The Dual of the Invariant Quintic

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CONTENTS

We have identified explicitly the dual of the hypersurface I5 defined as the unique quintic hypersurface in $P^5\mathbb{C}$ invariant under the natural action of the Weyl group $W(E_6)$ of the lattice E_6.

Let V be a finite-dimensional complex vector space and $X \subset P(V)$ a (reduced and irreducible) closed projective subvariety. Set $V^\perp = \mathrm{Hom}_\mathbb{C}(V, \mathbb{C})$ as usual; points of $P(V^\perp)$ correspond to hyperplanes in $P(V)$. The *dual* $X^\perp$ of X is the closure in $P(V^\perp)$ of the set of hyperplanes of $P(V)$ that are tangent to X at some smooth point.

Generally, given X as the zero-set of a polynomial, say, an explicit identification of $X^\perp$ is difficult, except in very special cases. This article studies the case where X is a remarkable hypersurface of degree five (quintic), I5 $\subset P^5\mathbb{C}$, the unique quintic that is invariant under the natural action of the Weyl group $W(\mathrm{E}_6)$ of the lattice E_6. This quintic was investigated in [Hunt 1996, Chapter 6], and a certain amount of information about its dual was obtained. However, an explicit determination of this dual — and even of its degree — seemed to be an unsolvable problem.

In [Freitag and Hermann 1998] a certain hypersurface I32 $\subset P^5\mathbb{C}$ of degree 32, also invariant under $W(\mathrm{E}_6)$, was discovered in connection with the investigation of certain modular varieties. Computer calculations in Macaulay led Hunt to conjecture that I32 is the dual of I5. Using Maple Freitag was able to verify this conjecture up to a linear transformation that intertwines the representation of $W(\mathrm{E}_6)$ with its contragredient.

Section 1 contains some more facts about dual varieties in general. Section 2 introduces I5 and Section 3 describes the calculation of the dual.

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1. DUALITY

We refer the reader to [Kleiman 1981] for a good comprehensive introduction to the concept of the dual variety. Here we mention only a few essential facts. Given X as above, the *biduality theorem* states that $(X^\perp)^\perp = X$. In general the dimensions of X and $X^\perp$ are different. But if $X \subset P(V)$ is a hypersurface, except for some exceptional cases (ruled hypersurfaces), the dual variety $X^\perp \subset P(V^\perp)$ is also a hypersurface and hence of the same dimension as X. The biduality theorem then implies that the Gauss map $X \to X^\perp$ (which attaches to every smooth point of X its tangent hyperplane) is birational.

We now assume that $V = \mathbb{C}^n$ and identify $V^\perp$ with $\mathbb{C}^n$ in the usual manner. The hypersurfaces $X, X^\perp$ are given as zero sets of irreducible polynomials $P, P^\perp \in \mathbb{C}[X_0, \dots, X_n]$, say of degrees $d, d^\perp$. These polynomials of course are determined only up to constant factors. We say that P and $P^\perp$ are dual polynomials. The duality relation is simply

$$P^\perp(\operatorname{Grad} P) \equiv 0 \mod P,$$

where

$$\operatorname{Grad} P = \left(\frac{\partial P}{\partial X_0}, \dots, \frac{\partial P}{\partial X_n}\right)$$

is the gradient of P. It is difficult to determine $P^\perp$ explicitly from this equation because this involves complicated elimination; even using computer algebra systems such as Macaulay one has a chance only in special cases of low dimension and degree. Moreover, with Macaulay what is calculated is a polynomial in some finite characteristic, whereas we are interested in the *actual* equation of the dual. For this other methods are needed. It helps if one has a guess about what $P^\perp$ might be, but even then it may be difficult to decide whether $P^\perp(\operatorname{Grad} P)$ is divisible by P. Our success in finding a big dual polynomial is due to unusually lucky circumstances.

2. THE LATTICE E_6 AND THE INVARIANT QUINTIC

For background on this section see [Helgason 1978]. Let $\mathfrak{t} = \mathbb{R}^6$ be a Cartan algebra of a compact form of the Lie algebra E_6 over $\mathbb{R}$. We denote by $Y_1, \dots, Y_6$ the standard basis of the dual space of $\mathfrak{t}$, so that $Y_i(x_1, \dots, x_6) = x_i$. All we have to know about E_6 is that the roots are $\pm(Y_i \pm Y_j)$ for $2 \le i < j \le 6$ and

$$\pm\tfrac{1}{2}(Y_1 \pm Y_2 \pm Y_3 \pm Y_4 \pm Y_5 \pm Y_6),$$

where there is an even number of minus signs inside the parentheses. A system of simple roots is

$$\begin{aligned}
W_1 &= -\tfrac{1}{2}(-Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6),\\
W_2 &= Y_2 - Y_3,\\
W_3 &= -\tfrac{1}{2}(Y_1 + Y_2 + Y_3 - Y_4 - Y_5 - Y_6),\\
W_4 &= Y_3 - Y_4,\\
W_5 &= Y_4 - Y_5,\\
W_6 &= Y_5 - Y_6.
\end{aligned}$$

The root lattice is the lattice $\Lambda \subset \mathfrak{t}^*$ generated by the roots. There exists a unique symmetric bilinear form such that all root vectors have norm 2. The Gram matrix of Λ with respect to the simple roots is

$$\begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0\\
0 & 2 & 0 & -1 & 0 & 0\\
-1 & 0 & 2 & -1 & 0 & 0\\
0 & -1 & -1 & 2 & -1 & 0\\
0 & 0 & 0 & -1 & 2 & -1\\
0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}.$$

The Weyl group $W(E_6)$ is a subgroup of index two in the full group of automorphisms of $(\Lambda, \langle\ ,\ \rangle)$, characterized by the fact that it does not contain $a \mapsto -a$. The intersection of $W(E_6)$ with the special linear group is a simple subgroup of index 2 and order 25,920. The dual lattice Λ^* of Λ is also considered a sublattice of $\mathfrak{t}^*$ using $\langle\ ,\ \rangle$. This lattice has 27 pairs of minimal vectors $\pm a$, each having norm $\frac{4}{3}$.

Proposition 1. *The following* 27 *fundamental weights are permuted under the Weyl group of* E_6. *Dividing them by* 6 *one obtains a system of representatives of the pairs of minimal vectors of* Λ^*.

$$\begin{aligned}
U_0 &= Y_1 + 3Y_2 + 3Y_3 + 3Y_4 + 3Y_5 + 3Y_6\\
U_1 &= Y_1 - 3Y_2 - 3Y_3 + 3Y_4 + 3Y_5 + 3Y_6\\
U_2 &= Y_1 - 3Y_2 + 3Y_3 - 3Y_4 + 3Y_5 + 3Y_6\\
U_3 &= Y_1 - 3Y_2 + 3Y_3 + 3Y_4 - 3Y_5 + 3Y_6\\
U_4 &= Y_1 - 3Y_2 + 3Y_3 + 3Y_4 + 3Y_5 - 3Y_6\\
U_5 &= Y_1 + 3Y_2 - 3Y_3 - 3Y_4 + 3Y_5 + 3Y_6\\
U_6 &= Y_1 + 3Y_2 - 3Y_3 + 3Y_4 - 3Y_5 + 3Y_6\\
U_7 &= Y_1 + 3Y_2 - 3Y_3 + 3Y_4 + 3Y_5 - 3Y_6\\
U_8 &= Y_1 + 3Y_2 + 3Y_3 - 3Y_4 - 3Y_5 + 3Y_6\\
U_9 &= Y_1 + 3Y_2 + 3Y_3 - 3Y_4 + 3Y_5 - 3Y_6\\
U_{10} &= Y_1 + 3Y_2 + 3Y_3 + 3Y_4 - 3Y_5 - 3Y_6
\end{aligned}$$

$$U_{11} = Y_1 + 3Y_2 - 3Y_3 - 3Y_4 - 3Y_5 - 3Y_6$$
$$U_{12} = Y_1 - 3Y_2 + 3Y_3 - 3Y_4 - 3Y_5 - 3Y_6$$
$$U_{13} = Y_1 - 3Y_2 - 3Y_3 + 3Y_4 - 3Y_5 - 3Y_6$$
$$U_{14} = Y_1 - 3Y_2 - 3Y_3 - 3Y_4 + 3Y_5 - 3Y_6$$
$$U_{15} = Y_1 - 3Y_2 - 3Y_3 - 3Y_4 - 3Y_5 + 3Y_6$$
$$U_{16} = 4Y_1$$
$$U_{17} = -2Y_1 + 6Y_2 \qquad U_{18} = -2Y_1 - 6Y_2$$
$$U_{19} = -2Y_1 + 6Y_3 \qquad U_{20} = -2Y_1 - 6Y_3$$
$$U_{21} = -2Y_1 + 6Y_4 \qquad U_{22} = -2Y_1 - 6Y_4$$
$$U_{23} = -2Y_1 + 6Y_5 \qquad U_{24} = -2Y_1 - 6Y_5$$
$$U_{25} = -2Y_1 + 6Y_6 \qquad U_{26} = -2Y_1 - 6Y_6$$

The polynomials

$$J_k := \sum_{i=0}^{26} U_i^k$$

are invariant under the Weyl group $W(\mathrm{E}_6)$. It is known (see [Hunt 1996, Section B.1.2.2] for general information on the invariants of unitary reflection groups) that the ring of all invariant polynomials is the polynomial ring

$$\mathbb{C}[Y_1, \dots Y_6]^{W(\mathrm{E}_6)} = \mathbb{C}[J_2, J_5, J_6, J_8, J_9, J_{12}].$$

The polynomial J_2 is just a multiple of the Killing form. The next one up, J_5, is the one that interests us; it defines the invariant quintic hypersurface

$$\mathrm{I5} := \{J_5 = 0\}.$$

A quick way to write it explicitly is:

Proposition 2. *Let E_i denote the i-th elementary symmetric polynomial in the squares of the five variables $Y_2, \dots, Y_6$. Then J_5 is given by*

$$720\big(Y_1^5 - 6Y_1^3E_1 - 27Y_1(E_1^2 - 4E_2) + 648Y_2Y_3Y_4Y_5Y_6\big).$$

3. THE DUAL OF THE INVARIANT QUINTIC

The invariant quintic I5 is an extremely interesting variety, mainly because, as shown in [Hunt 1996], it contains several subvarieties that are known to be modular, that is, quotients of bounded symmetric domains by certain arithmetic groups. It then follows from the general theory of Shimura that these subvarieties are *moduli spaces* for certain moduli problems. Furthermore, I5 is itself conjectured to be closely related to the moduli space of marked cubic surfaces. We briefly describe these connections.

I5 is singular; its singular locus consists of 120 lines meeting at 36 points: the projectivized roots of E_6. At these 36 points, I5 has multiplicity 3; blowing up the points we get a copy of the *Segre cubic threefold*, the unique cubic threefold with 10 ordinary double points (there are 10 of the 120 lines meeting at each of the 36 points). Dual to the 36 singular points are 36 hyperplane sections, each of which is isomorphic to the *Nieto quintic*. Results from [Hunt 1996] and unpublished investigations of the same author show that the Segre cubic, the Nieto quintic and I5 are closely related to ball quotients.

The (conjectural) relation with the moduli space of marked cubic surfaces has two descriptions. The moduli space of cubic surfaces is a ball quotient [Allcock et al. 1998], as is that of marked cubic surfaces. The latter space is a Galois cover of the former with a Galois group isomorphic to $W(\mathrm{E}_6)$. Since I5 is related to a ball quotient and has automorphism group isomorphic to $W(\mathrm{E}_6)$, we see the first connection. Next, consider the hyperplane sections of I5. These are quintic threefolds with 120 ordinary double points. Moreover, for a *tangent* hyperplane section, we have a 121-nodal quintic. It is well-known that this is a Calabi–Yau threefold (after resolution), and that it has a cubic form on the cohomology. In this case, the cubic form, taken projectively, is a cubic surface! This is the other connection.

Our goal is to find the dual polynomial. Its degree would be $5\cdot 4^4 = 1280$ if the quintic where nonsingular [Kleiman 1981]. However, singularities tend to lower the degree of the dual, in this case by a lot.

Theorem 3. *The dual polynomial of J_5 has degree 32. It is obtained from the polynomial J shown at the top of the next page by the replacement $Y_1 \mapsto 3Y_1$.*

We denote by $\tilde{J}$ the polynomial obtained from J replacing $Y_1 \mapsto 3Y_1$. Similarly the polynomial obtained from J_k by this transformation is denoted by $\tilde{J}_k$. The meaning of this transformation is that it intertwines the dual representation of $W(\mathrm{E}_6)$ with the original one. A simple observation states that $P(\operatorname{Grad} Q)$ is invariant under $W(\mathrm{E}_6)$ if Q is invariant under $W(\mathrm{E}_6)$ and P is invariant under the dual representation (i.e., invariant under the transposed matrices). We conclude that $\tilde{J}_k(\operatorname{Grad} J_5)$ can be written as a polynomial in the invariants J_2, J_5, J_6, J_8, J_9, J_{12}. As soon as one has found these polynomials explicitly for $k = 2, 5, 6, 8, 9, 12$ one has a representation of $\tilde{J}(\operatorname{Grad} J_5)$ as a polynomial in

$$
\begin{aligned}
J = {} & 2^{32}\,3^{16}\,5\cdot 7^2\,J_8^4 - 2^{34}\,3^{14}\,5\cdot 7^2\,J_5^2J_6J_8^2 + 2^{38}\,3^{13}\,5\cdot 7J_5^3J_8J_9 - 2^{39}\,3^{12}\,7^2\,J_5^4J_{12} + 2^{34}\,3^{10}\,7^2\,61J_5^4J_6^2 \\
& - 2^{35}\,3^{14}\,5\cdot 7^2\,J_2J_6J_8^3 - 2^{35}\,3^{14}\,5^2\,7J_2J_5J_8^2J_9 + 2^{38}\,3^{12}\,5\cdot 7^2\,J_2J_5^2J_8J_{12} + 2^{36}\,3^{10}\,5\cdot 7^3\,J_2J_5^2J_6^2J_8 \\
& - 2^{36}\,3^{11}\,5\cdot 7\cdot 13J_2J_5^3J_6J_9 + 2^{35}\,3^{10}\,7^2\,13J_2J_5^6 - 2^{34}\,3^{12}\,5^2\,7^2\,J_2^2J_8^2J_{12} + 2^{33}\,3^{10}\,5\cdot 7^2\,113J_2^2J_6^2J_8^2 \\
& + 2^{35}\,3^{11}\,5^2\,7\cdot 17J_2^2J_5J_6J_8J_9 - 2^{35}\,3^{11}\,5\cdot 7^3\,J_2^2J_5^2J_6J_{12} - 2^{34}\,3^9\,5^2\,7^2\,J_2^2J_5^2J_6^3 - 2^{35}\,3^{10}\,7^2\,29J_2^2J_5^4J_8 \\
& + 2^{35}\,3^{10}\,5^4\,J_2^3J_8J_9^2 + 2^{36}\,3^{10}\,5^2\,7^2\,J_2^3J_6J_8J_{12} - 2^{35}\,3^8\,5\cdot 7^2\,41J_2^3J_6^3J_8 - 2^{36}\,3^9\,5^3\,7J_2^3J_5J_9J_{12} \\
& - 2^{35}\,3^7\,5^2\,7\cdot 193J_2^3J_5J_6^2J_9 - 2^{31}\,3^{12}\,5\cdot 7^2\,J_2^3J_5^2J_8^2 + 2^{33}\,3^8\,7^2\,367J_2^3J_5^4J_6 + 2^{34}\,3^8\,5^3\,7^2\,J_2^4J_{12}^2 \\
& + 2^{26}\,3^{10}\,5^3\,7^4\,J_2^4J_8^3 - 2^{33}\,3^8\,5^4\,13J_2^4J_6J_9^2 - 2^{34}\,3^6\,5^2\,7^2\,41J_2^4J_6^2J_{12} + 2^{32}\,3^4\,5\cdot 7^2\,41^2\,J_2^4J_6^4 \\
& - 2^{27}\,3^{14}\,5\cdot 7^2\,J_2^4J_5^2J_6J_8 + 2^{30}\,3^7\,5^2\,7\cdot 11\cdot 31J_2^4J_5^3J_9 - 2^{27}\,3^8\,5^3\,7^2\,139J_2^5J_6J_8^2 - 2^{28}\,3^7\,5^4\,7\cdot 59J_2^5J_5J_8J_9 \\
& + 2^{30}\,3^6\,5^3\,7^2\,J_2^5J_5^2J_{12} + 2^{28}\,3^4\,5\cdot 7^2\,10903J_2^5J_5^2J_6^2 - 2^{27}\,3^6\,5^4\,7^4\,J_2^6J_8J_{12} + 2^{26}\,3^4\,5^3\,7^2\,11^2\,41J_2^6J_6^2J_8 \\
& + 2^{27}\,3^5\,5^4\,7\cdot 457J_2^6J_5J_6J_9 - 2^{24}\,3^4\,7^2\,179\cdot 593J_2^6J_5^4 + 2^{27}\,3^8\,5^5\,J_2^7J_9^2 + 2^{28}\,3^4\,5^4\,7^2\,41J_2^7J_6J_{12} \\
& - 2^{27}\,3^2\,5^3\,7^2\,41^2J_2^7J_6^3 + 2^{23}\,3^4\,5\cdot 7^2\,64937J_2^7J_5^2J_8 + 2^{17}\,3^4\,5^2\,7^2\,659947J_2^8J_8^2 - 2^{18}\,3^2\,5\cdot 7^2\,239\cdot 23747\times \\
& J_2^8J_5^2J_6 - 2^{19}\,3^2\,5^2\,7^2\,561947J_2^9J_6J_8 - 2^{19}\,3^5\,5^3\,7\cdot 17\cdot 179J_2^9J_5J_9 - 2^{18}\,3^4\,5^3\,7^2\,11\cdot 67J_2^{10}J_{12} \\
& + 2^{17}\,5^2\,7^2\,19\cdot 23\cdot 41\cdot 109J_2^{10}J_6^2 + 2^{16}\,3^2\,5^3\,7^2\,4373J_2^{11}J_5^2 + 2^{10}\,3^2\,5^4\,7^4\,11\cdot 67J_2^{12}J_8 - 2^{11}\,5^4\,7^2\,11\cdot 41\cdot 67\times \\
& J_2^{13}J_6 + 5^3\,7^2\,11^2\,67^2\,J_2^{16}.
\end{aligned}
$$

the invariants. The duality relation means that this polynomial is divisible by J_5. This means exactly that it vanishes after replacing J_5 by 0. Explicit expressions for $\tilde{J}_k(\operatorname{Grad} J_5)$ as polynomials in the invariants, for each of the six cases, have been found by computer; they can be found in [Freitag 1998].

The most involved calculations concern the case $k = 12$. The degree of $\tilde{J}_{12}(\operatorname{Grad} J_5)$ is 48 and there are 248 monomials of this degree. We chose 350 special values at random and had to solve 350 equations with 248 variables. The problem is that the solution consists of huge rational numbers with denominators. Therefore we solved the equations modulo several primes (actually 15 such beginning at 100003). To reconstruct the rational numbers we made an a priori assumption about their size. This a priori assumption is justified later because it is easy to verify the result by direct calculation. The vanishing of $\tilde{J}(\operatorname{Grad} J_5)$ after the substitution $J_5 = 0$ again can be verified by inserting enough special values.

Hunt [1996] has proved that J_5 defines a rational variety. We conclude that *the hypersurface* I32 *defined by J is rational.*

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