# Computer Graphics and a New Gibbs Phenomenon for Fourier–Bessel Series

Alfred Gray and Mark A. Pinsky

**CONTENTS**



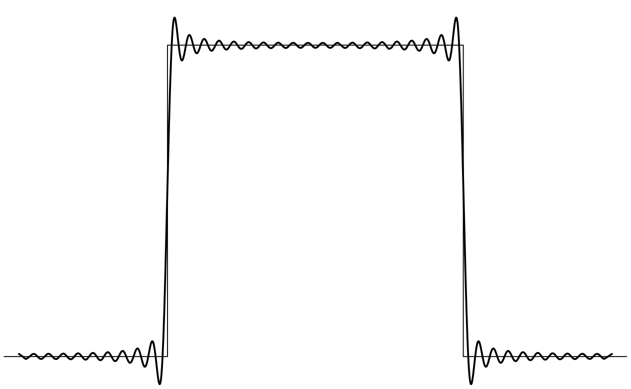
We report the existence of a Gibbs-like phenomenon at points of continuity in the expansion of functions in Fourier–Bessel series.

## 1. INTRODUCTION

It is well known that many functions can be expanded in Fourier–Bessel series. (A Fourier–Bessel series is one of the form

$$\sum_{k=1}^{\infty} a_k J_m(x x_k^{(m)}),$$

where $x_k^{(m)}$ denotes the $k$-th positive zero of the Bessel function $J_m$.) It is not surprising that partial sums of the Fourier–Bessel series of a piecewise continuous function have an overshoot at the points of discontinuity; this behavior is familiar from partial sums of ordinary Fourier series, as in the figure on the left, and is called the Gibbs phenomenon [Gibbs 1898; Weyl 1909].



Let's compare the graphs of the partial sums of the Fourier and Fourier–Bessel expansions of the function $f(x) = 1$ for the interval $-1 < x < 1$. The expansions are

$$\frac{1}{2} + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{(2k-1)\pi} \cos \frac{(2k-1)\pi x}{2}$$

and

$$2 \sum_{k=1}^{\infty} \frac{J_0(x x_k^{(0)})}{x_k^{(0)} J_1(x_k^{(0)})},$$

where we took $m = 0$ in the Fourier–Bessel expansion (we will consider other values of $m$ later). A partial sum (up to $k = 20$) for the ordinary Fourier series is what is shown on the left; the graph for


1058-6458/92 $0.50 per page

the Fourier–Bessel series is shown below, in Figure 1. Both series converge to 1 on the interval $-1 < x < 1$. At $x = \pm 1$ they vanish, so we expect the Gibbs phenomenon at these points, and indeed we observe it in the graphs.

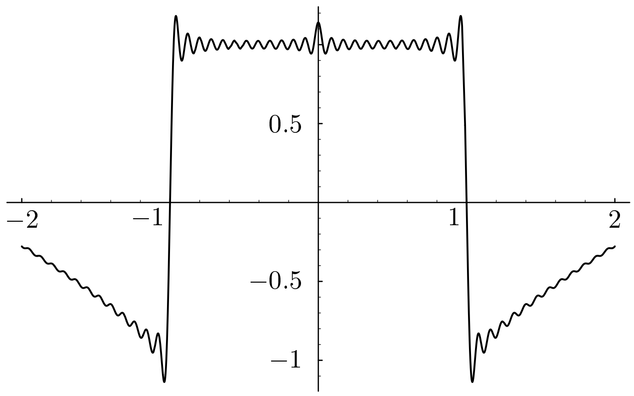



**FIGURE 1.** The graph of $2\sum_{k=1}^{20} \frac{J_0(x x_k^{(0)})}{x_k^{(0)} J_1(x_k^{(0)})}$

What is intriguing about the plot of the Fourier–Bessel series is that there also seems to be a Gibbs-like phenomenon at $x = 0$, which is a point of continuity. We shall see in Section 2 that this is due to the fact that the series converges more slowly at $x = 0$ than at surrounding points.

The first author observed this unusual behavior when he was using Mathematica to draw graphs for the appendix to [Pinsky 1991]. Before the advent of computer programs such as Mathematica, graphs of partial sums of Fourier–Bessel series were difficult to obtain. That Wilbraham published the first graphs of the ordinary Gibbs phenomenon in 1848 is quite remarkable [Hewitt and Hewitt 1980; Wilbraham 1848]. The Gibbs-like phenomenon described here is a good example of how computer graphics can suggest new mathematical results.

The ordinary Gibbs phenomenon for Bessel functions was investigated analytically by Cooke [1928], who showed that it is not present at $x = 0$. Cooke did not observe that a slower rate of convergence is possible.

A natural generalization of the expansion in $J_0$ just discussed is the expansion

$$2\sum_{k=1}^{\infty} \frac{J_m(x x_k^{(m)})}{x^m J_{m+1}(x_k^{(m)}) x_k^{(m)}} \qquad (1.1)$$

for $f(x) = 1$ in $-1 < x < 1$ [Pinsky 1991, p. 192]. Although $m$ may be any real number with $m > -1$, we shall limit ourselves to a discussion of the case when $m$ is an integer. If we were to plot (1.1), we would run into practical difficulties, since the sums become unbounded near $x = 0$ if $m > \frac{1}{2}$. In order to overcome this difficulty, we work instead with the associated convergent expansion

$$2\sum_{k=1}^{\infty} \frac{J_m(x x_k^{(m)})}{J_{m+1}(x_k^{(m)}) x_k^{(m)}}$$

for $x^m$ in $-1 < x < 1$. Figure 2 illustrates the slower convergence near $x = 0$ in the cases $m = 1, 2, 3, 4$. The theoretical analysis is given in Section 3.

## 2. ANALYSIS OF THE EXPANSION IN $J_0$

Let $f(x)$ be a function for which an expansion

$$f(x) = \sum_{k=1}^{\infty} a_k J_m(x_k^{(m)} x)$$

is valid on the interval $-1 < x < 1$. It follows from standard properties of Bessel functions (see [Watson 1966, Chap. XVIII], for example) that the coefficients $a_k$ are given by

$$a_k = \frac{2}{J_{m+1}(x_k^{(m)})^2} \int_0^1 t f(t) J_m(x_k^{(m)} t)\, dt.$$

We will use the following asymptotic expansions [Abramowitz and Stegun 1965, pp. 371, 364], both of which hold for fixed $m$:

$$J_m(x) = \sqrt{\frac{2}{\pi x}}\left(\cos\left(x - \frac{(2m+1)\pi}{4}\right) + O\left(\frac{1}{x}\right)\right) \qquad (2.1)$$

as $x \to \infty$, and

$$x_k^{(m)} = \left(k + \frac{2m-1}{4}\right)\pi + O\left(\frac{1}{k}\right) \qquad (2.2)$$

as $k \to \infty$. From these relations it is elementary to prove that the asymptotic behavior of

$$\frac{J_0(x x_k^{(0)})}{x_k^{(0)} J_1(x_k^{(0)})}$$

as $k \to \infty$ is

$$\frac{(-1)^{k-1}\cos(\frac{1}{4}\pi(1 + x - 4kx))}{(k - \frac{1}{4})\pi\sqrt{x}} + O\left(\frac{1}{k^2}\right)$$

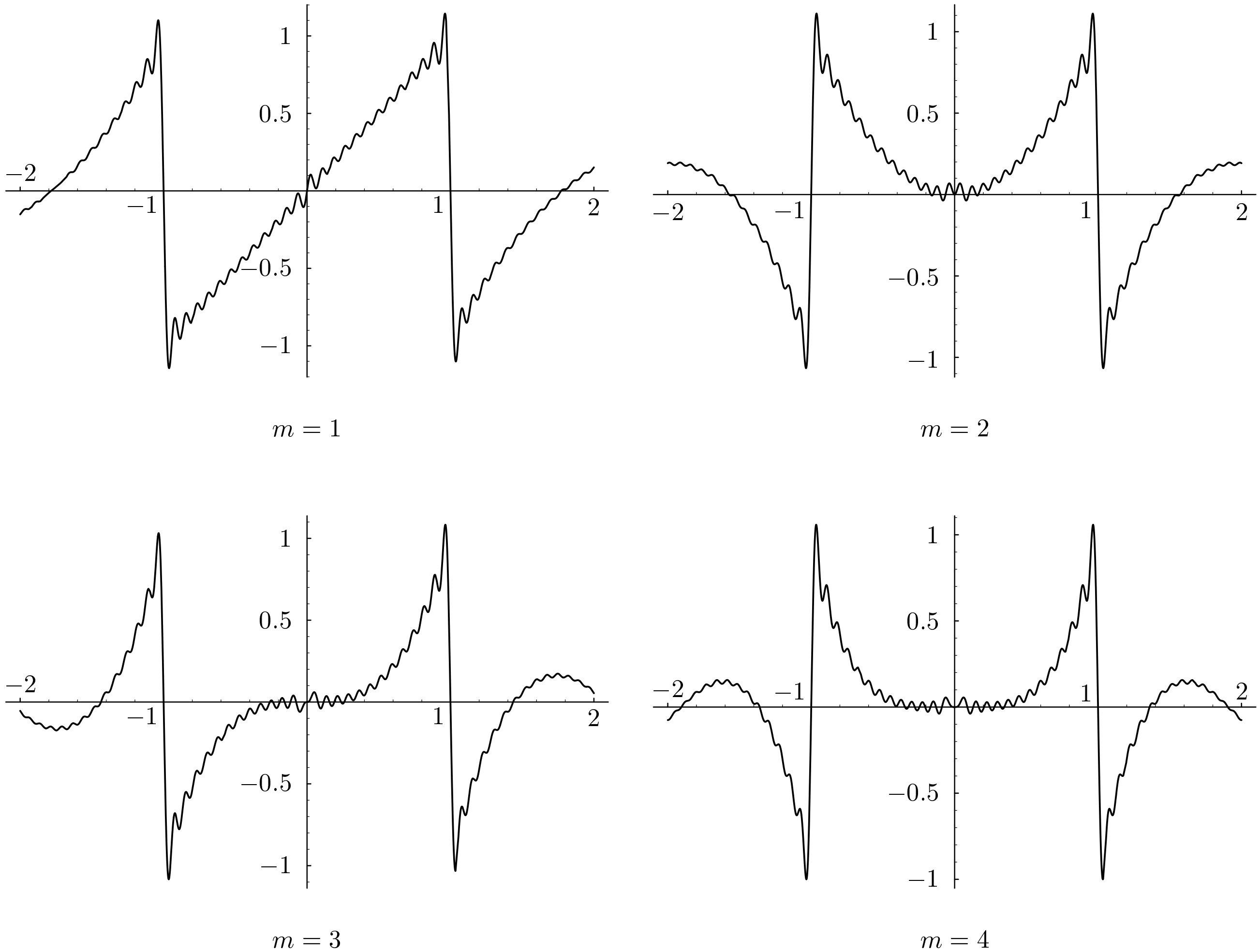



**FIGURE 2.** Partial sums of the $m$-th order Fourier–Bessel series $2\sum_{k=1}^{25}\dfrac{J_m(xx_k^{(m)})}{J_{m+1}(x_k^{(m)})x_k^{(m)}}$, which approximate $x^m$ in the interval $-1 < x < 1$.

for $x \neq 0$, but for $x = 0$ it is

$$\frac{(-1)^{k-1}\sqrt{2}}{\sqrt{4k-1}} + O\Big(\frac{1}{k^{3/2}}\Big).$$

Therefore, as mentioned above, the Gibbs-like phenomenon is caused by the slower rate of convergence at $x = 0$ than at surrounding points. More precisely:

**Theorem 1.** *The order-zero Fourier–Bessel expansion for $f(x) = 1$ in the interval $-1 < x < 1$ has the rate of convergence of the series*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\cos(\frac{1}{4}\pi(1+x-4kx))}{(k-\frac{1}{4})\pi\sqrt{x}}$$

*for $x \neq 0$, but for $x = 0$ it has the slower rate of convergence of the series*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\sqrt{2}}{\sqrt{4k-1}}.$$

## 3. ANALYSIS OF THE EXPANSION IN $J_m$, WITH $m > 0$

We now consider the expansion

$$2\sum_{k=1}^{\infty}\frac{J_m(xx_k^{(m)})}{x^m J_{m+1}(x_k^{(m)})x_k^{(m)}} \tag{3.1}$$

for the constant function $f(x) = 1$ in the interval $-1 < x < 1$. At $x = 0$ the terms of (3.1) are defined by continuity, using the relation

$$\lim_{x\to 0}\frac{J_m(ax)}{x^m}=\frac{a^m}{2^m\,m!},$$

so the series takes the form

$$2\sum_{k=1}^{\infty}\frac{(x_k^{(m)})^{m-1}}{J_{m+1}(x_k^{(m)})}.$$

By (2.1) and (2.2) we have

$$\frac{(x_k^{(m)})^{m-1}}{J_{m+1}(x_k^{(m)})}=(-1)^k\frac{k^{m-\frac{1}{2}}}{2^m\,m!}\Bigl(1+O\Bigl(\frac{1}{k}\Bigr)\Bigr)$$

as $k\to\infty$. On the other hand, for $x\neq 0$ we have

$$\frac{J_m(xx_k^{(m)})}{x^m J_{m+1}(x_k^{(m)})x_k^{(m)}}=\frac{(-1)^k}{kx^m 2^m\,m!}\Bigl(1+O\Bigl(\frac{1}{k}\Bigr)\Bigr).$$

To summarize:

**Theorem 2.** *The terms of the $m$-th order Fourier–Bessel series for the function $f(x)=1$ in the interval $-1<x<1$ are asymptotically equivalent to the terms of the convergent series*

$$\frac{1}{2^m\,m!\,x^m}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}$$

*when $x\neq 0$, and to the terms of the series*

$$\frac{1}{2^m\,m!\,x^m}\sum_{k=1}^{\infty}(-1)^k k^{m-\frac{1}{2}}$$

(*which diverges for $m\geq\frac{1}{2}$*) *when $x=0$.*

## NOTE ON THE PLOTS

All figures were generated using Mathematica. The requisite values of $x_k^{(m)}$ (see (1.1), for instance) were tabulated for each $k\leq 20$ and each $m$; for higher values of $k$ the asymptotic formula (2.2) is good enough, at least for small $m$. Due to the nature of the functions, the `PlotPoints` option to `Plot` had to be explicitly set to a relatively high value—around 200—or some oscillations would be missed. (`PlotPoints` controls the fineness of the initial subdivision of the domain; after that, an adaptive algorithm takes over.)

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Alfred Gray, Department of Mathematics, University of Maryland, College Park, Maryland 20742 (gray@athena.umn.edu)

Mark A. Pinsky, Department of Mathematics, Northwestern University, Evanston, Illinois, 60208 (m˙pinsky@math.nwu.edu)

Received March 21, 1992; accepted in revised form January 30, 1993