On a Conjecture of Carathéodory: Analyticity Versus Smoothness

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We show that, under mild nonflatness conditions, for any $r \geq 3$ and any C^r-immersion of a surface into $\mathbb{R}^3$ with an isolated umbilic point there exist an analytic surface with an isolated umbilic of the same index. The connection of this with Carathéodory's Conjecture on umbilics is discussed.

1. INTRODUCTION

The classical Carathéodory Conjecture states that every smooth convex embedding of a 2-sphere in $\mathbb{R}^3$ must have at least two umbilics. A well known approach to the problem is based on a "semi-local" argument. For any surface in $\mathbb{R}^3$, the eigenspaces of the second fundamental form define two orthogonal line fields (principal directions) whose singularities are exactly the umbilics. To each isolated umbilic we can attach the index of either one of the two fields, which is half of an integer, and the sum of those indexes is the Euler characteristic of the surface, if the surface is compact and all umbilics are isolated. So, if an embedded sphere has only one umbilic, this must have index two. We just observe that, up to an inversion in $\mathbb{R}^3$, we can always suppose that the curvature at a given umbilic is positive, and therefore the convexity hypothesis is not relevant for this argument.

Examples of umbilics of index j are known for all $j \leq 1$. A local conjecture stronger than Carathéodory's, known as the Loewner conjecture, states that there are no umbilics of index greater then one. This conjecture has been asserted to be true for analytic surfaces by several authors [Hamburger 1940–1941; Bol 1943–1944; Klotz 1959; Titus 1973], implying therefore Carathéodory's Conjecture for

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analytic surfaces. Very recently we where informed that Voss and Scherbel are trying to clarify some points in the above mentioned works. These points are explained in [Scherbel, Appendix B]; in this respect see also [Yau 1982, p. 684; Lang 1990, p. 19].

In this note we prove that for each umbilic on a C^r surface it is possible to construct, under a mild nondegeneracy condition, an analytic surface with an umbilic of the same index and therefore, in those cases, a positive answer to the the local C^r conjecture follows from a positive answer for the analytic case.

It is interesting to observe that generically the index of an umbilic is $\pm\frac{1}{2}$. In particular, a generically embedded compact surface S has at least $2\,|\chi(S)|$ umbilics. This was proved in [Feldman 1967] and subsequently in [Asperti 1980]. A nice geometric (generic) condition under which the index of an umbilic has to be at most one was given in [Smyth and Xavier 1992]. Their condition is a nonvanishing condition on the 3-jet of a suitable function. Our condition is somewhat better since, at least in the smooth case, it is a condition on the k-jet for some k, and is automatically verified in the analytic case.

The global configurations determined by the foliations tangent to the principal directions have already been studied. See for instance [Gutierrez and Sotomayor 1982; 1983; 1993; Ramírez-Galarza and Sánchez-Bringas 1995].

2. BONNET COORDINATES AND FUNCTIONS

Let $\mathcal{S} \subset \mathbb{R}^3$ be an oriented C^r-embedded surface, where $r \geq 3$. Suppose that the Gauss map $N : \mathcal{S} \to S^2$ takes $\mathcal{S}$ diffeomorphically and preserving orientation onto an open subset $N(\mathcal{S})$ of $S^2 \setminus \{(0,0,1)\}$. In particular, the gaussian curvature of $\mathcal{S}$ is positive everywhere. Let $\Pi : \mathbb{R}^2 \to S^2 \setminus (0,0,1)$ be the diffeomorphism given by the inverse map of the stereographic projection; that is,

$$\Pi(x,y) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{x^2+y^2-1}{1+x^2+y^2}\right).$$

Then the map

$$\Phi(x,y) = (X(x,y), Y(x,y), Z(x,y)) = N^{-1} \circ \Pi(x,y)$$

defined in $U = \Pi^{-1}(N(\mathcal{S}))$ provides a global C^{r-1} parametrization of $\mathcal{S}$, called *Bonnet chart* associated to $\mathcal{S}$ (and to the particular stereographic projection Π).

Let $\Lambda(x,y) = (1+x^2+y^2)\Pi(x,y)$; that is,

$$\Lambda(x,y) = (2x,\, 2y,\, x^2+y^2-1).$$

We define the *Bonnet function*

$$\beta(x,y) = \Lambda(x,y) \cdot \Phi(x,y),$$

where the dot stands for the usual inner product.

Proposition 2.1. *Let $\mathcal{S} \subset \mathbb{R}^3$ be an oriented C^r embedded surface, where $r \geq 3$. Suppose that the Gauss map $N : \mathcal{S} \to S^2$ takes $\mathcal{S}$ diffeomorphically and preserving orientation onto an open subset of $S^2 - \{(0,0,1)\}$. Then the Bonnet function $\beta = \beta(x,y)$ associated to $\mathcal{S}$ is of class C^r and the differential equation of the principal lines of curvature of $\mathcal{S}$, in its Bonnet chart, is given by*

$$\beta_{xy}\,dx^2 + (\beta_{yy} - \beta_{xx})\,dx\,dy - \beta_{xy}\,dy^2 = 0. \tag{2.1}$$

Proof. Since $\Lambda \cdot \Phi_x = \Lambda \cdot \Phi_y = 0$, where the subindex means the partial derivative with respect to this variable, we have $\Lambda_x \cdot \Phi = \beta_x$ and $\Lambda_y \cdot \Phi = \beta_y$. This, together with $\Lambda \cdot \Phi = \beta$, can be written in matrix notation as $M \cdot \Phi = \mathcal{B}$, where

$$M = \begin{pmatrix} 2x & 2y & x^2+y^2-1 \\ 2 & 0 & 2x \\ 0 & 2 & 2y \end{pmatrix},$$

$$\Phi = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} \beta \\ \beta_x \\ \beta_y \end{pmatrix}.$$

Since N is of class C^{r-1}, Φ is also of class C^{r-1}. Therefore, $M \cdot \Phi = \mathcal{B}$ implies that β is of class C^r. Since, for all $(x,y) \in \mathbb{R}^2$, the determinant of M is $-4(1+x^2+y^2) \neq 0$, we may write $\Phi = M^{-1} \cdot \mathcal{B}$. From this we can compute the first and second fundamental forms of Φ in terms of β and therefore we obtain (2.1). $\square$

Remark 2.2. There are several other proofs of (2.1); see, for example, [Bonnet 1860; Darboux 1896, pp. 285–300; Blaschke 1929, pp. 283–289]. Our direct approach has the merit that it can be easily checked using a symbolic computer system.

Proposition 2.3. *Let $\beta : \mathbb{R}^2 \to \mathbb{R}$ be a C^r function, with $r \geq 3$. Suppose that the 2-jet $j^2\beta_{(0,0)}$ of β at $(0,0)$ has the form*

$$j^2\beta_{(0,0)}(x,y) = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + 2a_{11}xy + a_{02}y^2,$$

and that

$$a_{00}^2 + a_{00}(a_{02} + a_{20}) - a_{11}^2 + a_{02}a_{20} \neq 0. \tag{2.2}$$

Then there exists an open neighborhood $U \subset \mathbb{R}^2$ of $(0,0)$ such that $\beta|_U$ is the Bonnet function of an oriented C^r surface embedded in $\mathbb{R}^3$.

Proof. The function $\beta = \beta(x,y)$ determines a function $\mathcal{B} = \mathcal{B}(x,y)$ as in the proof of Proposition 2.1; let $\Phi = M^{-1} \cdot \mathcal{B}$. Using (2.2), we can check that

$$\det \begin{pmatrix} \dfrac{\partial \Phi_1}{\partial x}(0,0) & \dfrac{\partial \Phi_1}{\partial y}(0,0) \\ \dfrac{\partial \Phi_2}{\partial x}(0,0) & \dfrac{\partial \Phi_2}{\partial y}(0,0) \end{pmatrix} \neq 0,$$

and therefore that (since $\mathcal{B}$ is of class C^{r-1}) there exists an open neighborhood $U \subset \mathbb{R}^2$ of $(0,0)$ such that $\Phi : U \to \mathbb{R}^3$ is a C^{r-1} regular parametrization of $S = \Phi(U)$. We want to show that there exists an open neighborhood of $\Phi(0,0)$ in S that is the inverse image of a regular value of a C^r function. Let $\varphi : U \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ be given by

$$\varphi(x,y,s) = \Phi(x,y) + s\Pi(x,y).$$

Then, by the Inverse Mapping Theorem, φ is a C^{r-1} local diffeomorphism around $(0,0,0)$. Let $V = \varphi(U \times (-\varepsilon, \varepsilon))$. By taking $U \times (-\varepsilon, \varepsilon)$ smaller if necessary, we shall proceed assuming that $\varphi : U \times (-\varepsilon, \varepsilon) \to V$ is a diffeomorphism. Let $g = \pi_3 \circ \varphi^{-1} : V \to \mathbb{R}$, where $\pi_3(x,y,s) \equiv s$. Then $S = g^{-1}(0)$. Since $(d\varphi^{-1})(\operatorname{grad} g) = (0,0,1)$, we have $\operatorname{grad} g = \Pi \circ \pi_{12} \circ \varphi^{-1}$, where $\pi_{12}(x,y,s) \equiv (x,y)$. Therefore $\operatorname{grad} g$ is of class C^{r-1}, so that g is of class C^r and 0 is a regular value. Finally, because $\Lambda \cdot \Phi = \Lambda \cdot M^{-1} \cdot \mathcal{B} = \beta$, we get that β is the Bonnet function of $\Phi(U)$. □

Remark 2.4. In the proof above, notice that the parametrization φ is essentially defined in terms of what will be the unit normal vector, so it has a class of differentiability one less than the (natural) class of differentiability of S.

Remark 2.5. Under the conditions of Proposition 2.1, if the Bonnet function β is defined in a neighborhood of $(0,0)$ and the 2-jet of β at $(0,0)$ is written as in the statement of Proposition 2.3, then (since $\Phi_x(0,0)$ and $\Phi_y(0,0)$ are linearly independent) inequality (2.2) of Proposition 2.3 is satisfied.

3. UMBILICS OF LOJASIEWICZ TYPE

We say that a C^r vector field ξ on $\mathbb{R}^2$ fulfills a *Lojasiewicz inequality at* $(0,0)$ if there exist $k \in \mathbb{N}^*$ and $\delta > 0$ such that $\|\xi(x,y)\| \geq \delta \|(x,y)\|^k$ on some neighborhood of $(0,0)$. Under these circumstances, we will also say that ξ satisfies a *Lojasiewicz inequality of order* k (with associated constant δ) at $(0,0)$.

Suppose that a C^r oriented surface $\mathcal{S} \subset \mathbb{R}^3$, with $r \geq 3$, has an isolated umbilic point $p \in \mathcal{S}$. We will say that p is an *umbilic of Lojasiewicz type* (of order k, with $1 \leq k \leq r-2$) if there is a local C^r diffeomorphism φ of a neighborhood of $p \in \mathbb{R}^3$ onto an open set of $\mathbb{R}^3$ such that the image surface $\varphi(\mathcal{S}) = \tilde{\mathcal{S}}$ satisfies the following properties:

1. $\tilde{p} = \varphi(p)$ is an isolated umbilic of $\tilde{\mathcal{S}}$ with the same index as p, and $\tilde{\mathcal{S}}$ has positive curvature in $\tilde{p}$ and unit normal vector $(0,0,-1)$.
2. The Bonnet function $\tilde{\beta}$ of $\tilde{\mathcal{S}}$ is such that the vector field $\tilde{\xi}(x,y) = (\tilde{\beta}_{xx} - \tilde{\beta}_{yy},\, 2\tilde{\beta}_{xy})$ satisfies a Lojasiewicz inequality of order k at $(0,0)$.

Remarks 3.1. (a) The composition of an appropriate rigid translation and the inversion $\mathcal{I}(p) = p/\|p\|^2$ preserves the principal lines of curvature, hence umbilics and their indexes as well. Thus an inversion may be used to transform a flat umbilic into an umbilic of positive curvature and, up to a conformal diffeomorphism, the first condition is always satisfied.

(b) With the notation above, the index of ξ at $(0,0)$ is twice the index of the umbilic point p [Smyth and Xavier 1992].

(c) If a vector field on $\mathbb{R}^2$ satisfies a Lojasiewicz inequality at the singular point $(0,0)$, then $(0,0)$ is an isolated singularity of the vector field.

(d) Suppose that $Y:(U,(0,0))\to(\mathbb{R}^2,(0,0))$ is an analytic vector field defined in an open set $U\subset\mathbb{R}^2$. Then $(0,0)$ is an isolated singular point of Y if, and only if, Y satisfies a Lojasiewicz inequality at $(0,0)$ [Dumortier 1977]. Therefore, using (a) above, an analytic surface immersed in $\mathbb{R}^3$ always satisfies a Lojasiewicz inequality at an isolated umbilic point.

Lemma 3.2. *If $\xi:(U,(0,0))\to(\mathbb{R}^2,(0,0))$ is a C^r vector field, where $r\geq 1$, defined in a neighborhood U of $(0,0)$ and satisfying a Lojasiewicz inequality of order k at $(0,0)$, where $1\leq k\leq r$, then:*

(a) *the k-jet $j^k\xi_0$ of ξ at $(0,0)$ satisfies a Lojasiewicz inequality of order k at $(0,0)$.*
(b) *both ξ and its k-jet $j^k\xi_0$ at $(0,0)$ have the same index at their common isolated singularity $(0,0)$.*

Proof. Let $\xi=j^k\xi_0+\varphi$. By assumption, there exists a constant $\delta>0$ such that $\|\xi(x,y)\|>\delta\|(x,y)\|^k$, for all (x,y) in a neighborhood V of $(0,0)$. Since φ is the Taylor remainder of order k, by shrinking V if necessary, we may find $\varepsilon>0$ such that $\varepsilon<\frac{1}{4}\delta$ and $\|\varphi(x,y)\|<\varepsilon\|(x,y)\|^k$ for all $(x,y)\in V$. This implies that, when restricted to V, each element of the family $\xi_\mu=j^k\xi_0+\mu\varphi$, with parameter $\mu\in[0,1]$, satisfies a Lojasiewicz inequality of order k with associated constant $\delta-\varepsilon$ at $(0,0)$. It is easy to see that the family ξ_μ provides a homotopy between ξ and $j^k\xi_0$ such that each $\xi_\mu|_V$ has a unique singularity, namely $(0,0)$. This implies the lemma, by index theory [Guillemin and Pollack 1974]. □

We are now in condition to prove the announced result:

Theorem 3.3. *If p is an umbilic point of a C^r surface $\mathcal{S}\subset\mathbb{R}^3$, where $r\geq 3$, that satisfies a Lojasiewicz inequality of order $1\leq k\leq r-2$, then there is an analytic surface $\tilde{\mathcal{S}}$ such that p is an isolated umbilic point of $\tilde{\mathcal{S}}$ and $\mathcal{S}$ having, in both cases, the same index.*

Proof. By definition, we may assume that the Gaussian curvature of a surface $\mathcal{S}$ at p is positive, and that the unit normal vector to $\mathcal{S}$ at p is $(0,0,-1)$. Let $\beta=\beta(x,y)$ be the C^r Bonnet function associated to $\mathcal{S}$ at a neighborhood of p. By assumption, the vector field

$$\xi(x,y)=(\beta_{xx}-\beta_{yy},\,2\beta_{xy})$$

satisfies a Lojasiewicz inequality of order k at the singularity $(0,0)$. Let $\gamma=\gamma(x,y)$ be the $(k+2)$-jet of β at $(0,0)$. Since β satisfies the properties mentioned in Remark 2.5, we can apply Proposition 2.3 to make a Bonnet function of γ with associated surface $\tilde{\mathcal{S}}$. Let

$$Y(x,y)=(\gamma_{xx}-\gamma_{yy},\,2\gamma_{xy}).$$

Since Y and the k-jet of ξ at $(0,0)$ coincide, it follows from the lemma above that p is an umbilic point of both $\mathcal{S}$ and $\tilde{\mathcal{S}}$ satisfying the conditions of the lemma. □

Theorem 3.4. *Assuming the truth of the Loewner conjecture for isolated umbilics on analytic surfaces, if a C^r surface $\mathcal{S}\subset\mathbb{R}^3$, with $r\geq 3$, satisfies a Lojasiewicz inequality at an umbilic point p, the index of p is at most 1. Therefore, if a C^r immersion of a sphere has one umbilic of Lojasiewicz type, it must have at least one more umbilic.*

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