,i}$ and $S^{\#}_{k,i}$ are printed `Ws[k,i]` and `Ss[k,i]`.

The program works as follows:

Step 1. Rename the input variables to `x[1]`, ..., `x[n]`. (We will represent them by $x_1,\dots,x_n$.)

Step 2. Check if $\nabla f(0,\dots,0)$ is zero. If so, the function is not singular at the origin: output A_0.

Step 3. Apply splitting lemma. This is done inductively by maintaining an index set S. The induction step is the following:

If x_i does not appear in the quadratic part of f, add the index i to S. Otherwise, let a be the coefficient of x_i^2. If necessary, change coordinates so that $a \neq 0$ (the change $x_j \mapsto x_j + x_i$, where $j > i$ and the coefficient of $x_i x_j$ is nonzero, will work). Write f as a polynomial in x_i and let P, a polynomial in $x_{i+1},\dots,x_n$, be the coefficient of the linear term in x_i. Perform the change of coordinates $x_i \mapsto x_i - P/(2a)$ to eliminate from f the linear part $x_i P$. At this point all terms in x_i in the resulting function are irrelevant to the determination of the $\mathcal{K}$-class of f, and therefore are dropped. This concludes the induction step.

Step 4. The number of elements in the index set S at the end of this induction is the corank of f, that is, the (initial) number of variables minus the rank of the Hessian of (the initial) f. We have the following cases:

- $\#S = 0$. The singularity is nondegenerate: output A_1.
- $\#S = 1$. Output A_k, where k is the order of f in the remaining variable.
- $\#S = 2$. Call `funrec` and output its result (see the next section).
- $\#S = 3$. Call `Funrec` and output its result (see the next section).
- $\#S > 3$. If this case occurs, the singularity is not simple or unimodal. The program issues the message "Higher modality", indicating that it cannot classify the singularity.

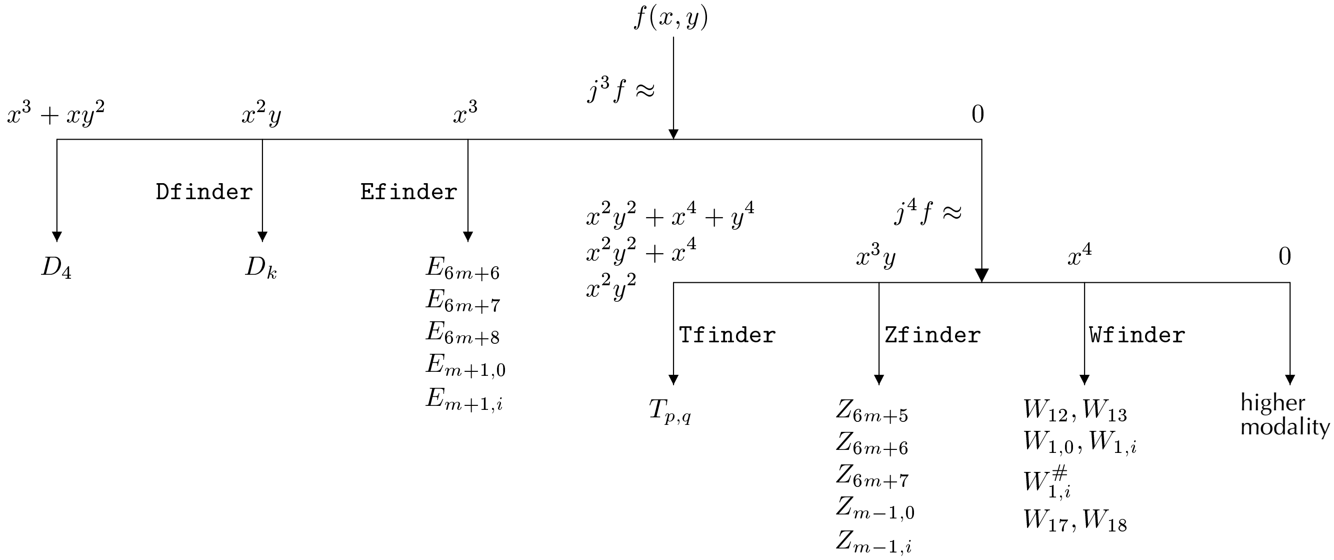

FIGURE 1. Algorithm used by `funrec` to find the $\mathcal{K}$-type of the singularity of a function f of corank 2. The notation $j^3f \approx$ means that the 3-jet of f (that is, its Taylor polynomial of degree 3) can be reduced to the given form by a linear change of coordinates.

In the preceding discussion, when a change of variables in f is called for, say $x \mapsto \varphi(x)$, the following code involving a dummy variable `xp` is employed internally:

```
x := phi(xp);
f := eval(collect(f,[xp],distributed));
x := 'x'; xp := x; f := eval(f); xp :='xp';
```

3. THE CASE OF CORANK 2 OR 3

We now turn to the functions `funrec` and `Funrec`, which perform the classification of f when, after the application of the splitting lemma, there are two or three variables left (see Step 4 above). The basic logic follows [Arnold 1976, pp. 101 ff.] and is complemented by code to find the explicit coordinate changes needed to put the 3-jet (or, in some cases, the 4-jet) of f in normal form.

The flowchart for `funrec` is shown in Figure 1. We first examine the 3-jet (cubic part) j^3f of f; a linear change of coordinates reduces it to one of the forms x^3+xy^2, x^2y, x^3 or 0. For example, if the term in x^3 is not zero, we reduce to the form $x^3+pxy^2+qy^3$ by replacing x with an appropriate linear combination of x and y, and then we test the discriminant $4p^3+27q^2$ to see whether or not the factorization of j^3f has a double factor. Or, if the term in x^3 is zero but the term in y^3 is not, we start by interchanging x and y, and so on.

If j^3f has the form x^2y or x^3 after this reduction, we call an auxiliary procedure, `Dfinder` or `Efinder`. If $j^3f=0$, we must examine the 4-jet j^4f of f. Again by linear changes of variables, we can reduce to one of the forms shown in Figure 1, and, if $j^4f \neq 0$, auxiliary procedures identify the singularity type of f. These procedures work mostly by searching for monomials of appropriate weights with nonzero coefficients, and making changes of variables as needed.

The flowchart for `Funrec` is shown in Figure 2. Again, we examine the cubic part j^3f or f, which now is trivariate and thus determines a cubic curve in $\mathbf{P}^2$. The classification of cubics in $\mathbf{P}^2$ is given in [Cayley 1856; Salmon 1879]. Using the `factor` command, we determine whether j^3f is reducible and what its factors are.

If j^3f is reducible, a linear change of variables puts it in one of the normal forms xyx, x^3+xyz, x^2z+z^2y or z^3+x^2z. Auxiliary procedures then identify the $\mathcal{K}$-class of f itself.

If j^3f is irreducible, the changes of variables needed to put it in one of the normal forms $x^3+y^3+z^3+axyz$ $(a^3+27\neq 0)$, x^3+y^3+xyz or x^3+yz^2 are harder to obtain. The number of flex points distinguishes between these orbits; it equals nine, three and one, respectively. If the number is nine, f has a singularity of type $T_{3,3,3}$. Otherwise, the cubic has a unique real flex point. By interchanging the variables, we can assume that the real flex point lies on the plane $z=1$. The real flex point and the associated tangent line are computed by the procedure `flex`.

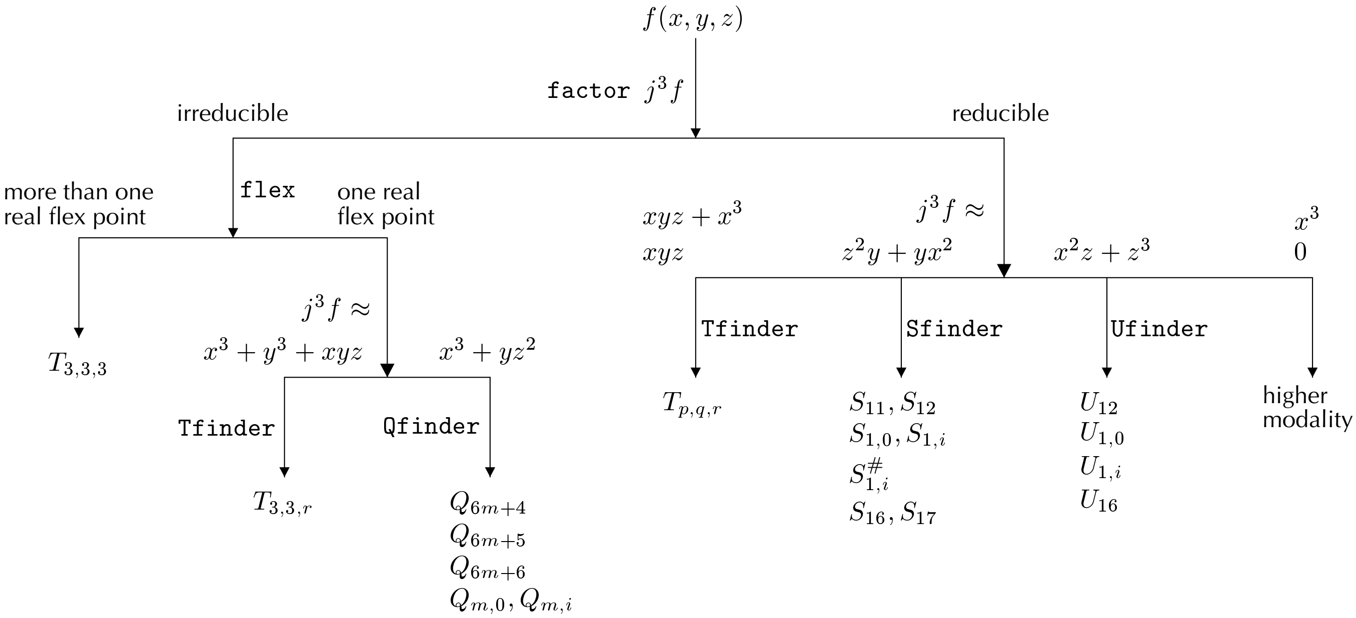

FIGURE 2. Algorithm used by `Funrec` to find the $\mathcal{K}$-type of the singularity of a function f of corank 3.

If we take for the x-axis the tangent line at the flexpoint and for the y-axis any other line through this point, j^3f takes the form

$$x(z^2 + (a_1x + a_2y)z + a_3x^2 + a_4xy + a_5y^2) + a_6y^3$$

[Salmon 1879, Art. 195]. The substitution $z \mapsto z - \frac{1}{2}(a_1x + a_2y)$ yields

$$j^3f = x(z^2 + a_1x^2 + a_2xy + a_3y^2) + a_4y^3$$

(with new coefficients a_i). The polynomial $a_1 + a_2y + a_3y^2 + a_4y^3$ has a repeated root; if it is a triple root we can reduce to the case $j^3f = x^3+yz^2$, and if it is a double root we can reduce to $j^3f = x^3 + y^3 + xyz$ [Salmon 1879, Art. 196]. After this reduction an auxiliary procedure gives the $\mathcal{K}$-class of f.

4. LIMITATIONS

Some of the Maple commands on which K-type relies work better when the input polynomial has rational coefficients. For speed, the input function should be a polynomial with rational coefficients.

Sometimes the program needs to make coordinate changes where the coefficients are roots of cubic or quartic equations. The presence of radicals then makes the resulting expression very unwieldy, and often execution breaks down for lack of memory space. It seems likely that this problem could be completely overcome by the use of a genuinely constructive classification algorithm. The search for such an algorithm is an interesting theoretical problem.

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SOFTWARE AVAILABILITY

A copy of K-type may be obtained from Andrew du Plessis at `matadp@mi.aau.dk`.

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