Solving the Sextic by Iteration:  
A Study in Complex Geometry and Dynamics

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We use the Valentiner action of $A_6$ on $\mathbb{C}P^2$ to develop an iterative algorithm for the solution of the general sextic equation over $\mathbb{C}$, analogous to Doyle and McMullen’s algorithm for the quintic.

1. INTRODUCTION

1A. Overview

Doyle and McMullen [1989] devised an iterative solution to the fifth degree polynomial. At the core of the method is a rational mapping $f$ of $\mathbb{C}P^1$ with the icosahedral symmetry of a general quintic. Algebraically, this means that $f$ commutes with a group of Möbius transformations that is isomorphic to the alternating group $A_5$. Moreover, this $A_5$-equivariant possesses reliable dynamics: for almost any initial point $a \in \mathbb{C}P^1$, the sequence of iterates $f^k(a)$ converges to one of the periodic cycles that comprise an icosahedral orbit; see [Doyle and McMullen 1989, p. 163] for a geometric description. This breaking of $A_5$-symmetry provides for a generally convergent quintic-solving algorithm: with almost any fifth-degree equation, associate a rational mapping that has reliable dynamics and whose attractor consists of a single orbit from which one computes a root.

An algorithm that solves the sixth-degree equation calls for a dynamical system with $S_6$ or $A_6$ symmetry. Since neither $S_6$ nor $A_6$ acts on $\mathbb{C}P^1$, attention turns to higher dimensions. Acting on $\mathbb{C}P^2$ is an $A_6$-isomorphic group of projective transformations found by Valentiner in the late nineteenth century. The present work exploits this two-dimensional $A_6$ soccer ball in order to discover a Valentiner-symmetric rational mapping of $\mathbb{C}P^2$ whose dynamics experimentally appear to be nice in the above sense—
transferred to the $\mathbb{CP}^2$ setting. This map provides the central feature of a conjecturally generally convergent sextic-solving algorithm analogous to that employed in the quintic case.

1B. Solving Equations by Iteration

For $n \leq 4$, the symmetric groups $S_n$ act faithfully on $\mathbb{CP}^1$. Corresponding to each action is a map whose nice dynamics provides for algorithmic convergence to roots of a given $n$th-degree equation. For instance, Newton's method provides a direct iterative solution to quadratic polynomials, but, due to a lack of symmetry, not to higher degree equations. My interests here are the geometric and dynamical properties of complex projective mappings rather than numerical estimates.

The search for elegant complex geometry and dynamics continues into degree five where $A_5$ is the appropriate group, since $S_5$ fails to act on the sphere. This reduction in the galois group requires the extraction of the square root of a polynomial's discriminant. Such root-taking is itself the result of a reliable iteration, namely, Newton's method. In practical terms, the Doyle-McMullen algorithm solves a family of fifth-degree resolvents the members of which possess $A_5$ symmetry. A map with icosahedral symmetry and nice dynamics plays the leading role.

Pressing on to the sixth-degree leads to the two-dimensional $A_6$ action of the Valentinier group $\mathcal{V}$. Here the problem shifts to one of finding a nice $\mathcal{V}$-symmetric mapping of $\mathbb{CP}^2$ from whose attractor one calculates a given sextic’s root. (The solution procedure follows that of the quintic algorithm: see Section 4.) Providing the overall framework is the two-dimensional $A_6$ analogue of the icosahedron.

1C. Proofs and Computations

At the moment, many of this work's results have only computational support; accordingly, I call them “facts”. Furthermore, its conjectural nature calls for a deeper understanding of Valentinier geometry and dynamics. As the theory of complex dynamics in several dimensions develops more sophisticated weaponry, the barricades to understanding might become assailable. For now, I hope that these discoveries provide a stimulus to such development.

2. VALENTINIER'S GROUP: THE $A_6$ ACTION ON $\mathbb{CP}^2$

In 1884, Klein wrote

If ... any equation $f(x) = 0$ is given, we will investigate what is the smallest number of variables with which we can construct a group of linear substitutions which is isomorphic with the Galois group of $f(x) = 0$. [Klein 1913, p. 138]

In the wake of the mid-nineteenth century abstractionist turn in mathematics the theory of group representations began to emerge. Part of the concrete yield from work on symmetric groups was Valentinier’s discovery [1889] of a complex projective group that is isomorphic to $A_6$ — the alternating group of six things. Then Wiman [1895] explored some of the geometric and invariant structure determined by this action on the complex projective plane. A more thorough exposition appeared in [Fricke 1926].

Here I take a new approach to the generation of the Valentinier group and then explore some of its rich combinatorial geometry. The core of this work involves the development of a combinatorially sensitive description of the basic geometric structures. In so doing, I reproduce some of the Wiman and Fricke results. The study culminates in an account of the system of Valentinier-invariant polynomials and, thereby, lays the algebraic foundation for constructing Valentinier-symmetric mappings of $\mathbb{CP}^2$.

2A. Basics of $A_6$

These are the nonidentity elements of $A_6$ classified by order and cycle-structure:

<table>
<thead>
<tr>
<th>order</th>
<th>structure</th>
<th># of elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$(ab)(cd)$</td>
<td>$45 = \frac{1}{2}(6^3) \cdot 3!$</td>
</tr>
<tr>
<td>3</td>
<td>$(abc)$</td>
<td>$40 = \frac{6}{3} \cdot 2$</td>
</tr>
<tr>
<td>3</td>
<td>$(abc)(def)$</td>
<td>$40 = \frac{6}{3} \cdot 2$</td>
</tr>
<tr>
<td>4</td>
<td>$(abcd)(ef)$</td>
<td>$90 = \frac{6}{3} \cdot 3!$</td>
</tr>
<tr>
<td>5</td>
<td>$(abcde)$</td>
<td>$144 = 6 \cdot 4!$</td>
</tr>
</tbody>
</table>

Sitting inside $A_6$ are twelve versions of $A_5$ that decompose into two conjugate systems of six:

1. the stabilizers $\text{Stab}\{k\}$ of one thing, and
2. the permutations of the six pairs of antipodal icosahedral vertices under rotation.
Acting by conjugation, $A_6$ permutes each of the two systems individually. A given $A_5$ subgroup fixes itself set-wise and permutes its five conjugates according to the rotational icosahedral group’s action on the five cubes found in the icosahedron. Meanwhile, the other system of six subgroups undergo the permutations of the six pairs of antipodal vertices. Consequently, the intersection of two $A_5$ subgroups in the same system is isomorphic to $A_4$ — the tetrahedral rotations — while two in different systems give a dihedral group $D_6$.

2B. Generating the Valentiner Group

An $A_5$ subgroup of $A_6$, say Stab{1}, extends to $A_6$ by addition of the generator (12)(3456). Furthermore, this order-four element generates an $S_4$ over the $A_4$ subgroup

$$\langle (35)(46), (456) \rangle \subset \text{Stab}\{1\}.$$  

This structure suggests a method for producing an $A_6$-isomorphic group $V$ in PGL$_3(\mathbb{C})$:
(1) take a tetrahedral subgroup $\mathcal{T}$ of an icosahedral group $\mathcal{I}$;
(2) by addition of an order-4 transformation $Q$, extend $\mathcal{T}$ to an octahedral group $\mathcal{O} = \langle \mathcal{T}, Q \rangle$;
(3) generate $\mathcal{V} = \langle \mathcal{T}, Q \rangle \simeq A_6$.

![Figure 1](https://via.placeholder.com/150)

**FIGURE 1.** The icosahedron in octahedral coordinates.

The 15 pairs of antipodal edges of the standard icosahedron decompose into five triples such that three lines joining antipodal edge-midpoints are mutually perpendicular. Stabilizing each such triple is one of the five tetrahedral subgroups of the icosahedral group. Alternatively, the lines in such a triple correspond to the two-fold rotational axes of a tetrahedron whose four vertices are face-centers of the icosahedron, as in Figure 1. With such a triple of lines as coordinate axes, in octahedral coordinates $\{x_1, x_2, x_3\}$, the points

$$A = \{[1, 1, 1], [-1, -1, 1], [1, -1, -1], [-1, 1, -1]\}$$  

constitute a set of tetrahedral vertices, where the brackets indicate homogeneous coordinates for projective space, as usual. The corresponding tetrahedral group $\mathcal{T} = \text{Stab}(A)$ consists of the identity and the 11 orthogonal transformations:

- $Z_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $Z_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $Z_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,
- $T_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $T_1^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$,
- $T_2 = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$, $T_2^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$,
- $T_3 = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $T_3^2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$,
- $T_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $T_4^2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Being orthogonal, $\mathcal{T}$ preserves the quadratic form $C(x) = x_1^2 + x_2^2 + x_3^2$, hence also the conic $\mathcal{C} = \{C = 0\}$ in $\mathbb{CP}^2$. This conic contains two sets of orbits of size four: two of the points fixed projectively by each of the threefold $T_k$. This is a manifestation of the pairs of antipodal tetrahedra found in the icosahedron, that is, tetrahedra obtained from one another by negating all coordinates. The third fixed point is one of the elements of the set $A$ above. With $\rho = e^{2\pi i/3}$ the respective points are

- $v_1 = [\rho, \rho^2, 1]$, $v_7 = [\rho^2, \rho, 1]$,
- $v_2 = [-\rho, -\rho^2, 1]$, $v_2 = [-\rho^2, -\rho, 1]$,
- $v_3 = [-\rho, \rho^2, 1]$, $v_3 = [-\rho^2, \rho, 1]$,
- $v_4 = [\rho, -\rho^2, 1]$, $v_1 = [\rho^2, -\rho, 1]$.

The barred notation $v_k$ derives from Fricke, being suggested by an antiholomorphic relationship between the two systems of tetrahedra. Indeed, in the $x$ coordinates chosen above, the conjugation map $x \to \bar{x}$ exchanges a tetrahedron and its antipode.
The order-four transformation

\[ Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \rho^2 \\ 0 & -\rho & 0 \end{pmatrix} \]

cyclically permutes the \( v_u \) but not the \( v_o \), while

\[ \tilde{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \rho \\ 0 & -\rho^2 & 0 \end{pmatrix} \]

cyclically permutes the \( v_o \) but not the \( v_u \). Also, \( Q^2 = \tilde{Q}^2 = Z_4 \). Since \( \text{PGL}_3(\mathbb{C}) \) is four-times transitive, \( Q \) and \( \tilde{Q} \) are the unique such projective transformations. Accordingly, the groups \( \mathcal{O} = \langle \mathcal{T}, Q \rangle \) and \( \overline{\mathcal{O}} = \langle \mathcal{T}, \tilde{Q} \rangle \) are octahedral, that is, isomorphic to \( S_4 \).

Extending \( \mathcal{T} \) to an icosahedral group \( \mathcal{J} \) requires a projective transformation \( P \) of order five that preserves \( \mathcal{C} \) and point-wise fixes a pair of antipodal icosahedral vertices. One way of producing such a \( P \) is to turn the icosahedron of Figure 1 as in Figure 2, so that a pair of antipodal vertices corresponds to the point \([0,1,0]\), i.e., to the affine points \((0, \pm 1, 0)\). In these icosahedral coordinates \( \{u_1, u_2, u_3\} \) the desired transformation of order five is

\[ P_u = \begin{pmatrix} \cos 2\pi/5 & 0 & -\sin 2\pi/5 \\ 0 & 1 & 0 \\ \sin 2\pi/5 & 0 & \cos 2\pi/5 \end{pmatrix}. \]

The change of basis from octahedral to icosahedral coordinates is

\[ u = Ax = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \]

where

\[ c = \sqrt{\frac{5 + \sqrt{5}}{10}}, \quad s = \sqrt{\frac{5 - \sqrt{5}}{10}}. \]

Thus, in icosahedral coordinates, the conic form preserved by \( \mathcal{J} \) is

\[ C(u) = u_1^2 + u_2^2 + u_3^2, \]

while \( P \) has the expression

\[ P_x = A^{-1} P_u A = \frac{1}{2} \begin{pmatrix} 1 & \tau^{-1} & -\tau \\ \tau^{-1} & \tau & 1 \\ -\tau & 1 & \tau^{-1} \end{pmatrix}, \]

with \( \tau = \frac{1 + \sqrt{5}}{2} \).

Finally, \( \mathcal{J} = \langle \mathcal{T}, P \rangle \). This produces two Valentin groups distinguished by chirality:

\[ \mathcal{V} = \langle \mathcal{J}, Q \rangle, \quad \overline{\mathcal{V}} = \langle \mathcal{J}, \tilde{Q} \rangle. \]

Our terms “octahedral coordinates” and “icosahedral coordinates” are from Fricke [1926, pp. 263 ff.]. His octahedral generators are nearly those above. In his icosahedral coordinates, the conic form is

\[ C_{\text{Fricke}}(z) = z_1 z_3 + z_2^2. \]

The change of coordinates \( B \) that yields \( C(Bz) = C_{\text{Fricke}}(z) \) is

\[ B = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ 1 & 0 & -i \end{pmatrix}. \]

Set \( \varepsilon = e^{2\pi i/5} \) and \( Z = Z_2 \). The generators \( Z_u = AZ_u A^{-1} \) and \( P_u = AP_u A^{-1} \) become Fricke’s icosahedral generators \( T \) and \( S \) [1926, p. 263]:

\[ Z_z = BZ_u B^{-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \varepsilon^2 + \varepsilon^3 & \varepsilon + \varepsilon^4 \\ \varepsilon^2 + \varepsilon^3 & 1 & \varepsilon^2 + \varepsilon^3 \\ \varepsilon + \varepsilon^4 & \varepsilon^2 + \varepsilon^3 & 1 \end{pmatrix} \]

and

\[ P_z = BP_u B^{-1} = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon^4 \end{pmatrix}. \]
2C. Valentiner Geometry

Icosahedral conics. Since a ternary icosahedral group has an orthogonal representation on $\mathbb{R}^3$, its complexification stabilizes the quadratic form

$$C(x) = x_1^2 + x_2^2 + x_3^2$$

over $\mathbb{C}$, hence also the conic $\mathcal{C} = \{C(x) = 0\}$ in $\mathbb{CP}^2$. For reasons that will become clear, the two systems of six icosahedral groups $\mathcal{I}_z$ and $\mathcal{I}_w$ in $\mathcal{V}$ receive the designations “barred” and “unbarred”, respectively. Corresponding to each of the twelve icosahedral subgroups of $\mathcal{V}$ is a quadratic form $C_\alpha$ or $C_a$ and conic $\mathcal{C}_\alpha$ or $\mathcal{C}_a$, that $\mathcal{I}_z$ or $\mathcal{I}_w$ preserves. Thereby, each such conic possesses the structure of an icosahedron.

Call the form above $C_1$. The action of $\mathcal{V}$ produces the five remaining forms. With $\eta = (3 + \sqrt{15})/4$, we have

- $C_1(x) = x_1^2 + x_2^2 + x_3^2$,
- $C_2(x) = C_1(Q^{-1}x) = x_1^2 + \rho^2 x_2^2 + \rho x_3^2$,
- $C_3(x) = C_2(P^{-1}x) = \eta (x_1^2 + \rho x_2^2 + \rho^2 x_3^2) + (\rho^2 x_1 x_2 + \rho x_1 x_3 - x_2 x_3))$,
- $C_4(x) = C_2(P^{-3}x) = \eta (\frac{1}{2} (x_1^2 + \rho^2 x_2^2 + x_3^2) + (-x_1 x_2 + \rho^2 x_1 x_3 + \rho x_2 x_3))$,
- $C_5(x) = C_2(P^{-2}x) = \eta (\frac{1}{2} (x_1^2 + \rho^2 x_2^2 + x_3^2) - (x_1 x_2 + \rho^2 x_1 x_3 + \rho x_2 x_3))$,
- $C_6(x) = C_2(P^{-1}x) = \eta (\frac{1}{2} (x_1^2 + \rho^2 x_2^2 + x_3^2) + (\rho^2 x_1 x_2 - \rho x_1 x_3 + x_2 x_3))$.

The action of $\mathcal{V}$ is given as follows (where, as before, $Z = Z_2$):

- $P : C_1 \leftrightarrow C_1$, $C_2 \leftrightarrow C_5$, $C_3 \leftrightarrow C_4$, $C_5 \leftrightarrow C_3$, $C_6 \leftrightarrow C_2$;
- $Z : C_1 \leftrightarrow C_1$, $C_2 \leftrightarrow C_2$, $C_3 \leftrightarrow \rho^2 C_1$, $C_4 \leftrightarrow \rho C_2$;
- $Q : C_1 \leftrightarrow C_2$, $C_3 \leftrightarrow C_5$, $C_6 \leftrightarrow \rho^2 C_1$, $C_4 \leftrightarrow C_3$.

Direct calculation yields this result:

**Proposition 2.1.** The quadratic forms $C_\alpha$ are linearly independent and so span the 6-dimensional space of ternary quadratic forms. In particular, an unbarred conic form $C_a$ is a linear combination of the $C_\alpha$ that is invariant under the icosahedral group $\mathcal{I}_a$.

To be specific, the “5-cycle” $P$ belongs to $\mathcal{I}_1$ and to an unbarred icosahedral group — say $\mathcal{I}_3$ — so that the indexing agrees with that of Fricke. (Recall that the intersection of two nonconjugate $A_5$ subgroups of $A_6$ is a $D_5$.) Relative or **projective** invariance under $P$ requires $C_3$ to take the form

$$C_3 = \alpha C_1 + C_2 + C_3 + C_4 + C_5 + C_6.$$ 

To determine the constant, apply to the $C_\alpha$ an element $T$ of $\mathcal{I}_3$ that does not belong to $\mathcal{I}_1$, say one of the 20 elements of order three in $\mathcal{I}_3$. Recalling the association between $\mathcal{I}_3$ and the permutations of the six pairs of antipodal icosahedral vertices labeled according to the action of $P$, such a transformation corresponds to a double 3-cycle of the form $(abc)(def)$. Specifically, the permutation $(164)(235)$ used below corresponds to an element of $\mathcal{I}_3$.

The action of the generators on the conics $C_\alpha$ is given by the permutation of the indices:

$$P : (26543),$$
$$Z : (134)(56),$$
$$Q : (12)(3654).$$

Computation in $A_6$ yields the correspondence

$$T = QP^2QPQ^3 : (164)(235).$$

Moreover, the action on the conic forms is

$$T : C_1 \rightarrow C_6 \rightarrow \rho C_4 \rightarrow C_1, C_2 \rightarrow \rho C_3 \rightarrow \rho^2 C_5 \rightarrow C_2,$$

so that

$$C_3(T^{-1}x) = \rho^2 C_1 + \rho(C_2 + C_3 + C_4 + C_5 + \alpha C_6).$$

The projective invariance under $T$ of the conic $C_3 = \{C_3 = 0\}$ requires $\alpha = \rho$. Accordingly,

$$C_3(T^{-1}x) = \rho^2 C_1 + \rho(C_2 + C_3 + C_4 + C_5 + \alpha C_6) = \rho C_3(x).$$

Just as the barred forms stem from $C_1$, the remaining unbarred conic forms arise from $C_3$ (again, our indices are chosen to agree with Fricke’s labels):

- $C_1(x) = C_2(P^{-1}x) = C_1 + \rho C_2 + C_3 + \rho C_4 + C_5 + \rho C_6$,
- $C_2(x) = C_3(Q^{-1}x) = C_1 + \rho C_2 + \rho C_3 + C_4 + \rho^2 C_5 + C_6$.

Recall that:

$$C_3 = \frac{1}{\sqrt{3}} [(x_1^2 + \frac{1}{2} \eta x_2^2 + \frac{1}{3} \eta x_3^2 + \rho x_3^2 - 2 \rho (\rho - 1)x_1 x_3),$$

$$C_2(x) = C_3(Q^{-1}x) = C_1 + \rho C_2 + \rho C_3 + C_4 + \rho^2 C_5 + C_6 = -\eta(x_1^2 + \frac{1}{3} \eta x_2^2 + \rho x_3^2 - 2 \rho (\rho - 1)x_1 x_3),$$

$$C_1(x) = C_2(P^{-1}x) = C_1 + \rho C_2 + C_3 + \rho^2 C_4 + C_5 + \rho C_6 = -\eta(x_1^2 + \frac{1}{3} \eta x_2^2 + \rho x_3^2 + 2 \rho (\rho - 1)x_1 x_3),$$

$$C_2(x) = C_3(Q^{-1}x) = C_1 + \rho C_2 + \rho C_3 + C_4 + \rho^2 C_5 + C_6.$$
\[ C_3(x) = \rho C_1 + C_2 + C_3 + C_4 + C_5 + C_6 \]
\[ = -\eta \rho (\rho^2 \eta x_1^2 + \rho \eta x_2^2 + \frac{1}{3} \eta^2 x_3^2 + 2 \rho (\rho - 1)x_1 x_2), \]
\[ C_4(x) = C_2(P^{-3}x) \]
\[ = C_1 + \eta \rho \eta \rho C_3 + C_4 + \rho C_5 + C_6 \]
\[ = -\eta (\rho^2 \eta x_1^2 + \rho \eta x_2^2 + \frac{1}{3} \eta^2 x_3^2 + 2 \rho (\rho - 1)x_1 x_2), \]
\[ C_5(x) = C_2(P^{-2}x) \]
\[ = C_1 + C_2 + \rho C_3 + C_4 + C_5 + \rho^2 C_6 \]
\[ = -\eta (\frac{1}{3} \eta^2 x_1^2 + \rho \eta x_2^2 + \frac{1}{3} \eta^2 x_3^2 - 2 \rho (\rho - 1)x_1 x_2). \]

Application of \( \mathcal{V} \) yields
\[ P : C_3 \leftrightarrow C_3, \quad C_1 \to C_5 \to C_4 \to C_6 \to C_2 \to C_1, \]
\[ Z : C_1 \leftrightarrow C_1, \quad C_2 \leftrightarrow C_2, \quad C_3 \leftrightarrow \rho C_4, \quad C_5 \leftrightarrow C_5, \]
\[ Q : C_5 \leftrightarrow C_6, \quad C_1 \to C_3 \to C_2 \to \rho C_4 \to C_1. \]

**Antiholomorphic symmetry.** The one-dimensional icosahedral group \( S_{\text{dodeca}} \) acts on two sets of five tetrahedra each of which corresponds to a quadruple of points in \( \mathbb{C}P^1 \). However, no element of the group sends the tetrahedra of one set to those of the other. Such an exchange occurs by means of antiholomorphic maps of degree one. Of these, 15 correspond to reflections through the 15 great circles of reflective icosahedral symmetry; the remaining 45 are the various “odd” compositions of the 15 basic reflections — e.g., the antipodal map. Extending the holomorphic \( S_{\text{dodeca}} \) by such an “anti-involution” produces the group \( S_{\text{icosahedron}} \) of all 120 symmetries of the icosahedron. The 15 icosahedral reflections generate this extended group while their even products result in \( S_{\text{dodeca}} \). In coordinates where one of the great circles corresponds to the real axis, the associated anti-involution is complex conjugation: in homogeneous coordinates,
\[ [x_1, x_2] \to [\overline{x}_1, \overline{x}_2]. \]

The Valentin analogues of the tetrahedra are the two systems of conics. Are there ternary anti-involutions that exchange the barred and unbarred conics? If so, can they take the form
\[ [x_1, x_2, x_3] \to [\overline{x}_1, \overline{x}_2, \overline{x}_3]? \]

Fricke answered both questions affirmatively [1926, pp. 270–271, 286–289]. (See below for a combinatorial geometric computation of this additional symmetry.) In the current octahedral coordinates, this bar-unbar map is
\[ \text{bub}[x_1, x_2, x_3] = [\rho^2 \overline{x}_1 - \rho \overline{x}_3, -\rho (\rho + \tau) \overline{x}_2, -\rho \overline{x}_1 - \overline{x}_3]. \]

The action on the conic forms is:
\[ C_1(\text{bub}(x)) = \alpha \rho^2 C_1(x), \]
\[ C_2(\text{bub}(x)) = \alpha \rho C_2(x), \]
\[ C_3(\text{bub}(x)) = \alpha C_3(x), \]
\[ C_4(\text{bub}(x)) = \alpha^2 C_4(x), \]
\[ C_5(\text{bub}(x)) = \alpha \rho^2 C_5(x), \]
\[ C_6(\text{bub}(x)) = \alpha C_6(x), \]
where \( \alpha = \frac{1}{2}(3 + \sqrt{15} i). \) (The match between \( C_6 \) and \( C_4 \) is no accident. Fricke used this map to dub the unbarred conics.)

**Proposition 2.2.** The group \( \overline{\mathcal{V}}_{2;360} = \langle \mathcal{V}, \text{bub} \rangle \) is a degree two extension of \( \mathcal{V} \).

**Proof.** For \( T \in \mathcal{V} \), the composition \( T' = \text{bub} \circ T \circ \text{bub} \) is a projective transformation that permutes the conics within a system. Therefore it belongs to \( \mathcal{V} \).

Concerning the form of a bub map, there are coordinate systems in which its expression is conjugation of each coordinate. While interesting in their own right, such coordinates also yield computational benefits. Some of the Valentin structure suggests a means of achieving this diagonalization. I will take up the topic once the relevant framework is in place.

**Special orbits.** Some of the special icosahedral points on a conic \( \mathcal{C}_d \) occur at the intersections of \( \mathcal{C}_d \) and the other 11 conics.

**Fact 2.3.** Within a system, \( \mathcal{C}_d \) meets each \( \mathcal{C}_m \) (where \( m \neq d \)) in four tetrahedral points; this gives the 20 face-centers on \( \mathcal{C}_d \).

The overall result is a 60 point \( \mathcal{V} \)-orbit \( \mathcal{O}_{20} \). Similarly, the unbarred intersections yield \( \mathcal{O}_{10} \). Alternatively, each member of \( \mathcal{O}_{20} \) (or \( \mathcal{O}_{10} \)) is a point fixed by one of the 20 barred (or unbarred) cyclic subgroups of order three in \( \mathcal{V} \). (In \( A_6 \), the barred-unbarred splitting manifests itself in the two struc-
Fact 2.4. Across systems the intersection of $\mathcal{C}_\alpha$ with the $\mathcal{C}_\delta$ gives six pairs of antipodal icosahedral vertices \( \{p_{\sigma\delta}, p_{\bar{\sigma}\delta}\} \).

These total to \(72 = 6 \times 12\) points each of which is fixed under one of 36 order-five cyclic groups \( \langle p_{\sigma\delta}\rangle \). Since an icosahedral group is transitive on its vertices — indeed, some element $J_0$ of order two exchanges $p_{\sigma\delta}$ and $p_{\bar{\sigma}\delta}$, the 72 points

\[ \mathcal{O}_{72} = \{ p_{\sigma\delta} | \alpha, b = 1, \ldots, 6 \} \cup \{ p_{\bar{\sigma}\delta} | \alpha, b = 1, \ldots, 6 \} \]

form a $V$-orbit. A Valentiner exchange of $p_{\sigma\delta}$ and $p_{\bar{\sigma}\delta}$ also transposes the lines $\mathcal{L}_{\sigma\delta}$ and $\mathcal{L}_{\bar{\sigma}\delta}$ tangent to $\mathcal{C}_\alpha$ and $\mathcal{C}_\delta$ at $p_{\sigma\delta}$ and $p_{\bar{\sigma}\delta}$. (The labeling of these lines in the subscript is arbitrary and is done so as to agree with the natural cases in which a point does not reside on its associated line.) Hence, the intersection of these lines belongs to a 36-point orbit, as shown in Figure 3. We will call them “36-points”; in general, we will refer to special points and lines in terms of the size of their orbits. Each 36-point $p_{\sigma\delta}$ corresponds to the 36-line $\mathcal{L}_{\sigma\delta} = \{ L_{\sigma\delta} = 0 \}$ passing through $p_{\sigma\delta}$, and $p_{\bar{\sigma}\delta}$. Furthermore, a dihedral $\mathcal{D}_5$ stabilizes the triangle \( \{p_{\sigma\delta}, p_{\bar{\sigma}\delta}, p_{\delta\delta}\} \):

\[ \mathcal{D}_{\sigma\delta} = \text{Stab} \{ p_{\sigma\delta} \} = \text{Stab} \{ p_{\sigma\delta}, p_{\bar{\sigma}\delta} \} = \langle p_{\sigma\delta}, Z_{\sigma\delta} \rangle \simeq \mathcal{D}_5. \]

An explanation of the indices attached to the element $Z_{\sigma\delta}$ of order two occurs below.

As for other special orbits, each of the 45 involutions $Z$ in $V$ is conjugate to

\[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \]

In $\mathbb{CP}^2$ each such $Z$ fixes a point $p_Z$ and point-wise fixes a line $\mathcal{L}_Z$. Furthermore, $Z$ is the square of an element $Q$ of order four. The three fixed points of $Q$ consist of $p_Z$ and two points $(p_Q)_1$ and $(p_Q)_2$ on $\mathcal{L}_Z$. Each of the latter points have a $\mathbb{Z}/4$ stabilizer and so belong to a 90-point orbit $\mathcal{O}_{90}$. The points $p_Z$, having $\mathcal{D}_4$ stabilizers, give an orbit $\mathcal{O}_{45}$. Since $Z$ acts trivially on $\mathcal{L}_Z$, the $\mathcal{D}_4$ action restricted to $\mathcal{L}_Z$ reduces to that of a Klein-four group. Finally, the generic points on a “45-line” lie in four-point orbits and, overall, provide $V$-orbits of size 180.

The configuration of 45 lines and points. The intersections of 45-lines yield special orbits of size less than 180. Furthermore, the number of lines meeting at such a site $p$ corresponds to the number of involutions in the stabilizer of $p$. Being $\mathcal{D}_5$-stable, a 36-point lies on five of the 45-lines. Similarly, four of the 45-lines meet at a 45-point while three concur at each of the 60 and 60-points. This accounts for all intersections of the 45-lines:

\[ 36 \left( \begin{smallmatrix} 5 \\ 2 \end{smallmatrix} \right) + 45 \left( \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \right) + 60 \left( \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \right) = \left( \begin{smallmatrix} 15 \\ 2 \end{smallmatrix} \right). \]

On a 45-line there are four points of each type so that the corresponding clusters of lines complete the remaining set of 44 lines:

\[ 4 \left( \begin{smallmatrix} 4 \\ 3 \end{smallmatrix} + 3 + 2 + 2 \right) = 44. \]

\[ \begin{array}{cccc}
36\text{-points} & 45\text{-points} & 60\text{-points} & 60\text{-points}
\end{array} \]

Furthermore, only one involution fixes a 90-point. Hence, such points lie on just one of the 45-lines. Since a 72-point has a $\mathbb{Z}/5$ stabilizer, $\mathcal{O}_{72}$ acquires its exceptional status as the only special orbit that is not a subset of the 45-lines.

Valentin-speak. Given that the combinatorial relationships among special objects derive from those at the level of conics, the special orbits should admit description in a conic-based terminology. We start with the observation that, by the Valentin duality between special points and lines, whatever holds for such points also holds for the associated lines. Thus, I will usually suppress reference to one of the other.

![Figure 3](image-url)
Already evident is the natural designation of 36, 72-points in terms of one barred and one unbarred conic:

\[ \{ p_{\alpha b}, p_{\beta a} \} = C_\alpha \cap C_\beta \]

and \( p_{\alpha b} \) is the pole of \( \{ p_{\alpha b}, p_{\beta a} \} \) with respect to \( C_\alpha \) and \( C_\beta \). Turning to the 45-points, a given involution \( Z \) belongs to two tetrahedral subgroups of the barred and unbarred icosahedral systems alike. If \( Z \) preserves cones \( C_\alpha, C_\beta, C_\gamma, \) and \( C_\delta \), then it does so uniquely (see below). Accordingly, \( Z_{a\beta c\delta}, p_{a\beta c\delta}, \)

and \( L_{a\beta c\delta} \) denote the transformation, corresponding point and line respectively. Alternatively, the 45-line \( L_{a\beta c\delta} \) contains the 36-points \( p_{\alpha c}, p_{\beta d}, p_{\gamma e}, p_{\delta f} \) or by duality, the 45-point \( p_{a\beta c\delta} \) lies at the intersection of the 36-lines \( L_{\alpha c}, L_{\beta d}, L_{\gamma e}, L_{\delta f} \). Furthermore, \( L_{a\beta c\delta} \) is an icosahedral axis of order two for \( C_\alpha, C_\beta, C_\gamma, \) and \( C_\delta \), while the intersections

\[ L_{a\beta c\delta} \cap C_\alpha, L_{a\beta c\delta} \cap C_\beta, L_{a\beta c\delta} \cap C_\gamma, L_{a\beta c\delta} \cap C_\delta \]

correspond to the antipodal pairs of edge midpoints. Of course, the labels for the pair of 90-points—each of which \( Z_{a\beta c\delta} \) fixes—should be \( p_{a\beta c\delta} \) and \( p_{a\beta c\delta} \).

<table>
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**FIGURE 4.** Combinatorial scheme for the Valentiner group.

Now, given a 45-line \( L_{a\beta c\delta} \), which four of the 45-points belong to it? Since there are \( 15 = \binom{15}{5} \) choices each for the prefix \( ab \) and suffix \( cd \), a \( 15 \times 15 \) array with 45 distinguished entries depicts the configuration of 45-points and lines: see Figure 4. Being \( \mathcal{V} \)-equivalent, the rows and columns each contain three marked entries. Associated with each of the 15 barred rows \( ab \) and its triple of 45-things indexed by \( \{ \text{abcd}, \text{abef}, \text{abgh} \} \), we have

\[ \{ c, d, e, f, g, h \} = \{ 1, \ldots, 6 \}, \]

is a tetrahedral group

\[ \mathcal{T}_{ab} = J_a \cap J_b \]

whose three involutions are \( \text{Z}_{\text{abcd}}, \text{Z}_{\text{abef}}, \text{Z}_{\text{abgh}} \). The analogous state of affairs obtains for the unbarred columns where the tetrahedral group

\[ \mathcal{T}_{cd} = J_c \cap J_d \]

contains involutions \( \text{Z}_{\text{abcd}}, \text{Z}_{\text{cd}} \), \( \text{Z}_{\text{bcd}} \). Each of these 15 tetrahedral groups extends to an octahedral subgroup of \( \mathcal{V} \),

\[ \mathcal{O}_{ab} = \text{Stab} \{ \text{p}_{\text{abcd}}, \text{p}_{\text{abef}}, \text{p}_{\text{abgh}} \}, \]

\[ \mathcal{O}_{cd} = \text{Stab} \{ \text{p}_{\text{abcd}}, \text{p}_{\text{bcd}} \}. \]

Hence, the stabilizer of \( \text{p}_{\text{abcd}} \) is the intersection of octahedral groups

\[ \mathcal{O}_{ab} \cap \mathcal{O}_{cd} = \text{Stab} \{ \text{p}_{\text{abcd}} \} = \text{Stab} \{ L_{\text{abcd}} \} \simeq \mathbb{D}_4. \]

Furthermore, the involution \( \text{Z}_{\text{abcd}} \) associates canonically with the pair of barred and unbarred tetrahedral groups

\[ \mathcal{T}_{ab} \cap \mathcal{T}_{cd} = \langle \text{Z}_{\text{abcd}} \rangle \simeq \mathbb{Z}/2. \]

This 45-array, a graphical version of which appears in [Wiman 1895, p. 542], encodes a wealth of combinatorial geometry including an answer to the query of the preceding paragraph. At a 45-point \( p_{a\beta c\delta} \) there are four concurrent 45-lines whose references have the form \( L_{\alpha b = 1}, \text{L}_{\text{abgh}}, \text{L}_{\text{abcd}}, \) and \( \text{L}_{\text{abcd}} \). To find these lines read along the \( ab \) row and the \( cd \) column. By way of example,

\[ p_{\text{T234}} \in \mathcal{L}_{\text{23}, \text{12}} \cap \mathcal{L}_{\text{23}, \text{65}} \cap \mathcal{L}_{\text{35}, \text{34}} \cap \mathcal{L}_{\text{56}, \text{34}}. \]

Duality gives

\[ \{ p_{\text{T234}}, p_{\text{T26}, \text{P33}, \text{P53}}, p_{\text{T53}}, p_{\text{T43}} \} \subset \mathcal{L}_{\text{T234}}. \]

The \( \mathbb{D}_5 \) stabilizer of a 36-point \( p_{ab} \) contains five involutions whose indices have a prefix \( a \) and a suffix \( b \). For \( p_{35} \) this gives \( 1335, 2315, 3445, 3556, 5625 \). Hence,

\[ p_{35} \in \mathcal{L}_{\text{13}, \text{35}} \cap \mathcal{L}_{\text{23}, \text{15}} \cap \mathcal{L}_{\text{34}, \text{45}} \cap \mathcal{L}_{\text{55}, \text{56}} \cap \mathcal{L}_{\text{56}, \text{25}} \]

and

\[ \{ p_{\text{13}, \text{35}}, p_{\text{23}, \text{15}}, p_{\text{34}, \text{45}}, p_{\text{55}, \text{56}}, p_{\text{56}, \text{25}} \} \subset \mathcal{L}_{\text{35}}. \]
The array also supplies a connection to \( A_6 \). For instance, the involution \( Z_{1234} \) fixes \( L_{1234} \) pointwise while preserving \( L_{1212}, L_{1326}, L_{3434}, \) and \( L_{5534} \) as sets. Accordingly, it permutes the conics by

\[
C_3 \leftrightarrow C_6, \quad C_1 \leftrightarrow C_5, \quad C_1 \leftrightarrow C_2, \quad C_5 \leftrightarrow C_6.
\]

Finally, which three involutions fix a \( 60, \) 60-point \( p \)? (The stabilizer of \( p \) is a \( D_3 \).) Unlike the other special orbits, \( O_{30} \) and \( O_{60} \) have a bias toward one or the other system of conics. Recalling that \( p \) is an icosahedral face-center for two conics, say \( C_3 \) and \( C_6 \), an involution \( Z \) that fixes \( p \) cannot preserve the two conics individually; such an action would have order three. Hence, \( Z \) exchanges the conics and lacks a prefix \( ab \). For \( ab = 25 \) the array indicates six such involutions:

\[
\begin{align*}
Z_{134} : & \ (25)(13), & \ Z_{314} : & \ (25)(13), \\
Z_{625} : & \ (25)(14), & \ Z_{215} : & \ (25)(36), \\
Z_{336} : & \ (25)(36), & \ Z_{336} : & \ (25)(34).
\end{align*}
\]

The six associated lines pass through the four points in \( C_2 \cap C_5 \) as edges of the tetrahedron whose stabilizer is \( T_7 \). They naturally fall into four sets of three lines; among these triples a given index in \( \{ \bar{1}, \bar{3}, \bar{4}, \bar{6} \} \) appears twice. The intersection of the three lines occurs at the three-fold 60-points (see Figure 5), thereby suggesting appropriate names:

\[
\begin{align*}
L_{134} \cap L_{125} \cap L_{136} = & \ \{ p_{1345} \} \\
L_{134} \cap L_{336} \cap L_{525} = & \ \{ p_{3145} \} \\
L_{125} \cap L_{336} \cap L_{514} = & \ \{ p_{4135} \} \\
L_{336} \cap L_{525} \cap L_{515} = & \ \{ p_{6134} \}.
\end{align*}
\]

Similarly, I will call the 60-points \( p_{a+bcd} \).

To finish off the description of the special Valentin points on a 45-line: Which four of the \( 60, \) 60-points lie on \( L_{ab} \)? Since \( L_{ab} \) contains 60-points whose indices satisfy \( c*dst \) and \( d*axy \), the matter comes down to finding values of \( s, t, x, y \) that are “Valentin consistent.” This means that they fill out the scheme

\[
\begin{align*}
p_{c*dst} & \in L_{c*d} \cap L_{e*c} \cap L_{d*d}, \\
p_{c*dev} & \in L_{c*de} \cap L_{c*ev} \cap L_{d*ev}, \\
p_{d*axy} & \in L_{d*ax} \cap L_{d*ay} \cap L_{d*gy}, \\
p_{d*xzw} & \in L_{d*xz} \cap L_{d*zw} \cap L_{d*wy},
\end{align*}
\]

FIGURE 5. Intersecting conics within a system: a tetrahedral configuration.

where each triple of prefixes exhausts \( \{ \bar{1}, \bar{3}, \bar{4}, \bar{6} \} \) and \( \{ s, t, u, v \} = \{ x, y, z, w \} = \{ 1, \ldots, 6 \} \setminus \{ c, d \} \).

For \( L_{1234} \),

\[
\begin{align*}
p_{3456} & \in L_{1234} \cap L_{5345} \cap L_{536} \\
p_{34125} & \in L_{1234} \cap L_{53145} \cap L_{536} \\
p_{41356} & \in L_{1234} \cap L_{5345} \cap L_{5346} \\
p_{41235} & \in L_{1234} \cap L_{53245} \cap L_{5346} \\
p_{12356} & \in L_{1234} \cap L_{53145} \cap L_{5315} \\
p_{13456} & \in L_{1234} \cap L_{53145} \cap L_{5346} \\
p_{52136} & \in L_{1234} \cap L_{53145} \cap L_{5315} \\
p_{52135} & \in L_{1234} \cap L_{53145} \cap L_{5336}
\end{align*}
\]

and

\[
\begin{align*}
p_{1256} & \in L_{1234} \cap L_{5316} \cap L_{5315} \\
p_{12315} & \in L_{1234} \cap L_{53125} \cap L_{5315} \\
p_{21356} & \in L_{1234} \cap L_{53215} \cap L_{5326} \\
p_{21235} & \in L_{1234} \cap L_{53215} \cap L_{5325}.
\end{align*}
\]

Collected below are the various data for \( L_{1234} \):

<table>
<thead>
<tr>
<th>Orbit</th>
<th>Special points on ( L_{1234} )</th>
<th>Multiplicity of 45-lines</th>
</tr>
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<td>( O_{36} )</td>
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<td>( \binom{5}{2} = 10 )</td>
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<td>( O_{60} )</td>
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<td>( \binom{3}{2} = 3 )</td>
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For subsequent reference, we summarize the geometric terminology.

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<th>orbit stabilizer</th>
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<td>$p_{\alpha \beta}/\mathcal{L}_{\alpha \beta}$</td>
<td>$O_{36}$ $D_5$</td>
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<tr>
<td>${p_{\alpha \beta}, p_{\alpha \beta}}/{\mathcal{L}<em>{\alpha \beta}, \mathcal{L}</em>{\alpha \beta}}$</td>
<td>$O_{72}$ $Z/5$</td>
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<tr>
<td>$p_{\alpha \beta \gamma \delta}/\mathcal{L}_{\alpha \beta \gamma \delta}$</td>
<td>$O_{45}$ $D_4$</td>
</tr>
<tr>
<td>${p_{\alpha \beta \gamma \delta}, p_{\alpha \beta \gamma \delta}}/{\mathcal{L}<em>{\alpha \beta \gamma \delta}, \mathcal{L}</em>{\alpha \beta \gamma \delta}}$</td>
<td>$O_{90}$ $Z/4$</td>
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<tr>
<td>$p_{\alpha \beta \gamma \delta \epsilon}/\mathcal{L}_{\alpha \beta \gamma \delta \epsilon}$</td>
<td>$O_{180}$ $D_3$</td>
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<tr>
<td>$p_{\alpha \beta \gamma \delta \epsilon \zeta}/\mathcal{L}_{\alpha \beta \gamma \delta \epsilon \zeta}$</td>
<td>$O_{60}$ $D_3$</td>
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</table>

**2D. Computing a diagonal bub involution**

One way to approach the matter of an antiholomorphic symmetry that exchanges systems of conics is to look for points that such a symmetry should fix. Given three such points $a, b, c$ in coordinates $y$ where

$$a = [1, 0, 0], \quad b = [0, 1, 0], \quad c = [0, 0, 1],$$

the associated bub map would have the diagonal form

$$\text{bub} [y_1, y_2, y_3] = [\alpha y_1, \beta y_2, \gamma y_3].$$

Fixing a fourth point determines appropriate values for the inhomogeneous parameters $\alpha, \beta$.

But, which points should such a map fix? Moreover, how many such anti-involutions should there be?

**A heuristic for bub-symmetry.** The basic Valentinian object involving a mixture of barred and unbarred conics is the $D_5$ structure consisting of a pair of conics $\{C_a, C_b\}$ that intersect in the pair of 72-points $\{p_{\alpha \beta}, p_{\alpha \beta}\}$. A bub map that exchanges these two conics, must preserve the set $\{p_{\alpha \beta}, p_{\alpha \beta}\}$ as well as the associated 36-point $p_{\alpha \beta}$. To put some flesh on the skeletal configuration of Figure 3, consider two icosahedra that share a five-fold axis and, about this axis, are one-tenth of a revolution away from each other; see Figure 6. The poles where the axis passes through the icosahedra correspond to the pair of five-fold points $p_{\alpha \beta, \gamma \delta}$. A reflection through the equatorial plane preserves this arrangement while exchanging the icosahedra and the poles. The icosahedra also transpose under reflection through five planes that include the polar axis. In these cases, the two poles are fixed. This model hints that for each pair $C_a$ and $C_b$, there is a distinguished bub-involution and five of a secondary nature. This makes for a total of 36 maps $\text{bub}_{\alpha \beta}$.

For the primary reflection relative to the pair $\{C_2, C_2\}$, this heuristic demands that $p_{22}$ and $p_{22}$ exchange while $p_{22}$ remains fixed. What other points should “bub$_{22}$” fix? Since five other bub maps switch $C_2$ and $C_2$, symmetry requires that $\text{bub}_{22}$ provide a secondary reflection for each barred-unbarred pair of icosahedra associated with the $22$ configuration. This being so, $\text{bub}_{22}$ fixes the corresponding poles of 72-points. Accordingly, the correspondence between the five remaining barbed and unbarred conics determines these five pairs of points. Now, the five pairs of nonpolar antipodal vertices on the $C_2$ icosahedron correspond to the points of intersection of $C_2$ with the five conics $C_1, C_3, C_4, C_5$, and $C_6$ while the nonpolar vertices on the $C_2$ icosahedron correspond to the intersections of $C_2$ with the five conics $C_1, C_3, C_4, C_5$, and $C_6$. Let these sets of five pairs correspond to vertices of two pentagons that are one-tenth of a revolution away from each other with antipodal pairs of points being $\text{bub}_{22}$ symmetric (see Figure 7 and imagine looking down, from above a pole, on the intersecting icosahedra of Figure 6). The $D_5$...
provided by the above heuristic, make a parametrized change of icosahedral to octahedral coordinates, of the form

\[
A = \begin{pmatrix}
    a p_{111}^T & b p_{112}^T & p_{111}^T
\end{pmatrix}
\]

\[
= \begin{pmatrix}
    a \begin{pmatrix}
        \frac{1 + \sqrt{5}}{2} \\
        -\sqrt{\frac{5+\sqrt{5}}{2}} i \\
        1
    \end{pmatrix}
    & b \begin{pmatrix}
        \frac{1 - \sqrt{5}}{2} \\
        \sqrt{\frac{5-\sqrt{5}}{2}} i \\
        1
    \end{pmatrix}
    & \begin{pmatrix}
        \frac{1 + \sqrt{5}}{2} \\
        0 \\
        1
    \end{pmatrix}
\end{pmatrix}.
\]

In these icosahedral coordinates \( y \), the \( \tilde{1} \) triangle is

\[
p_{111} = [1, 0, 0], \quad p_{112} = [0, 1, 0], \quad p_{111} = [0, 0, 1].
\]

The candidate \( \text{bub}_{22} \) map \( K(y) = \tilde{y} \) fixes each of these. In octahedral coordinates the \( \tilde{2} \) triangle consists of

\[
p_{221} = \left[ \frac{1 + \sqrt{5}}{2} \rho^2, \sqrt{\frac{5+\sqrt{5}}{2}} i \rho, 1 \right]
\]

\[
p_{222} = \left[ \frac{1 + \sqrt{5}}{2} \rho^2, -\sqrt{\frac{5+\sqrt{5}}{2}} i \rho, 1 \right]
\]

\[
p_{22} = \left[ \frac{1 - \sqrt{5}}{2} \rho^2, 0, 1 \right].
\]

The hope here is that, when transformed to \( y \) coordinates, some choice of \( a, b \) results in

\[
K(A^{-1} p_{221}) = p_{222},
\]

\[
K(A^{-1} p_{222}) = p_{221},
\]

\[
K(A^{-1} p_{22}) = p_{22}.
\]

Satisfying these conditions are the values

\[
a = b = \frac{\sqrt{3} - \sqrt{5} i}{8}.
\]

The change of coordinates becomes

\[
A = \begin{pmatrix}
    \frac{1 - \sqrt{5}}{4} (\frac{1 - \sqrt{5}}{4}) & \frac{1 - \sqrt{5}}{4} (\frac{3 - \sqrt{5}}{4}) & \frac{1 + \sqrt{5}}{4} \\
    \frac{\sqrt{5} - \sqrt{5}}{2 \sqrt{2}} (\frac{1 - \sqrt{5}}{4}) & \frac{\sqrt{5} - \sqrt{5}}{2 \sqrt{2}} (\frac{3 - \sqrt{5}}{4}) & 0 \\
    \frac{3 - \sqrt{5}}{2 \sqrt{2}} (\frac{3 - \sqrt{5}}{4}) & \frac{3 - \sqrt{5}}{2 \sqrt{2}} (\frac{3 - \sqrt{5}}{4}) & 1
\end{pmatrix}.
\]

As for the conic forms, they satisfy the desired condition:

\[
C_k(y) = \overline{C_k(y)}.
\]

A further change of coordinates given by a real diagonal matrix leaves \( \text{bub}_{22}(y) = \tilde{y} \) undisturbed. In \( y \) coordinates the one-point orbit \( p_{22} \) under the \( D_5 \) for \( \tilde{2} \) is

\[
[1, 1, \sqrt{6}/\tau],
\]

where as before \( \tau = \frac{1}{2}(1 + \sqrt{5}) \). For a final simplification, the additional coordinate change

\[
B = \begin{pmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & \sqrt{6}/\tau
\end{pmatrix}
\]

arranges for this point to be \([1, 1, 1]\). In these adjusted \( y \) coordinates, the \( \tilde{1} \) and \( \tilde{2} \) triangles are

\[
p_{111} = [1, 0, 0], \quad p_{221} = [3, 2 \eta^2, -\eta],
\]

\[
p_{112} = [0, 1, 0], \quad p_{222} = [3, 2 \bar{\eta}^2, -\bar{\eta}],
\]

\[
p_{111} = [0, 0, 1], \quad p_{22} = [1, 1, 1],
\]

\[
\text{FIGURE 7.} \quad \text{A heuristic for bub-symmetry. Regard each of the five pairs of antipodal vertices on the } D_5 \text{ union of conics } C_2 \text{ and } C_2 \text{ as a vertex of one of two pentagons whose arrangement corresponds to that of the remaining icosahedra. The primary bub}_{22} \text{ reflection interchanges the pentagons as well as antipodal vertices. The secondary reflections are } \{ \text{bub}_{a \nu} : a \neq 2 \} \text{ and transpose the vertices } \tilde{2} a \text{ and } \tilde{a} \text{ respectively.}
\]
where as before $\eta = \frac{1}{2}(3 + \sqrt{15} \, i)$; in these $\text{bub}_{22}$ coordinates, the normalized conic forms for $1$ and $1$ are

$$C_1(y) = \left(\frac{\eta}{\eta}\right)^2 y_1 y_2 + y_3^2$$

$$C_1(y) = \left(\frac{\eta}{\eta}\right)^2 y_1 y_2 + y_3^2.$$ 

The unwieldy expressions for the remaining forms are not recorded here.

Since $\text{bub}_{22}$ restricts to the identity on the $\mathbb{R}P^2$, given by

$$\mathbb{R}_{22} = \{[t_1, t_2, t_3] : t_k \in \mathbb{R}\},$$

symmetry provides for such a fixed set $\mathbb{R}_{a}$ for each of the $36$ maps $\text{bub}_{a}$. Figure 8 provides a geometric interpretation of these $\mathbb{R}^2$’s. One consequence of the extra symmetry is that the Molien series and $\mathbb{V}_{2,360}$ symmetric maps are, when expressed in special $\text{bub}$-coordinates, given by polynomials with real and even, in special cases, rational coefficients.

This is discussed in a later section on basic invariants; see page 221. We remark that although Wiman [1895, pp. 548-550] and Fricke [1926, pp. 286-289] mention these coordinates, they seem not to have made much use of them.

2E. Invariant Structure

For a group action on a vector space the Molien series provides one of the basic tools of classical invariant theory. Given a finite group $S$ acting faithfully on $\mathbb{C}^n$, the dimension of the space $\mathbb{C}[x]_m$ of invariant homogeneous polynomials of degree $m$ appears as the coefficient of the $m$th degree term in the Molien series for $S$:

$$M(\mathbb{C}[x]^S) = \sum_{m=0}^{\infty} \left( \dim \mathbb{C}[x]^S_m \right) t^m.$$ 

In the Valentin case the space is $\mathbb{C}^3$ while the group is a 1-to-3 lift of $\mathbb{V}$ to a subgroup $\mathbb{V}_{3,360}$ of $\text{SU}_3$. As a result of the character $\langle \rho \rangle$ that appears under $\mathbb{V}$’s action on the icosahedral conic forms, this lift of $\mathbb{V}$ to a linear group has minimal order. (A lift of a projective group $\mathbb{S}$ to a linear group $\mathbb{S}'$ has minimal order if any other lift $\mathbb{H}$ of $\mathbb{S}$ satisfies $|\mathbb{H}| \geq |\mathbb{S}'|$. See [Fricke 1926, pp. 267-268] for details.) A further consequence of minimality is that the Molien series for the $\mathbb{V}_{3,360}$ gives complete information concerning the invariants of the projective group $\mathbb{V}$.

FIGURE 8. A geometric interpretation of $\text{bub}$-maps. Since $\text{bub}_{a}$ interchanges the pair of conics $\mathbb{E}_{a}$ and $\mathbb{E}_{b}$ as well as the lines $\mathbb{L}_{a}(p) = \{L_{a}(p) = 0\}$ and $\mathbb{L}_{b}(\text{bub}_{a}(p)) = \{L_{b}(p) = 0\}$, tangent to $\mathbb{E}_{a}$ at $p$ and to $\mathbb{E}_{b}$ at $\text{bub}_{a}(p)$, it also fixes the $\mathbb{R}P^2$’s worth of points $\mathbb{R}_{a} = \{\mathbb{L}_{a}(p) \cap \mathbb{L}_{b}(\text{bub}_{a}(p)) : p \in \mathbb{E}_{a}\}$.

**Proposition 2.5.** The invariants of $\mathbb{V}$ and $\mathbb{V}_{3,360}$ are in one-to-one correspondence.

**Proof.** Trivially, a $\mathbb{V}_{3,360}$-invariant gives a $\mathbb{V}$-invariant. Conversely, let $F(x)$ be a $\mathbb{V}$-invariant with

$$F(T^{-1}x) = \alpha(T) F(x)$$

for $T \in \mathbb{V}_{3,360}$. The kernel of the multiplicative character

$$\alpha : \mathbb{V}_{3,360} \rightarrow \mathbb{C} - \{0\}$$

is the normal subgroup $\text{Stab}(F) \subset \mathbb{V}_{3,360}$ that stabilizes $F$. Since the projective image $[\text{Stab}(F)] \simeq \mathbb{V}_{3,360}/\langle \rho \rangle$ is normal in the simple group $\mathbb{V}$, $[\text{Stab}(F)]$ is either trivial or $\mathbb{V}$. In the former case, $\text{Stab}(F)$ would be either trivial or $\langle \rho \rangle$ so that $\mathbb{V}_{3,360}/\text{Stab}(F)$ would be nonabelian. Thus, $[\text{Stab}(F)] = \mathbb{V}$. Since $\mathbb{V}_{3,360}$ includes no subgroup of order $360$, $\text{Stab}(F) = \mathbb{V}_{3,360}$ and $F$ is $\mathbb{V}_{3,360}$-invariant.

Also, $\mathbb{V}$ lifts 1-to-6 to a so-called unitary reflection group $\mathbb{V}_{6,360}$ generated by $45$ involutions on $\mathbb{C}^5$; see
Molien's theorem and its application to $\mathcal{V}$. By projecting $\mathbb{C}[x]_n$ onto $\mathbb{C}[x]_n^3$, one arrives at a generating function for the Molien series.

Theorem 2.6. For a finite group action $\mathcal{G}$ on $\mathbb{C}^n$,

$$M(\mathcal{G}) = \frac{1}{|\mathcal{G}|} \sum_{c \in \mathcal{G}} \frac{|C_T|}{\det(I - iT^{-1})},$$

where $C_T$ are conjugacy classes.

For the proof, see [Benson 1993, pp. 21–22].

Proposition 2.7. For the Valentin groups $\mathcal{V}_{3,300}$ and $\mathcal{V}_{6,300}$, the Molien series are given by

$$M(\mathcal{V}_{3,300}) = \frac{1 + t^{45}}{(1 - t^6)(1 - t^{12})(1 - t^{30})} = 1 + t^6 + 2t^{12} + 2t^{18} + \cdots + t^{45} + \cdots,$$

$$M(\mathcal{V}_{6,300}) = \frac{1}{(1 - t^6)(1 - t^{12})(1 - t^{30})} = 1 + t^6 + 2t^{12} + 2t^{18} + \cdots.$$

Proof. With $k = 0, 1, 2$, the matrices

$$\pm \rho I, \pm \rho^k P, \pm \rho^k Z, \pm \rho^k Q, \pm \rho^k PZ, \pm \rho^k T$$

represent distinct conjugacy classes in $\mathcal{V}_{6,300}$. For $\mathcal{V}_{3,300}$, the three matrices of each type corresponding to the $+ \rho^k$ do the job. Substitution into the formula of Molien’s theorem produces the indicated generating functions.

The basic invariants themselves. We know from the theory of complex reflection groups that there are three algebraically independent basic forms that generate the ring of $\mathcal{V}_{6,300}$-invariants. The generating function for the Molien series indicates that these occur in degrees 6, 12, and 30. Techniques of classical invariant theory provide for the computation of the forms in degrees 12 and 30 from that of degree 6. But, how does the latter arise? Although $\mathcal{V}_{6,300}$ permutes the conics, its action on the conic forms is not simple — a nontrivial character appears. However, the cubes of the forms do receive simple treatment by $\mathcal{V}_{6,300}$. Hence, summing the cubes of either system of conic forms and normalizing the coefficients yields a $\mathcal{V}_{6,300}$-invariant:

$$F(x) = \alpha \sum_{n=1}^{6} C_n(x)^3 = \alpha \sum_{n=1}^{6} C_n(x)^3$$

$$= x_1^6 + x_2^6 + x_3^6 + 3(5 - \sqrt{15}i)x_1^2x_2^2x_3^2$$

$$+ \frac{3}{4}(2\sqrt{5} - 5)\rho)(x_1^4x_2^2 + x_2^4x_3^2 + x_3^4x_1^2)$$

$$- \frac{3}{4}(2\sqrt{5} + 5)\rho^2)(x_1^4x_2^2 + x_2^4x_3^2 + x_3^4x_1^2).$$

By uniqueness, $F$ is also $\mathcal{V}_{3,300}$-invariant. Expressed and normalized in $\mathcal{b}_{23}$, coordinates,

$$F(y) = 10y_1^3y_2^2 + 9y_1^5y_3 + 9y_2^3y_3$$

$$- 45y_1^2y_2^3y_3 - 135y_1y_2y_3 + 27y_2^6.$$

The form $\Phi$ of degree 12 arises from the determinant of the Hessian $H(F)$ of $F$:

$$\Phi(y) = \alpha \phi |H(F(y))|$$

$$= 6y_1^{11}y_2^2 - 38y_1^6y_2^6 + 6y_1^{11}y_2 + 90y_1^8y_2^3y_3$$

$$+ 90y_1^3y_2y_3^3 - 90y_1^{10}y_3^2 - 468y_1^5y_2^5y_3 - 9y_2^10y_3$$

$$+ 1080y_1^7y_2^3y_3 + 3375y_1^4y_2^4y_3$$

$$- 324y_1^6y_2^3y_3^5 - 324y_1^5y_2y_3^5 - 1080y_1^3y_2^3y_3^6$$

$$+ 2916y_1^5y_2^3y_3^7 + 1215y_1^2y_2^2y_3^8$$

$$+ 4374y_1y_2y_3^9 + 729y_1y_3^{12}.$$

Similarly, the form $\Psi$ of degree 30 arises from the determinant of the “bordered Hessian” $BH(F, \Phi)$ of $F$ and $\Phi$:

$$\Psi(y) = \alpha \phi |BH(F(y), \Phi(y))|$$

$$= \alpha \phi \begin{vmatrix} H(\Phi(y)) & F_{y_1} & F_{y_2} & F_{y_3} \\ F_{y_1} & F_{y_2} & F_{y_3} & 0 \end{vmatrix}$$

$$= 3y_1^{30} + 3y_2^{30} + 57395628y_3^{30}.$$

The constants $\alpha_\phi = -1/20250$ and $\alpha_\phi = 1/24300$ remove the highest common factor among the coefficients.

Finally, the product of the 45 linear forms that correspond to the generating involutions is a relative $\mathcal{V}_{3,300}$-invariant but an absolute $\mathcal{V}_{3,300}$-invariant, and hence a projective $\mathcal{V}$-invariant. As a specific instance of a general result [Shephard and Todd 1954, pp. 278, 287]. The elements of $\mathcal{V}_{6,300}$ satisfy $\det T = \pm 1$ while those of $\mathcal{V}_{3,300}$ satisfy $\det T = 1$. 
p. 283], this degree-45 form is given by the Jacobian determinant
\[
X(y) = \alpha_X \left| J(F(y), \Phi(y), \Psi(y)) \right| 
= \alpha_X \begin{vmatrix}
F_{y_1} & F_{y_2} & F_{y_3} \\
\Phi_{y_1} & \Phi_{y_2} & \Phi_{y_3} \\
\Psi_{y_1} & \Psi_{y_2} & \Psi_{y_3}
\end{vmatrix} 
= \beta \prod \{p(y)\}
= y_1^{45} + \cdots - y_2^{45} + \cdots + 3570467226624 y_2^5 y_3^4,
\]
where \(\alpha_X = -1/4860\) and \(\beta\) is a constant. Being \(V_{3\cdot3\cdot0}\)-invariant, \(X^2\) is a polynomial in \(F, \Phi, \Psi:\)
\[
3^3 X^2 = 4 F^{13} \Phi + 80 F^{11} \Phi^2 + 816 F^9 \Phi^3 \\
+ 4376 F^7 \Phi^4 + 13084 F^5 \Phi^5 + 12312 F^3 \Phi^6 \\
+ 5616 F^2 \Phi^7 + 18 F^1 \Phi^8 + 198 F \Phi^9 \\
+ 954 F^6 \Phi^2 - 198 F^4 \Phi^3 - 5508 F^2 \Phi^4 \\
- 1944 \Phi^5 - 162 F^2 \Phi^2 - 1944 F \Phi^3 \\
- 1458 F \Phi^2 \Phi^2 + 729 \Phi^3.
\]
(2-1)

\(V\)-symmetric maps and the sextic. The system of invariants provides a foundation on which to construct mappings of \(\mathbb{C}^3\) or \(\mathbb{CP}^2\) that are symmetric or equian-variant under the action of \(V_{3\cdot3\cdot0}\) or \(V\). Algebraically, this means that the map commutes with the action. Given such a map that also possesses reliable dynamics, the sixth-degree equation has an iterative solution.

3. RATIONAL MAPS WITH VALENTINER SYMMETRY

An iterative solution to the sextic utilizes a parametrized family of dynamical systems having \(A_6\) symmetry. In practice, a given sixth-degree polynomial \(p\) with galois group \(A_6\) specifies a projective transformation
\[
S_p : \mathbb{CP}^2 \to \mathbb{CP}^2 
\]
and thereby hooks up to a rational map
\[
S_p^{-1} \circ f \circ S_p
\]
that has \(A_6\) symmetry. Accordingly, the fixed map \(f\) is the centerpiece of a sextic-solving algorithm.

3A. Finding Equivariant Maps

A linear group \(\mathbb{G}\) acts on the exterior algebra \(\Lambda(\mathbb{C}^n)\) by
\[
(T(\alpha))(x) = \alpha(T^{-1}x),
\]
where \(T \in \mathbb{G}, \alpha\) is a 1-form, and \(x \in \mathbb{C}^n\). As in the case of \(V\)-invariant polynomials, \(V\)-invariant \(p\)-forms associate one-to-one with \(V_{3\cdot3\cdot0}\)-invariant \(p\)-forms. Hence, the search for symmetric maps can take place within the regime of the linear action.

Define a map \(f\) to be relatively \(\mathbb{G}\)-equivariant if, for all \(T \in \mathbb{G}\), we have \(T f = \chi_T f \circ T\) (for an appropriate choice of \(\chi_T\). If the character \(\chi_T\) is trivial, \(f\) is absolutely equivariant. As we look for equivariants, the \(V_{3\cdot3\cdot0}\) action on \(\Lambda(\mathbb{C}^3)\) provides guidance, thanks to a correspondence between \(V_{3\cdot3\cdot0}\)-equivariants and \(V_{3\cdot3\cdot0}\)-invariant 2-forms. Set
\[
dX_2 = (dx_2 \wedge dx_3, dx_3 \wedge dx_1, dx_1 \wedge dx_2)
\]
and let \(\cdot\) signify a formal dot product.

Proposition 3.1. For a given finite action \(\mathbb{G} \subset U_3\) and a \(\mathbb{G}\)-invariant 2-form
\[
\varphi(x) = f(x) \cdot dX_2 = f_1(x) dx_2 \wedge dx_3 \\
\quad + f_2(x) dx_3 \wedge dx_1 + f_3(x) dx_1 \wedge dx_2,
\]
the map \(f = (f_1, f_2, f_3)\) is relatively \(\mathbb{G}\)-equivariant. If \(\mathbb{G} \subset SU_3\), the equivariance of \(f\) is absolute.

Proof. Given \(T \in \mathbb{G}\), set \(T = (t_{11}, t_{12}, t_{13}) \quad \text{and} \quad (t_{21}, t_{22}, t_{23}) \quad \text{and} \quad (t_{31}, t_{32}, t_{33})\)
\[
T^{-1} = |T|^{-1} \begin{pmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{pmatrix}.
\]
We wish to show that \(T f(T^{-1}x)\) is a multiple of \(f(x)\). Let \((e_1, e_2, e_3)\) be the canonical basis of \(\mathbb{C}^3\), and evaluate the forms \(T f(T^{-1}x) \cdot dX_2\) and \(f(x) \cdot dX_2\) on \((e_2, e_3)\). By definition and the invariance of \(\varphi\),
\[
(f(x) \cdot dX_2)(e_2, e_3) 
= \varphi(x)(e_2, e_3) = (T(\varphi))(x)(e_2, e_3) = \varphi(T^{-1}x)(e_2, e_3) 
= |T|^{-2} (f_1(T^{-1}(x))C_{22}C_{33} - C_{23}C_{32}) \\
+ f_2(T^{-1}(x))(C_{23}C_{13} - C_{33}C_{12}) \\
+ f_3(T^{-1}(x))(C_{12}C_{33} - C_{13}C_{22}) 
= |T|^{-3} (f_1(T^{-1}(x))t_{11} + f_2(T^{-1}(x))t_{21} \\
+ f_3(T^{-1}(x))t_{31}).
\]
(Here \(C_{22}C_{33} - C_{23}C_{32}\) comes from \(d(\sum_i C_{2i}x_i) \wedge d(\sum_i C_{3i}x_i)\) applied to \((e_2, e_3)\), and so on.) The preceding expression, apart from the factor \(|T|^{-3}\), is exactly the result of evaluating \((T f(T^{-1}x)) \cdot dX_2\).
on \((e_2, e_3)\). The same equality holds for other pairs of basis vectors, showing that \(T f(T^{-1}x) = |T|^3 f(x)\), and \(f\) is relatively equivariant. If \(T \in \text{SU}_3\), we have \(|T| = 1\) and absolute equivariance occurs.

Conversely, an absolute equivariant corresponds to a relatively invariant 2-form, with absolute invariance holding in case \(\mathcal{S} \subset \text{SU}_3\).

For invariant exterior forms, there is a 2-variable “exterior Molien series”

\[
M(\Lambda^2) = \sum_{p=0}^{n} \sum_{m=0}^{\infty} \left( \dim \Lambda_p \left( \mathbb{C}^n \right)_m \right) t^m s^p,
\]

in which the variables \(s\) and \(t\) index respectively the rank of the form and the polynomial degree, and the \(\Lambda_p(\mathbb{C}^n)_m\) are the \(\mathcal{S}\)-invariant homogeneous \(p\)-forms of degree \(m\); see [Benson 1993, p. 62] or [Smith 1995, pp. 265 ff.]. Projection of \(\Lambda_p(\mathbb{C}^n)_m\) onto \(\Lambda_p(\mathbb{C}^n)\) yields the analogue to Molien’s theorem.

**Theorem 3.2.** The exterior Molien series for a finite group action \(\mathcal{S}\) is given by the generating function

\[
M(\Lambda^2) = \frac{1}{|\mathcal{S}|} \sum_{v \in \mathcal{S}} \det \left( I + s T^{-1} \right) \det \left( I - t T^{-1} \right)
\]

where the \(C_v\) are conjugacy classes.

As in the case of invariants, this has the following consequence:

**Proposition 3.3.** For \(\mathcal{G}_{3,300}\), the exterior Molien series \(M(\Lambda^{\mathcal{G}_{3,300}})\) is given by

\[
1 + t^{45} + (t^5 + t^{11} + t^{20} + t^{29} + t^{44})s + (t + t^{19} + t^{25} + t^{27} + t^{44})s^2 + (1 + t^{45})s^3
\]

which results in

\[
= 1 + t^0 + 2t^{12} + 2t^{18} + 3t^{24} + 4t^{30} + \ldots
\]

By uniqueness, up to scalar multiplication, of the 3-form \(X\) associated with the 45 complex planes of reflection, the exterior product of “dual” forms must yield a multiple of this form.

This duality between invariant \(p\) and \((n-p)\)-forms also appears in the *ternary* icosahedral group \(\mathcal{I}_{60}\). The generating 0-forms for the full reflection group \(\mathcal{I}_{60}\) have degrees 2, 6, and 10 [Shephard and Todd 1954, p. 301, line 23 of table]. From the discussion of the Valentin conics the invariant of degree 2 is familiar, while those of degrees 6 and 10 are products of linear forms that \(\mathcal{I}_{60}\) preserves. Here the duality occurs in degree 15 which is the number of reflection-planes for the icosahedron:

\[
M(\Lambda^{\mathcal{I}_{60}}) / M(\Lambda^{\mathcal{I}_{60}}) = (1 + t^{45}) + (t + t^5 + t^6 + t^9 + t^{10} + t^{14})s + (t + t^6 + t^8 + t^{10} + t^{14})s^2 + (1 + t^{15})s^3.
\]

Which finite reflection groups have series with this property? Is this duality connected to that described by Orlik and Terao [1992, p. 286]?
On \(V\)-equivariance and special orbits. Suppose \(a\) is fixed by an element \(T \in \mathcal{G}\). Since a \(\mathcal{G}\)-equivariant map \(f : \mathbb{C}P^n \rightarrow \mathbb{C}P^n\) satisfies
\[
f(a) = f(Ta) = Tf(a),
\]
\(T\) also fixes \(f(a)\). Hence, special orbits map to special orbits. For the Valentin action, the points fixed by an involution are a 45-point and its line while the 3, 4, 5-fold fixed points come in triples. Table 1 summarizes the matter. Thus, under the Valentin action must fix the orbit point-wise.

### Proposition 3.5. Up to scalar multiplication, the only \(V_{3,360}\)-invariant 2-form of degree 16 is
\[
dF(x) \wedge d\Phi(x) = (\nabla F(x) \times \nabla \Phi(x)) \cdot dX.
\]

Consequently, the unique degree 16 \(V\)-equivariant is
\[
\psi_{16}(x) = \nabla F(x) \times \nabla \Phi(x).
\]

Here \(\nabla\) is a formal gradient \(\nabla F = \left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \frac{\partial F}{\partial x_3}\right)\) and \(x\) is the cross-product. Geometrically, \(\psi\) associates a point \(y \in \mathbb{C}P^2\) with the intersection of the pair of lines
\[
\psi(y) = \{\nabla F(y) \cdot x = 0\} \cap \{\nabla \Phi(y) \cdot x = 0\}.
\]

### Dynamics of the 16-map. Given a generic point \(a\) that lies on just one 45-line \(L_{\text{abcd}}\) the lines
\[
\{\nabla F(a) \cdot x = 0\} \quad \text{and} \quad \{\nabla \Phi(a) \cdot x = 0\}
\]
pass through the point \(p_{\text{abcd}}\). Thus, \(\psi\) collapses \(L_{\text{abcd}}\) to its companion point. Taking \(p_{\text{abcd}} = [1, 0, 0]\) and \(L_{\text{abcd}} = \{x_1 = 0\}\),
\[
\psi(a) = [\psi_1(a), 0, 0]
\]
where \(\psi_1(a)\) has degree 16 in the homogeneous coordinates \(a = [a_0, a_2, a_3]\). The 16 roots of \(\psi_1(a)\) correspond to the 16 points where \(L_{\text{abcd}}\) intersects the 44 remaining 45-lines. These occur at the 36, 45, 60, 60-points of which there are four each on \(L_{\text{abcd}}\). The blowing-down of the 45-lines forces the blowing-up of their intersections: To which 45-point could the intersection go?

Their collapsing behavior makes \(\psi\) critical on the 45-lines. Since the Jacobian determinant \(|J_\psi|\) has degree 3 \((16 - 1) = 45, \{X = 0\}\) is exactly the critical set. Thus, the 45-lines are superattracting. But, in approaching \(L_{\text{abcd}}\) a trajectory inevitably gets carried near the point \(p_{\text{abcd}}\) which is blowing up onto \(L_{\text{abcd}}\); this means that the \(\mathbb{C}P^1\) of directions through \(p_{\text{abcd}}\) maps, by the agency of the Jacobian transformation \(J_\psi\), to points on \(L_{\text{abcd}}\) (see below). Conversely, \(J_\psi\) associates a point on \(L_{\text{abcd}}\) with a

### Table 1. Fixed points of \(V\)

<table>
<thead>
<tr>
<th>Order</th>
<th>Number and Type</th>
<th>Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-fold</td>
<td>1 45-point, 1 45-line</td>
<td>({p_{\text{abcd}} L_{\text{abcd}}})</td>
</tr>
<tr>
<td>3-fold</td>
<td>3 60-points</td>
<td>({p_{\text{def}}, p_{\text{hde}}, p_{\text{cde}}})</td>
</tr>
<tr>
<td>3-fold</td>
<td>3 90-points</td>
<td>({p_{\text{def}} r, p_{\text{def}} r, p_{\text{def}} r})</td>
</tr>
<tr>
<td>4-fold</td>
<td>1 45-point, 2 90-points</td>
<td>({p_{\text{abcd}}, p_{\text{def}}, p_{\text{ghk}}})</td>
</tr>
<tr>
<td>5-fold</td>
<td>1 36-point, 2 72-points</td>
<td>({p_{abc}, p_{abd}, p_{abc}})</td>
</tr>
</tbody>
</table>

### Section 3C. The Degree 16 Map

Returning to the exterior Molien series for \(V_{3,360}\), the coefficient of \(t^m s^2\) gives the dimension of the space of degree \(m\) equivariants. The series in \(t\) begins
\[
t + t^7 + 2t^{13} + t^{16} + \cdots.
\]
The first term \(t\) is due to the identity map, while \(t^7\) occurs through promotion of the identity to the degree-7 map \(F\). (Note that although the identity map is always absolutely equivariant, its 2-form counterpart \(x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2\) will not be absolutely invariant if the group is not in \(SU_3\).) Two dimensions worth of invariants in degree 12 account for the \(2t^{13}\) term. The occurrence in degree 16 of the first nontrivial equivariant finds explanation in exterior algebra. Since exterior differentiation and multiplication preserve invariance, the 2-form \(dF \wedge d\Phi\) is invariant, and hence corresponds to an equivariant map whose coordinate functions are given by the coefficients of the 2-form basis
\[
\{dx_2 \wedge dx_3, dx_3 \wedge dx_1, dx_1 \wedge dx_2\}.
\]
direction through \( p_{abcd} \). The next two iterations of \( \psi \) first return the trajectory to the vicinity of \( p_{abcd} \) and then send it back to \( \mathcal{L}_{abcd} \) somewhere.

Observation of trajectories that start near \( p_{abcd} \) or \( \mathcal{L}_{abcd} \) reveals a rapid attraction on every other iteration. What about the 3 and 5-fold intersections of 45-lines? Are they attracting? While these points do indeed blow-up, their “image” under \( \psi \) is not a curve that blows-down to the point; such is the sole propriety of the 45-points. Hence, \( \psi \) draws a generic point near a 60, 60, 36-point into one of three or five point-line cycles.

This “every-other” dynamics poses the following problem: which every-other iterate do we watch? For some initial positions, the even iterates converge to a 45-point while the trajectory spends the odd times around the corresponding 45-line. For others, the process is reversed. Moreover, experiment indicates that the dynamics at the 45-line eventually settles down; trajectories end up at one of the 45-points on the line. Hence, the iteration outputs a pair of 45-points each of which lies on the line of the other. But, there are, for each 45-point, four possible pairs of this sort so that taking every other iterate amounts to a neglect of information.

The dynamics of \( \psi \) appears to come down to what takes place on the critical 45-lines. Given a point \( x \) on a 45-line \( \mathcal{L}_{abcd} \), the derivative \( J_\psi \) associates with \( x \) a \( \mathbb{CP}^1 \) through \( p_{abcd} \), namely, the image \( \mathcal{L}_\psi (x) \) of \( J_\psi (x) \). In turn, \( J_\psi (p_{abcd}) \) sends the line \( \mathcal{L}_\psi (x) \) to the point \( (J_\psi (p_{abcd}) \mathcal{L}_\psi (x)) \) on \( \mathcal{L}_{abcd} \). The degree-15 map

\[
x \to (J_\psi (p_{abcd})) \mathcal{L}_\psi (x)
\]

gives \( \psi \) on \( \mathcal{L}_{abcd} \).

Figure 9 on page 237 shows a basins-of-attraction plot illustrating the dynamics on a 45-line. The union of the patches in a given color indicates a basin of attraction for one of the four attracting 45-points. The black dots contain points that the 45-points might not attract. Does this set have interior or positive measure? (For more details about the figure, see the Appendix.)

On a 45-line the map is not critically finite (see [Fornæss and Sibony 1994, pp. 223ff.] for this concept); in particular, its critical points are not periodic. Indeed, there might be a wandering critical point there, as suggested by Curt McMullen (private communication). Hence, establishing convergence almost everywhere would be difficult should the map even possess this property. Moreover, this behavior hardly reveals the geometric elegance whose prospective discovery motivates the present enterprise.

Unlike the one-dimensional case of the icosahedral group, for which the nontrivial equivariant of lowest degree provides an elegant dynamical system [Doyle and McMullen 1989, pp. 152–153] for the purposes of solving the quintic, the higher-dimensional Valentiner action fails to bear similar fruit. The failure occurs in spite of the 16-map’s being obtained by a procedure analogous to that employed by Doyle and McMullen in producing the degree-11 icosahedral map:

### 1 dimension

\[
F: \text{3-invariant of degree 12}
\]
\[
f_{11} = \begin{vmatrix}
\hat{x} & \hat{y} \\
F_x & F_y
\end{vmatrix}
\]

### 2 dimensions

\[
F, \Phi: \text{7-invariants of degrees 6, 12}
\]
\[
\psi_{16} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
F_x & F_y & F_z \\
\Phi_x & \Phi_y & \Phi_z
\end{vmatrix}
\]

where \( \hat{x}, \hat{y}, \hat{z} \) represent unit coordinate vectors. But in one dimension all 3-symmetric maps arise as combinations of three others each of which are constructed in the manner of \( f_{11} \) but with different basic invariants standing in for \( F \). A richer stock of equivariants inhabits the Valentiner waters. In contrast, the next higher degree offers promise as well as a bit of mystery.

### 3D. A Family of 19-Maps

The Molien series for Valentiner equivariants

\[
t + t^7 + 2t^{13} + t^{16} + 3t^{19} \ldots
\]

specifies three dimensions worth of maps in degree 19 of which two are due to promotion of the identity by degree-18 invariants. Hence, there are, as the exterior Molien quotient (Section 3B) indicates, nontrivial \( \mathcal{V} \)-symmetric maps in degree 19. How do these arise? Since there is no apparent exterior algebraic means of producing such a map, the more practical matter of computing them takes priority. (Peter Doyle and Anne Shepler appear to have found
in the sense that the resulting linear conditions tends to the remaining. Forcing den t, contrary to the theory of complex re/connections Since dF computation is straigh tforw ard/. In at least one case/, there is a nonzero in v arian ts groups /Shephard and T o dd /1954/, p/. /282/]/. These dimensions of degree in v arian ts to promote (b) dimensions of degree 24 in v arian ts to promote f40 = \nabla \Phi \times \nabla \Psi.

Proposition 3.6. These 14 maps span the space of degree 64 equivariants.

Proof. If not, then for some \( \alpha_k \) not all 0,

\[
(\alpha_1 F^8 + \cdots + \alpha_7 F \Phi \Psi) dF \wedge d\Phi \\
+ (\alpha_8 F^5 + \cdots + \alpha_{11} \Psi) dF \wedge d\Psi \\
+ (\alpha_2 F^4 + \alpha_3 F^2 \Phi + \alpha_4 \Phi^2) dF \wedge d\Psi = 0.
\]

Since \( dF \wedge d\Phi \wedge d\Psi = \beta X \text{vol} \) for some constant \( \beta \neq 0 \), at least one of the following equalities hold:

\[
(\alpha_1 F^8 + \cdots + \alpha_7 F \Phi \Psi) \beta X = 0,
\]

\[
(\alpha_8 F^5 + \cdots + \alpha_{11} \Psi) \beta X = 0,
\]

\[
(\alpha_2 F^4 + \alpha_3 F^2 \Phi + \alpha_4 \Phi^2) \beta X = 0.
\]

In at least one case, there is a nonzero \( \alpha_k \). Then the invariants \( F, \Phi, \Psi \) are not algebraically independent, contrary to the theory of complex reflection groups [Shephard and Todd 1954, p. 282]. \( \square \)

In this event, \( Xf \) is a combination of maps whose computation is straightforward.

Reasoning in the other direction, a 64-map

\[
f_{64} = F_{48} \psi_{16} + F_{30} \varphi_{34} + F_{24} f_{40}
\]

that "vanishes" on the 45-lines—i.e., each coordinate function of \( f_{64} \) vanishes—must have a factor of \( X \). The quotient is a degree 19 equivariant

\[
f_{19} = \frac{f_{64}}{X_{45}}.
\]

Arranging for the vanishing of \( f_{64} \) on the 45-lines requires consideration of only one line; symmetry tends to the remaining 44. Forcing \( f_{64} \) to vanish at 12 independent points on a 45-line—inde pent in the sense that the 12 resulting linear conditions in the 14 undetermined coefficients of \( f_{64} \) are independent—we obtain a two-parameter family of 64-maps each member of which vanishes on \( \{ X = 0 \} \). The two inhomogeneous parameters reflect the three dimensions (i.e., homogeneous parameters) of degree-19 \( \mathcal{V} \)-equivariants. In \( \mathfrak{b}_{22} \) coordinates, setting these two parameters equal to 0 and normalizing the coefficients yields

\[
f_{64}(y) = (10 F(y)^6 \Phi(y) + 100 F(y)^4 \Phi(y)^2 \\
+ 45 F(y)^2 \Phi(y)^3 + 156 \Phi(y)^4 \\
+ 39 F(y)^3 \Psi(y) + 51 F(y) \Phi(y) \Psi(y) \psi(y) \\
- 27 \Phi(y) \varphi(y) + 54 \Phi(y)^2 f(y). \tag{3-1}
\]

The two-parameter family of nontrivial 19-maps is then

\[
g_{10}(y; a, b) = f_{10}(y) + (a F(y)^3 + b F(y) \Phi(y)) y. \tag{3-2}
\]

Are any of these maps dynamically “special”? Indeed, what might it mean to be special in this sense?

Extended symmetry in degree 19. Since \( f_{19} \) is a nontrivial \( \mathcal{V}_{2300} \)-equivariant—note the integer coefficients in (3-1), each member of the two real parameter family

\[
f_{19} + F(a B_{12} + a U_{12}) id
\]

is impartial towards the two systems of conics and so, enjoys the additional symmetry. Here \( B_{12} = \prod C_7 \) and \( U_{12} = \prod C_5 \) are the degree-12 invariants given by the product of the respective six conic forms. To honor the doubled symmetry a member of this family must preserve each \( \mathcal{R}_{45} \) that \( \mathfrak{b}_{45} \) fixes point-wise.

3E. The 19-Map

The icosahedron again. An intriguing aspect of the 19-maps is the degree itself. Since 19 is one of the special equivariant numbers [Doyle and McMullen 1989, p. 166] for the binary icosahedral group, there arises the prospect of finding a \( \mathcal{V} \)-equivariant that restricts to self-mappings of the conics. By symmetry, an equivariant that fixes (setwise) a conic of a given system also fixes the other five. Might there be a map that preserves each of the 12 conics?

From [Doyle and McMullen 1989, p. 163] comes a geometric description of the canonical degree-19 icosahedral mapping of the round Riemann sphere:
stretches each face $F$ over the 19 faces in the complement of the face antipodal to $F$ while making a half-turn in order to place the three vertices and edges of $F$ on the three antipodal vertices and edges. By symmetry, the 20 face-centers are fixed and repelling. Since the resulting map is critical only at the 12 period-2 vertices, it has reliable dynamics [Doyle and McMullen 1989, p. 156]. A conic-fixing $V$-equivariant of degree 19 would, when restricted to a conic $C_2$ or $C_3$, give the unique map in degree 19 with binary icosahedral symmetry. This would determine its effect on the special icosahedral orbits:

1. fix the face-centers: $\tilde{60}$, 60-points of the appropriate system;
2. exchange antipodal vertices: pairs of 72-points;
3. exchange antipodal midpoints of edges: intersections of conics $C_2$ and $C_3$ with a 45-line indexed by $\tilde{a}$ and $b$.

General $V$-equivariants satisfy condition (1). (We say “general” because, while some maps can be set to blow-up at the 60-points, such a circumstance is rare.) Since

$$\mathcal{O}_{22} = \{F = 0\} \cap \{\Phi = 0\},$$

the image of a 72-point under each $g_{19}$ in (3–2) is the same as that under $f_{19}$. Propitiously, $f_{19}$ exchanges pairs of 72-points. Finally, a 19-map cannot blow-down a 45-line; this would make it critical there — a condition precluded by the critical set’s being an invariant of degree $54 = 3(19 - 1)$. Consequently, each 45-line must map to itself so that arranging for 3) costs one parameter for each system of conics. Fortunately, there are two parameters to spend and their expenditure purchases a canonical $V$-equivariant $h_{19}$ that maps each of the 12 conics onto itself. See below for an explicit expression.

**Fact 3.7.** There is a unique degree-19 $V$-equivariant $h_{19}$ that preserves each of the 12 icosahedral conics.

By favoring neither system of conics, $h_{19}$ possesses bub-symmetry and so, self-maps each of the $\mathcal{R}_{19}$. Expressing the family of 19-maps by

$$g_{19} = h_{19} + F(aB_{12} + bU_{12})\text{id}$$

makes evident the one-parameter collections that fix the barred ($b = 0$) and the unbarred ($a = 0$) conics.

Unlike the 16-map $\psi_{16}$, the 19-map $h_{19}$ does not blow up somewhere.

**Proposition 3.8.** The conic-preserving map $h_{19}$ is holomorphic on $\mathbb{CP}^2$.

**Proof.** By equivariance, the set of points on which $h_{19}$ blows up is empty or a union of $V$-orbits. Direct calculation shows that

$$h_{19}(p) \neq 0$$

for

$$p \in \mathcal{O}_{36} \cup \mathcal{O}_{45} \cup \mathcal{O}_{60} \cup \mathcal{O}_{90} \cup \mathcal{O}_{300}.$$

The remaining possibilities are that $h_{19} = 0$ on a 180 or 360 point orbit.

First, take the case of a 180 point orbit and recall that each such point belongs to one 45-line. Also, let $h_{19} = [h_1, h_2, h_3]$. Since $h_{19}$ preserves each 45-line $L$, the only way that $h_1 = h_2 = h_3 = 0$ is for the coordinates of the restriction $h_{19}|_L$ to have a common factor. In particular, we have $L_{12} = \{y_1 - y_2 = 0\}$, so that

$$h_{19}|_{L_{12}} = [f, f, g].$$

But the resultant of $f$ and $g$ does not vanish. Hence, $f$ and $g$ do not have a common factor.

Finally, suppose that $h_{19} = 0$ at a 360 point orbit and that $[0, 0, 1]$ is a 36-point $p_{36}$. Since

$$|\{h_1 = 0\} \cap \{h_2 = 0\} \cap \{h_3 = 0\}| \leq 19 \cdot 19 = 361,$$

there is only one member of $h_{19}^{-1}(p_{36})$ in $\mathbb{CP}^2$. Of course, this holds for every 36-point of which there are four on a 45-line $L$. Moreover, $h_{19}(p_{36}) = p_{36}$. Thus, the one-dimensional rational map $h_{19}|_L$ has four exceptional points — a state of affairs that requires the map to be of degree one [Beardon 1991, p. 52]. Since the restricted map is not of degree one, $h_{19}$ does not vanish at a 360 point orbit. (Indeed, no degree 19 equivariant can blow up a 360 point orbit.)

**Fact 3.9.** The conic-fixing equivariant has the bubble given at the top of the next page.

**Dynamical behavior.** The discovery of $h_{19}$ supplies the unique degree-19 $V$-equivariant that self-maps, in addition to the 45-lines and the 36 bubble $\mathbb{R}P^2$s, the 12 conics. The dynamics on each conic is well-understood. A plot showing the basins of attraction appears in Figure 10 on page 327.
Proposition 3.10. Under \( h_{19} \), the trajectory of almost any point on an icosahedral conic tends to an antipodal pair of the superattracting vertices.

Moreover, the conics themselves are attracting.

Proposition 3.11. The Jacobian \( J_{h_{19}} \) has rank one at the superattracting 72-points. Thus, \( h_{19} \) attracts on a full \( \mathbb{C}P^2 \) neighborhood of such a point. Furthermore, the Fatou components of the restricted map \( h_{19}|_e \) are the intersections with \( \mathbb{C}_3 \) of Fatou components of the map on \( \mathbb{C}P^2 \).

Is this attracting behavior of the conics pervasive in the measure-theoretic sense? What about the restricted dynamics on the 45-lines and \( \mathbb{R}P^2 \)?

Perhaps the place to begin is at a 72-point, say \( p_{11} \), which lies at the hub of much Valentinian activity. Passing through \( p_{11} \) are many special objects:

1. the pair \( \{ C_1, C_2 \} \) of conics, which meet tangentially;

2. the 36-line \( L_{11} \), which gives a 10-fold \( D_5 \) axis about which \( C_1 \cup C_2 \) turns;

3. the 72-line \( L_{11} \), which is stable under the cyclic half of the \( D_5 \) stabilizer of \( L_{11} \) and thereby tangent to \( C_1 \) and \( C_2 \);

4. the sixth-degree curve \( \{ F = 0 \} \), which is tangent to \( L_{11} \);

5. the twelfth-degree curve \( \{ \Phi = 0 \} \), which is tangent to \( L_{11} \);

6. the five \( \mathbb{R}P^2 \)'s \( \{ R_{22}, R_{34}, R_{35}, R_{36}, R_{35} \} \), each of which intersect \( C_1 \) and \( C_2 \) only at \( p_{11} \) and \( p_{12} \).

In addition, a 72-point situates itself at the intersection of two components \( \{ F_0 = 0 \} \) and \( \{ G_{48} = 0 \} \) of \( h_{19} \)’s critical set.

Fact 3.12. The Jacobian determinant \( |J_{h_{19}}| = F_0 G_{48} \) where the invariant \( G_{48} \) distinguishes itself by lacking a \( \Psi \) term when decomposed into an expression in the basic invariants \( F, \Phi, \Psi \).
\[ G_{48}(y) = -13718 \left(14F(y)^6 + 180F(y)^4 \Phi(y) 
+ 1701F(y)^4 \Phi(y)^2 + 3402F(y)^3 \Phi(y)^3 + 5103 \Phi(y)^4 \right). \]

Since \( G_{48} \) is a polynomial in \( F \) and \( \Phi \) alone, its curve satisfies
\[
\{F = 0\} \cap \{G_{48} = 0\} = \{F = 0\} \cap \{\Phi = 0\} = \emptyset_{72}.
\]

Furthermore, the special invariant structure of \( G_{48} \) has an alternative expression in terms of \( B_{12} \) and \( U_{12} \) alone. From the identities
\[
B_{12}(y) = 2(5 - \sqrt{15i})\rho F(y)^2 - 2 \sqrt{15}(1 + \sqrt{15i})\rho^2 \Phi(y),
\]
\[
U_{12}(y) = 2(5 + \sqrt{15i})\rho^2 F(y)^2 - 2 \sqrt{15}(1 - \sqrt{15i})\rho^2 \Phi(y),
\]

it follows that
\[
G_{48}(y) = 2^{3}3^{2}4^{5}5^{19}3^{3}(6 - 3 - \sqrt{15i})\rho B_{12}(y)^4 
+ 4(32 + 3\sqrt{15i})\rho^2 B_{12}(y)^3 U_{12}(y)
- 333 B_{12}(y)^2 U_{12}(y)^2 
+ 4(32 - 3\sqrt{15i})\rho B_{12}(y)U_{12}(y)^3
- 6(3 + \sqrt{15i})\rho^2 U_{12}(y)^4).
\]

Consequently, the degree-48 component of the critical set meets the conics only at the 72-points:
\[
\{G_{48} = 0\} \cap \{B_{12} = 0\} = \{G_{48} = 0\} \cap \{U_{12} = 0\} = \emptyset_{72}.
\]

Accounting for the multiplicity at these eighth-order intersections is the singularity of \( \{G_{48} = 0\} \) at \( \emptyset_{72} \), a result that follows directly from the invariant decomposition.

\( \mathbb{RP}^2 \) dynamics. On each of the five bub-\( \mathbb{RP}^2 \)s that are mutually tangent at \( p_{11} \) and \( p_{12} \), these 72-points are super-attracting for the restricted maps
\[
h_{19}|_{\mathbb{R}^{72}}, \quad \bar{a}b = 52, 54, 43, 56, 65.
\]

Are there attracting sites on \( \mathbb{R}^{6} \) other than the five pairs of 72-points? Are there sets of positive measure or open sets on which \( h_{19}|_{\mathbb{R}^{48}} \) fails to converge to a pair of 72-points? The experimental evidence strongly suggests that

1. the 72-points are the only attractors;
2. there is no region with thickness or positive measure that remains outside of their influence;
3. the set of 45-lines \( \{X = 0\} \) is repelling.

The Appendix exhibits basin plots of \( h_{19}|_{\mathbb{R}^{72}} \), which, of course, is dynamically equivalent to each \( h_{19}|_{\mathbb{R}^{48}} \). What significance does the \( \mathbb{RP}^2 \)-dynamics hold for that on \( \mathbb{CP}^2 \)? Extensive trials [Crass 1997a] in \( \mathbb{CP}^2 \) have not revealed behavior contrary to that observed on the \( \mathbb{RP}^2 \)s.

**Conjecture 3.13.** The only attracting periodic points for \( h_{19} \) are the elements of \( \emptyset_{72} \). Moreover, the union of the basins of attraction for \( \emptyset_{72} \) has full measure in \( \mathbb{CP}^2 \).

The 45-lines present a problem in that they map to themselves but do not contain the 72-points. (Again, this is a feature peculiar to the 72-points. They form the only special \( V \)-orbit that does not lie on the 45-lines.) The basin plot in Figure 11 (page 238) reveals repelling behavior along the \( \mathbb{RP}^1 \) where the \( \mathbb{RP}^2 \) meets one of the 45-lines left invariant by bub_{22}.

**Conjecture 3.14.** On the 45-lines \( h_{19} \) is repelling, and hence \( \{X = 0\} \) resides in the Julia set \( J_{h_{19}} \).

Is \( J_{h_{19}} \) the closure of the backward orbit of the 45-lines?

## 4. Solving the Sextic

By means of various algebraic manipulations, a general sixth-degree polynomial reduces to a member of a two-parameter family of Valentinier resolvents. Such a reduction requires the extraction of square and cube roots. Furthermore, a certain set of sextics transforms into a special one-parameter collection of resolvents. These resolvents are especially suited for solution by an iterative algorithm that exploits Valentinier symmetry and symmetry-breaking.

### 4A. General Sixth-Degree Valentinier Resolvents

At the core of Klein's program for equation-solving is the "form problem" relative to a particular action of a given equation's symmetry group; for prescribed values \( a_1, \ldots, a_n \) of the generating invariants \( F_1, \ldots, F_n \) find a point \( p \) common to the inverse images \( F_i^{-1}(a_1), \ldots, F_i^{-1}(a_n) \). As with the quintic, solving the general sextic is tantamount to solving the corresponding form-problem; see [Fricke 1926, pp. 308–310; Coble 1911]. This circumstance has a projectively equivalent formulation in terms of ra-
tional functions in the basic invariants. In the Valentiner setting this concerns

\[ Y_1 = \alpha \frac{\Phi}{F^2} \quad \text{and} \quad Y_2 = \beta \frac{\Psi}{F^5}, \]

where \( \alpha \) and \( \beta \) are chosen so that \( Y_1 = 1 \) and \( Y_2 = 1 \) at a 36-point. (In bub_{22} coordinates \( \alpha = 1 \) and \( \beta = \frac{1}{2} \)) Given values \( a_1 \) and \( a_2 \) of \( Y_1 \) and \( Y_2 \), the task is to find a point \( z \) in \( \mathbb{C}P^2 \) that belongs to the \( \mathcal{V} \)-orbit

\[ Y_1^{-1}(a_1) \cap Y_2^{-1}(a_2). \]

Accordingly, the general 6-parameter sextic \( p(x) \) reduces to a resolvent that depends on the two parameters \( Y_1 \) and \( Y_2 \). Such a reduction requires the extraction of a cube root [Fricke 1926, p. 285] in addition to the square root of \( p \)'s discriminant. This cube root is a so-called “accessory irrationality” — its adjunction to the coefficient field does not reduce the galois group. The 1-to-3 correspondence between the projective and linear Valentiner groups \( \mathcal{V} \) and \( \mathcal{V}_{3,360} \) accounts for its appearance. In the one-dimensional icosahedral case, the projective group lifts 1-to-2 to a linear group, thereby producing the need for an accessory square root; see [Klein 1913, pp. 172–173].

As for the derivation of a two-parameter resolvent, the map

\[ Y : \mathbb{C}P^2 \setminus \{ F(z) = 0 \} \rightarrow (\mathbb{C}P^2 \setminus \{ F(z) = 0 \})/\mathcal{V} \]

given by

\[ Y(z) = [F(z)^3 \Phi(z), \Psi(z), F(z)^5] = [Y_1(z), Y_2(z), 1] \]

provides the \( \mathcal{V} \)-quotient of \( \mathbb{C}P^2 \setminus \{ F(z) = 0 \} \) in that the fibers are \( \mathcal{V} \)-orbits. The exceptional status of the sixth-degree curve is due to its being the fiber above the single point \([0, 1, 0]\). Furthermore, under the icosahedral function

\[ U_1(z) = \frac{C_1(z)^3}{F(z)}, \]

a fiber \( Y^{-1}[a_1, a_2, 1] \) maps to six points

\[ \{U_n(z) = C_n(z)^3/F(z) : n = 1, \ldots, 6\}, \]

where \( z \in Y^{-1}[a_1, a_2, 1] \). The \( U_n(z) \) are the roots of the sixth-degree polynomial

\[ R_z(u) = \prod_{n=1}^{6} (u - U_n(z)). \]

As \( z \) varies in \( \mathbb{C}P^2 \setminus \{ F(z) = 0 \} \), \( R_z(u) \) yields a family of sextic resolvents. Since \( \mathcal{V}_{3,360} \) permutes the \( C_n(z)^3 \) simply — no multiplicative character appears, \( R_z \) is \( \mathcal{V} \)-invariant in \( z \), and hence so is each \( u \)-coefficient. Expressing the coefficients in terms of the basic invariants \( F(z), \Phi(z), \Psi(z) \) and then converting to \( Y_1 \) and \( Y_2 \) yields, in bub_{22} coordinates, the resolvents \( R_Y(u) = R_{\{Y_1, Y_2\}}(u) \). With \( \omega = \sqrt{15}i \), their expression is

\[
R_Y(u) = u^6 + \frac{5+\omega}{90} u^5 + \frac{11(1-\omega)-3(3+\omega)Y_1}{2\omega^5} u^4 \\
\quad + \frac{(100+57\omega)+9(30+\omega)Y_1}{3\omega^5} u^3 \\
\quad + \frac{-(152+17\omega)+(42+2\omega)Y_1+27(-4+\omega)Y_2}{2\omega^5} u^2 \\
\quad + \frac{(425+103\omega)+(675+183\omega)Y_1+27(25+33\omega)Y_2}{2\omega^5} u \\
\quad + \frac{-(5+3\omega)+(9(15-7\omega)Y_1+81(25-3\omega)Y_2^2+81(45+11\omega)Y_2^3}{2\omega^5}.
\]

This makes explicit the fact that the solution of \( R_Y(u) \) follows from inversion of \( Y \).

For the unbarred functions

\[ U_n(z) = \frac{C_n(z)^3}{F(z)}, \]

one obtains the associated resolvents \( S_Y \) from \( R_Y \) by complex conjugation of the \( u \)-coefficients:

\[ S_Y(u) = \overline{R_Y(\overline{u})}. \]

### 4B. Special Sixth-Degree Resolvents

For the resolvents \( R_Y \) the parameter space is an affine plane \([Y_1, Y_2, 1]\) that lifts to \( \mathbb{C}P^2 \setminus \{ F(z) = 0 \} \). There is a complementary set of resolvents parametrized by a \( \mathbb{C}P^1 \) that lifts to \{\( F(z) = 0 \)\}.

The sixth-degree \( \mathcal{V} \)-invariant curve \{\( F(z) = 0 \)\} is a genus 10 surface that contains three special Valentiner orbits:

\[ \mathcal{O}_{72} = \{ F(z) = 0 \} \cap \{ \Phi(z) = 0 \}, \]
\[ \mathcal{O}_{90} = \{ F(z) = 0 \} \cap \{ \Psi(z) = 0 \}, \]
\[ \mathcal{O}_{180} = \{ F(z) = 0 \} \cap \{ X(z) = 0 \} = \mathcal{O}_{90}. \]

Set

\[ V(z) = \alpha \frac{\Phi(z)^5}{\Psi(z)^2}, \]

constant \( \alpha \) is chosen so that \( V = 1 \) at a 180-point. (In bub_{22} coordinates, \( \alpha = \frac{3}{8} \).) On \{\( F(z) = 0 \)\} this
rational map gives the 2-4-5 quotient of \( \{ F(z) = 0 \} \) under \( V \):
\[
V: \{ F(z) = 0 \} \to \{ F(z) = 0 \} / V.
\]
Furthermore, the icosahedral function
\[
S_1(z) = \frac{\Phi(z)^2}{\Psi(z)^3} C_1(z)^3
\]
divides \( \{ F(z) = 0 \} \) by the icosahedral subgroup \( S_1 \):
\[
S_1: \{ F(z) = 0 \} \to \{ \text{icosahedral subgroup} \} / S_1.
\]
A value \( V_0 \neq 0, 1, \infty \) of \( V \) has an inverse image on \( \{ F(z) = 0 \} \) that consists of a \( V \)-orbit of size 360 while the image of \( V^{-1}(V_0) \cap \{ F(z) = 0 \} \) under \( S_1 \) is the set of six points
\[
\{ S_n(z) = \Phi(z)^2 C_n(z)^3 / \Psi(z) : n = 1, \ldots, 6 \},
\]
where \( z \in V^{-1}(V_0) \cap \{ F(z) = 0 \} \). The \( S_n(z) \) supply roots of a \( V \)-parametrized family of sextic resolvents
\[
T_z(s) = \prod_{n=1}^{6} (s - S_n(z)).
\]
As above, \( T_z \) is \( V \)-invariant in \( z \), and hence so is each \( s \)-coefficient. Expressing the coefficients in terms of the basic invariants \( F(z), \Phi(z), \) and \( \Psi(z) \), restricting to \( \{ F(z) = 0 \} \), and converting to \( V \) gives the one-parameter resolvents
\[
T_V(s) = s^6 - \frac{-3 + \sqrt{15} i}{2^{5}3^{5}5^{2}} V s^4 - \frac{4 + \sqrt{15} i}{2^{8}3^{6}5^{5}} V^2 s^2 + \frac{\sqrt{15} i}{2^{5}3^{3}5^{8}} V^3 + \frac{45 - 11 \sqrt{15} i}{2^{9}3^{5}5^{8}} V.
\]
Again, with the unbarred functions
\[
S_n(z) = \frac{\Phi(z)^2}{\Psi(z)^3} C_n(z)^3,
\]
conjugation of the coefficients of \( T_V \) yields the unbarred resolvents.

**Invariant building-blocks.** Success in finding a 19-map for arbitrary \( y \) or \( V \) (with the exception of the singular values \( Y(\{ X(z) = 0 \}) \) of \( y \) and the values 0, 1, \infty of \( V \)) requires provision only of basic invariant forms
\[
F_y, \Phi_y, \Psi_y, X_y \text{ or } F_V, \Phi_V, \Psi_V, X_V,
\]
parametrized by \( y \) and \( V \). In turn, the latter three of each type depend on the single forms \( F_y \) and \( F_V \).

Much of this development amounts to keeping track of coordinates. The Valentin actions \( V_z \) and \( V_y \) on the respective planes \( \mathbb{CP}^2 \) and \( \mathbb{CP}^2 \) are the same — the parameter \( z \) merely replaces \( y \). Think of these as a parameter and reference space respectively.

To obtain a parametrized sixth-degree form in the general case, we proceed as follows:

1. Compose \( F(z) \) with a certain family of maps \( \tau_z(w) \) each of which is \( V_z \)-equivariant and linear in \( w \).
   (The \( w \)-space \( \mathbb{CP}^2 \) is the iteration space.)
2. Express the coefficients of the \( w \) monomials in terms of \( F(z), \Phi(z), \) and \( \Psi(z) \).
3. Convert these coefficients to expressions in \( Y_1 \) and \( Y_2 \) — as in the derivation of \( R_Y \).

The special case requires more care.

1. Compose \( F(z) \) with a select family of maps \( \sigma_z(w) \) each of which is \( V_z \)-equivariant and linear in \( w \).
2. Restrict to \( \{ F(z) = 0 \} \) and express the coefficients of the \( w \) monomials in terms of \( \Phi(z), \Psi(z) \).
   (The choice of \( \sigma_z(w) \) becomes significant at this stage; see below.)
3. Divide through by any overall factors in \( \Phi \) and \( \Psi \) to obtain a polynomial whose degree in \( z \) is a multiple of 60 — the degree of \( V \) — and then express the result in terms of \( V \).

**Sixth-degree forms in the two-parameter case.** Consider the family of maps
\[
y = \tau_z(w) = (F(z)^4 z) w_1 + (F(z) h_{19}(z)) w_2 + k_{25}(z) w_3,
\]
which are of degree 25 in \( z \) and projective transformations in \( w \). Here \( k_{25}(z) \) is the equivariant whose expression in "Hermitian coordinates" is
\[
k_{25}(z) = \nabla F(N F(z)).
\]

**4C. Parametrized Families of Valentin Groups**

The algorithms that solve given resolvents \( R_V \) or \( T_V \) employ an iteration of a dynamical system \( h_V(w) \) or \( h_V(w) \) that belongs to a family of maps parametrized by \( Y = (Y_1, Y_2) \) or \( V \), and each member of which is conjugate to \( h_{19}(y) \). The first task is to parametrize by \( Y \) and \( V \) families of Valentin groups. Each such group supports a conic-fixing 19-map the computation of which follows that of its conjugate, \( h_{19}(y) \).
(We use the term “Hermitian” because each \( T \) in \( V_{3,300} \) satisfies \( T^T \overline{T} = I \).

With \( \tau_z \), one constructs a \( z \)-parametrized family of Valentiner groups \( V_w = \tau_z^{-1} V_y \tau_z \) each member of which acts on \( \mathbb{CP}^2 \). By construction, this family possesses an equivariance property: for \( T \in V_z, V_y \),

\[
\tau_Tz(w) = T\tau_z(w).
\]

Hence,

\[
F(\tau_Tz(w)) = F(\tau_z(w))
\]

so that the \( w \)-coefficients of \( F(\tau_z(w)) \) are \( V_z \)-invariant and thereby expressible in terms of the basic forms \( F(z), \Phi(z), \Psi(z) \), and \( X(z) \). However, since the degree in \( z \) of each \( w \)-coefficient is \( 6 \cdot 25 = 150 \), an odd power of \( X(z) \) cannot appear in the decomposition of these coefficients into polynomials in the basic invariants. Being \( V_{3,300} \)-invariant, \( X^2 \) decomposes into a polynomial in \( F, \Phi, \) and \( \Psi \); see equation (2-1) on page 222. Thus, each coefficient is a combination of the forms of degrees 6, 12, and 30. After division by an appropriate power of \( F(z) \) as well as a simplifying numerical factor \( \alpha \), the result is expressible in terms of \( Y_1 \) and \( Y_2 \):

\[
F_Y(w) = \frac{F(\tau_z(w))}{\alpha F(z)^{25}}
\]

(4-1)

The coefficients of the \( Y \) monomials in \( w \)-coefficient are solutions to a system of linear equations. The five-page explicit expression for this fundamental form can be found in [Crass 1997a; 1997b, pp. 64-68].

An important matter concerns the degeneration of \( \tau_z(w) \) where the determinant \( |\tau_z| \) vanishes. Taking \( z^T, h_{19}(z)^T, \) and \( k_{25}(z)^T \) to be column vectors, we can write

\[
|\tau_z| = \begin{vmatrix}
F(z)^4 z^T & F(z) h_{19}(z)^T & k_{25}(z)^T \\
F(z)^5 & z^T & h_{19}(z)^T & k_{25}(z)^T \\
-1458 F(z)^5 X(z)
\end{vmatrix}
\]

The final equality follows by uniqueness of \( X \) as a degree-45 invariant and evaluation of \( |\tau_z|, F(z), \) and \( X(z) \) at a single point. Thus, the square of \( |\tau_z| \) is expressible in \( F(z), \Phi(z), \) and \( \Psi(z) \) alone. In terms of \( Y \),

\[
|\tau_z|^2 = 1458^2 F^{10} X^2
\]

\[
= 432 F^{25} (Y_1 + 20 Y_2^2 + 204 Y_3^3 + 1094 Y_4^4 + 3271 Y_5^5 + 3078 Y_6^6 + 1404 Y_7^7 + 18 Y_2)
\]

\[
+ 198 Y_1 Y_2 + 954 Y_2^2 Y_2 - 198 Y_3^3 Y_2 - 5508 Y_1 Y_2^2 + 1944 Y_1^5 Y_2 - 648 Y_2^2 - 7776 Y_1 Y_2^2
\]

\[
- 5832 Y_3 Y_2^2 + 11664 Y_2^3).
\]

(4-2)

The special case. Take the family of maps

\[
\sigma_z(w) = (72 \Phi(z)^4 z) w_1 + (\Psi(z) h_{19}(z) w_2 + (24 \Phi(z)^2 k_{25}(z) w_3,\]

having \( z \)-degree 49 and \( w \)-degree one. The integer coefficients have been chosen so that, in \( \text{bub22-coordinates}, \) the point

\[
[w_1, w_2, w_3] = [1, 1, 1]
\]

corresponds to the map

\[
\nabla F(z) \times \nabla X(z)
\]

associated with the 2-form \( dF \wedge dX \). As in the general situation,

\[
\sigma_Tz(w) = T\sigma_z(w)
\]

so that

\[
F(\sigma_Tz(w)) = F(\sigma_z(w)) .
\]

Thus the \( w \)-coefficients are \( V_z \)-invariant and thereby expressible in terms of the basic forms \( F(z), \Phi(z), \Psi(z), \) and \( X(z) \). Since the degree in \( z \) of \( F(\sigma_z(w)) \) is \( 6 \cdot 49 = 294 \), odd powers of \( X(z) \) cannot take part in the basic invariant decomposition of these coefficients. Furthermore, restriction of the parameter space \( \mathbb{CP}^2 \) to \( \{ F(z) = 0 \} \) yields coefficients in \( \Phi(z) \) and \( \Psi(z) \) alone. Finally, since \( 294 = 22 \cdot 12 + 30 = 2 \cdot 12 + 9 \cdot 30 \), the restriction \( F(\sigma_z(w))|_{\{ F(z) = 0 \}} \) is divisible by \( \Phi(z)^2 \Psi(z) \). Hence,

\[
F(\sigma_z(w))|_{\{ F(z) = 0 \}} = \eta \Phi(z)^2 \Psi(z)^3 F_Y(w),
\]

where \( \eta \) is a simplifying numerical factor and \( F_Y(w) \) is a polynomial that is degree four in \( V \) and degree six in \( w \). The expression for \( F_Y(w) \) also appears in [Crass 1997b, pp. 68-69].

Since the parameter space gets restricted to

\[
\{ F(z) = 0 \},
\]
\[|\sigma_z| \text{ should not vanish there. In fact, 49 is the lowest degree in which this fails to occur for three projectively distinct maps. Explicitly,}
\]
\[
|\sigma_z| = \begin{vmatrix} 72\Phi^4(z)z^T \Psi(z)h_{19}(z)^T & 24\Phi^2(z)k_{25}(z)^T \end{vmatrix}
\]
\[
= 24 \cdot 72\Phi^4(z)\Psi(z)\Phi^2(z)z h_{19}(z) k_{25}(z)
\]
\[
= 2^73^3\Phi(z)^6\Psi(z)X(z).
\]

Furthermore, on \{F(z) = 0\}, the expression (2-1) for \(X^2\) reduces to
\[
X^2 = \frac{1}{81} (8\Phi^5\Psi - 3\Psi^3) = -\frac{\Psi^3}{81} (8\Phi^5 - 3)
\]
\[
= -\frac{\Psi^3}{81} (8 \frac{3V}{8} - 3) = -\frac{\Psi^3}{27} (V - 1).
\]

Consequently,
\[
|\sigma_z|^2 = -2^{14}3^{15}\Phi^{12}\Psi^5(V - 1)
\]
\[
= -2^{14}3^{15}\Phi^{2}\Psi^9\frac{\Phi^{10}}{\Psi^4}(V - 1)
\]
\[
= -2^{14}3^{15}\Phi^{2}\Psi^9\left(\frac{3V}{8}\right)^2(V - 1)
\]
\[
= -2^{8}3^{17}\Phi^{2}\Psi^9V^2(V - 1).
\]

**The remaining basic invariants.** The forms of degrees 12, 30, and 45 arise from the sixth-degree invariant as before. However, a parametrized change of coordinates requires special handling. Under \(y = Ax\), the Hessian \(H_x\), the bordered Hessian \(BH_x\), and the Jacobian \(J_x\) transform as
\[
H_x(F(y)) = A^T H_y(F(y))A,
\]
\[
BH_x(F(y), G(y)) = \begin{pmatrix} A^T & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix},
\]
\[
J_x(F(y), G(y), K(y)) = J_y(F(y), G(y), K(y))A,
\]
where the subscripts \(x, y\) indicate the differentiation variable. As for transformation of the respective determinants:
\[
|H_x(F(y))| = |A|^2 |H_y(F(y))|,
\]
\[
|BH_x(F(y), G(y))| = |A|^2 |BH_y(F(y), G(y))|,
\]
\[
|J_x(F(y), G(y), K(y))| = |A| |J_y(F(y), G(y), K(y))|.
\]
For the parametrized change of coordinates \(y = \tau_z(w)\), let
\[
\Phi_y(w) = \alpha_y [H_y(F_y(w))],
\]
\[
\Psi_y(w) = \alpha_y [BH_y(F_y(w), \Phi_y(w))],
\]
\[
X_y(w) = \alpha_x [J_y(F_y(w), \Phi_y(w), \Psi_y(w))],
\]
where the constants \(\alpha_y, \alpha_y, \alpha_x\) are those defined in Section 2E (page 221). Then we have
\[
\Phi(y) = \alpha_y [H_y(F_y(y))]
\]
\[
= \alpha_y [H_y(F_y(\tau_z(w)))]
\]
\[
= \alpha_y [\tau_z]^{-2} [H_y(\alpha F(z)^{25}F_y(w))]
\]
\[
= \frac{\alpha}{|\tau_z|^2} (\alpha F(z)^{25}3 |H_y(F_y(w))|
\]
\[
= \frac{(\alpha F(z)^{25})^3}{|\tau_z|^2} \Phi_y(w),
\]
(4-3)
\[
\Psi(y) = \alpha_y [BH_y(F_y(y), \Phi_y(y))]
\]
\[
= \alpha_y [BH_y(F(\tau_z(w)), \Phi(\tau_z(w)))]
\]
\[
= \alpha_y [\tau_z]^{-2} [BH_y(\alpha F(z)^{25}F_y(w),
\]
\[
= \frac{\alpha}{|\tau_z|^2} (\alpha F(z)^{25})^3 |H_y(F_y(w))|
\]
\[
= \frac{(\alpha F(z)^{25})^8}{|\tau_z|^6} \Psi_y(w),
\]
(4-4)
and
\[
X(y) = \alpha_x [J_y(F_y(y), \Phi_y(y), \Psi_y(y))]
\]
\[
= \frac{(\alpha F(z)^{25})^12}{|\tau_z|^12} X_y(w).
\]
(4-5)
With \(y = \sigma_z(w)\), similar calculations lead to the one-parameter forms:
\[
\Phi(y) = \frac{(\eta\Phi(z)^3 \Psi (z)^9)^3}{|\sigma_z|^2} \Phi_V(w)
\]
\[
\Psi(y) = \frac{(\eta\Phi(z)^3 \Psi (z)^9)^8}{|\sigma_z|^6} \Psi_V(w)
\]
\[
X(y) = \frac{(\eta\Phi(z)^3 \Psi (z)^9)^12}{|\sigma_z|^12} X_V(w).
\]
**The 19-maps.** With an invariant system in place for each (nonsingular) value of \(Y\) (or \(V\)), production of the degree-19 map that preserves all 12 of the conics proceeds as before:
(1) determine a 64-map \((f_{64})_Y(w)\) that vanishes at \(\{X_y(w) = 0\};

(2) compute \((f_{19})_Y(w) = (f_{64})_Y(w)/X_Y(w)\); 
(3) compute the conic-fixing map \(h_Y(w)\).

In fact, the previous calculations in the coordinates \(\{y_1, y_2, y_3\}\) provide a framework for those at hand. Intuitively, the expressions involving \(F(y), \Phi(y), \Psi(y)\), and \(X(y)\) as well as the maps \(\psi_{16}(y), \varphi_{34}(y)\), and \(f_{40}(y)\) substitute the appropriate terms in \(Y\) and \(w\), namely, \(F_Y(w), \Phi_Y(w), \Psi_Y(w), X_Y(w), \psi_Y(w), \varphi_Y(w)\), and \(f_Y(w)\). The substitutions for the invariants appear in equation (41) on page 232 and in (4-3), (4-4), and (4-5) on the preceding page. Concerning maps, they transform as the coefficients of 2-forms. In terms of the cross-product, for \(u = Ax\),

\[
\nabla_x(F(u)) \times \nabla_x(G(u)) = A^T \nabla_u F(u) \times A^T \nabla_u G(u)
\]

\[
= |A|^2 (\nabla_u F(u) \times \nabla_u G(u)) = |A|^3 (\nabla_u F(u) \times \nabla_u G(u)).
\]

Accordingly,

\[
\psi(y) = \nabla_y F(y) \times \nabla_y \Phi(y)
\]

\[
= \frac{\tau_y}{|\tau_y|} \left( \nabla_y \left( \Phi(z) F_Y(w) \right) \nabla_y \left( \left( \frac{\alpha F(z)^{25}}{|\tau_y|^2} \Phi_Y(w) \right) \right) \right)
\]

\[
= \frac{(\alpha F(z)^{25})^4}{|\tau_y|^4} \tau_y \nabla_y F_Y(w) \times \nabla_y \Phi_Y(w)
\]

\[
= \frac{(\alpha F(z)^{25})^4}{|\tau_y|^4} \tau_y \psi_Y(w),
\]

\[
\varphi(y) = \nabla_y F(y) \times \nabla_y \Psi(y)
\]

\[
= \frac{1}{|\tau_y|^2} \left( \nabla_y \left( \Phi(z) F_Y(w) \right) \nabla_y \left( \left( \frac{\alpha F(z)^{25}}{|\tau_y|^2} \Psi_Y(w) \right) \right) \right)
\]

\[
= \frac{(\alpha F(z)^{25})^8}{|\tau_y|^8} \tau_y \varphi_Y(w),
\]

and

\[
f(y) = \nabla_y \Phi(y) \times \nabla_y \Psi(y)
\]

\[
= \frac{1}{|\tau_y|^2} \left( \nabla_y \left( \Phi(z) F_Y(w) \right) \nabla_y \left( \frac{\alpha F(z)^{25}}{|\tau_y|^2} \Psi_Y(w) \right) \right)
\]

\[
= \frac{(\alpha F(z)^{25})^{11}}{|\tau_y|^9} \tau_y (f_Y(w)).
\]

The one-parameter maps transform as follows:

\[
\psi(y) = \frac{(\eta \Phi(z) \Psi(z) |\sigma_z|)}{\alpha F(z)^{25}} \sigma_z (\psi_Y(w))
\]

\[
\varphi(y) = \frac{(\eta \Phi(z) \Psi(z) |\sigma_z|)}{\alpha F(z)^{25}} \sigma_z (\varphi_Y(w))
\]

\[
f(y) = \frac{(\eta \Phi(z) \Psi(z) |\sigma_z|)}{\alpha F(z)^{25}} \sigma_z (f_Y(w)).
\]

Making the substitutions into the expression for \(f_{40}(y)\) yields a collection of maps \((f_{40})_Y(w)\) each of which is divisible by \(X_Y(w)\). Finally, substitution into the formula for the canonical map \(h_{19}(y)\) supplies the conic-fixing family \(h_Y(w)\). The explicit calculations appear in [Crass 1997b, pp. 68–69].

**General expressions for the canonical 19-map.** Each \(u\)-coefficient in

\[
h_z(w) = \tau_z^{-1} (h_{19}(\tau_z(w)))
\]

is a polynomial in \(Y_1\) and \(Y_2\) with \(z\)-degree

\[
\deg_z h_z(w) \leq 21 \cdot 25 = 19 \cdot 25 + 2 \cdot 25 = 12 \cdot 40 + 45 = 30 \cdot 16 + 45.
\]

The factor \(|\tau_z|^{-1}\) due to \(\tau_z^{-1}\) does not affect the map on \(\mathbb{CP}^2\), so we neglect it. After dividing away a factor of \(X(z)\), the polynomial map \(h_Y(w)\) satisfies

\[
\deg_{Y_1} h_Y(w) \leq 40 \quad \deg_{Y_2} h_Y(w) \leq 16.
\]

Hence, one finds the coefficients of the \(w\) monomials by solving, for each term, 353 linear equations.

In the one-parameter case,

\[
h_z(w) = \sigma_z^{-1} (h_{19}(\sigma_z(w)))
\]

and

\[
\deg_z h_z(w) \leq 21 \cdot 49 = 12 \cdot 82 + 45 = 12 \cdot 2 + 30 \cdot 32 + 45.
\]

Thus, on \(\{F(z) = 0\}\), the map \(h_z(w)\) is divisible by \(\Phi(z)^2 X(z)\), so that

\[
\deg_z \left( \frac{h_z(w)}{\Phi(z)^2 X(z)} \right) = 60 \cdot 16, \quad \deg_{Y_1} h_Y(w) \leq 16.
\]

**4D. Symmetry Lost, a Root Found**

Under Conjecture 3.13, the trajectory \(h_{19}(y_0)\) converges to a pair of 72-points for almost any \(y_0 \in \mathbb{CP}^2 \). Being conjugate to \(h_{19}(y)\), the maps \(h_Y(w)\) share this property for points in \(\mathbb{CP}_w^2\). Breaking the
$A_6$ symmetry of $R_Y(u) = 0$ qualifies $h_Y(w)$ for a role in root finding. An analogous treatment applies in the one-parameter case.

**Root selection.** Consider the rational function

$$J_z(w) = \frac{\Gamma_z(w)^3}{F(z)\Psi(\tau_z(w))},$$

where

$$\Gamma_z(w) = \sum_{m=1}^{6} \prod_{n \neq m} C_n(\tau_z(w)) C_m(z).$$

At a 72-point pair $\{q_1, q_2\} = \tau_z^{-1}(\{p_{a_1}, p_{a_2}\})$ in $w$ space, five of the six terms in $\Gamma_z(w)$ vanish. The result is a “selection” of one of the six roots $U_n(z)$ of $R_Y(u)$:

$$J_z(\{q_1, q_2\}) = \frac{(\prod_{n \neq a} C_n(\{p_{a_1}, p_{a_2}\}))^3}{\Psi(\{p_{a_1}, p_{a_2}\})} \frac{C_a(z)^3}{F(z)} = \frac{U_n(z)}{\mu}.$$

Here $\mu$ is the value of

$$\frac{\Psi(\{p_{a_1}, p_{a_2}\})}{(\prod_{n \neq a} C_n(\{p_{a_1}, p_{a_2}\}))^3},$$

which is constant on $\Xi_2$:

$$\mu = \frac{6561 (279 + 145 \sqrt{15} i)}{2}.$$

In light of the $\mathcal{V}_z$-invariance of $\Gamma_z(w)$, $J_z$ also enjoys this property and so, presents a second face which expresses itself in $Y$ and $w$. Since

$$\text{deg}_z \Gamma_z(w) = 252 = 10 \cdot 25 + 2 = 6 \cdot 42,$$

it transforms by

$$\Gamma_Y(w) = \frac{\Gamma_z(w)}{\beta F(z)^2},$$

where $\beta$ is a simplifying factor for the coefficients over $w$. Each coefficient is a polynomial in $Y_1$ and $Y_2$ whose coefficients satisfy a system of linear equations. (See [Crass 1997a] for the lengthy expression of the polynomial $\Gamma_Y(w)$.) Now, let $T_Y$ be the polynomial in $Y_1$ and $Y_2$ that satisfies

$$|\tau_z|^2 = F(z)^2 T_Y;$$

see (4-2) on page 232. Then, from (4-4),

$$\Psi(\tau_z(w)) = \frac{(\alpha F(z)^{25})}{(F(z)^{29 T_Y})^3} \Psi_Y(w) = \frac{\alpha^8 F(z)^{125}}{T_Y^3} \Psi_Y(w).$$

Finally,

$$J_Y(w) = \frac{\beta^3 (T_Y \Gamma_Y(w))^3}{\Psi_Y(w)}.$$ 

In the one-parameter case, the above development goes through with $J_z(w)$ and $\Gamma_z(w)$ replaced by

$$\bar{K}_z(w) = \Phi(z)^2 \Theta_z(w)^3 \frac{\Psi(z) \Psi(\tau_z(w))}{F(z)}$$

and

$$\bar{\Theta}_z(w) = \sum_{m=1}^{6} \prod_{n \neq m} C_n(\tau_z(w)) C_m(z).$$

First of all, there is the transformation of $\bar{\Theta}_z(w)$ under $\sigma_z$ on $\{F(z) = 0\}$:

$$\bar{\Theta}_V(w) = \frac{\bar{\Theta}_z(w)[(F(z)=0)]}{\gamma \Phi(z) \Psi(z)^{16}}.$$

(See [Crass 1997a] for the expression of this polynomial.) Furthermore, restricting to $\{F(z) = 0\}$ yields, on the one hand, a root of $T_V(s)$

$$\bar{K}_z(\{q_1, q_2\}) = \frac{S_n(z)}{\mu}$$

and, on the other,

$$\bar{K}_V(w) = \frac{\Phi(z)^2 (\gamma \Phi(z) \Psi(z)^{16} \bar{\Theta}_V(w))^3 |\sigma_z|^6}{(\eta \Phi(z)^2 \Psi(z)^{9} \Psi_Y(w)} = \frac{2^{9} \alpha^{10} \gamma^3 V^5 (V-1)^3 \bar{\Theta}_V(w)}{\eta \Phi(z) \Psi(w)}.$$

**The algorithm.** Now within reach are the ingredients required for preparation of a root-finding algorithm; see [Crass 1997a] for an implementation. To summarize the procedure:

1. Select a value $A = (A_1, A_2)$ of $Y = (Y_1, Y_2)$ and, thereby, a sixth-degree resolvent $R_A(u)$. (For sake of description, let $z \in Y_1^{-1}(A_1) \cap Y_2^{-1}(A_2).$

   The algorithm actually finds a root without explicitly inverting $Y_1$ or $Y_2$.)

2. From an initial point $w_0 \in \mathbb{C} P^2$, iterate the map $h_A(w)$ to convergence:

   $$h_A^n(w_0) \rightarrow \{q_1, q_2\} \in (O_{T_2}) \subset \mathbb{C} P^2(w).$$

   As output take the pair of approximate 72-points in $\mathbb{C} P^2$:

   $$\{p_1, p_2\} \approx \{q_1, q_2\} = \{\tau_z^{-1}(p_{a_1}), \tau_z^{-1}(p_{a_2})\}.$$
(3) Using \( p_1 \) or \( p_2 \), approximate a root of \( R_A(u) \):

\[
U_a(z) \approx \mu J_A(p_1) = \frac{\mu^{3^2}(T_A \Gamma_A(p_1))^{3}}{\alpha^3 \Psi_A(p_1)} = 2^{76}3^{24} \frac{(11 + 3 \sqrt{15} \pi)}{\Psi_A(p_1)}.
\]

Performing the corresponding steps in the special case produces an approximate solution to the \( V \)-resolvent \( T_{V_0}(s) \) given an initial choice \( V_0 \) of \( V \):

\[
S_a(z) \approx \mu R_{V_0}(p_1) = -2^{75}3^{30} \frac{\mu \gamma^3 V_0 \times (V_0 - 1)^3 \Theta V_0(p_1)}{\eta^5 \Psi V_0(p_1)} = -2^{75}3^{30} \frac{(11 + 3 \sqrt{15} \pi)}{\Psi V_0(p_1)}.
\]

4E. Getting All the Roots

Because a pair of 72-points lies on one barred conic, the algorithm above determines just one of the six roots of the selected resolvent. This manifests the iteration’s failure to break the galois symmetry completely; the stabilizer of a 72-point is a \( D_5 \). From the coefficient field of the resolvent \( R_A(u) \), the algorithm leads to an extended field \( K \) whose galois group \( G(\Sigma/K) \) is \( D_5 \). Of course, \( \Sigma \) is the splitting field for \( R_A \).

Finding all six roots calls for a dynamical system that converges to 6-cycles in a 360 point orbit and a root selector function that gives equations in all the roots. This would likely complicate the associated formulas.

APPENDIX: SEEING IS BELIEVING

Figures 9–12 provide empirical dynamical information for some of the special maps discussed in the text. They were created using the program Dynamics [Nusse and Yorke 1994] running on a Silicon Graphics Indigo-2, and have a resolution of \( 720 \times 720 \).

The first three images are the product of the BAS routine, which colors a grid-cell if the trajectory of the cell’s center gets close enough to a specified attractor to guarantee ultimate attraction to it. The color depends upon the destination. If, in a specified number of iterations, the center’s trajectory fails to converge to an attractor, the cell’s color is black.

For Figure 9, which shows the basins of \( \psi_{16} \) restricted to a 45-line, the attractors are the 45-points, which lie in the large basins near the center. The figure suggests that most points converge to one of these attractors. Does this happen for almost every point? Or do the black specks contain sets of positive measure whose forward orbits fail to converge to one of the four attractive 45-points? The BAS algorithm checked 60 iterates before giving up on determining convergency.

Figure 10 shows the basins of the conic-fixing map \( h_{19} \) restricted to one of the invariant conics. Six attracting orbits are known, each consisting of a pair of 72-points. Each basin is the one-dimensional intersection of a two-dimensional basin in \( \mathbb{CP}^2 \). Does the backward orbit of these basins fill out \( \mathbb{CP}^2 \) in measure? (This is the second part of Conjecture 3.13.)

Figure 11 shows the same map, but this time restricted to one of the bub-RP²’s—specifically \( \mathcal{R}_{22} \), with the 1-point orbit \( p_{22} \) at the origin and the 1-line orbit \( L_{22} \) at infinity. The chosen coordinates make evident the map’s \( D_5 \) symmetry. Along the unit circle is the 10-point orbit of 72-points \( p_{2a,2}, \) with \( a = 1, 3, 4, 5, 6 \). The five lines of reflective symmetry passing through (0,0) are affine lines in the five \( \mathbb{RP}^1 \) intersections with \( \mathcal{R}_{22} \) of both the 45-lines \( L_{20}, \) and the \( \mathcal{R}_{a}, \) for \( a = 1, 3, 4, 5, 6 \). The figure reveals repelling behavior along the \( \mathbb{RP}^1 \) where the \( \mathbb{RP}^2 \) meets one of the 45-lines left invariant by bub₂₂. Only the five period-2 cycles of 72-points appear as attractors. No black specks are seen; their presence would compromise Conjecture 3.13. Several magnified views of this basin plot appear in [Crass 1997a].

Figure 12 is a partial plot using the BA algorithm of Dynamics, which colors whole trajectories of cells, thereby manifesting some aspects of the dynamics. (See [Nusse and Yorke 1994, pp. 269 ff.] for a more thorough description of BA.)

Many of the points in the strip are mapped inside the hazy pentagon whose vertices lie on the 45-lines—the inner star is nearly filled. Around this pentagon is the outer star-like piece of the critical set shown in Figure 15. Furthermore, the pentagon seems to be the image of the inner pentagonal oval. Accordingly, the map folds the plane along the pentagon’s edges just outside of which the 72-points make their presence seen in the dense streaks at
FIGURE 9. Basin of attraction of the degree-16 map $\psi_{16}$ restricted to a 45-line; each color represents one of the four attracting 45-points. The black specks indicate points for which no attraction to one of these four points was detected after 60 iterations.

FIGURE 10. Dynamics of the degree-19 map with icosahedral symmetry on $\mathbb{C} \cup \{\infty\}$. Almost all points are attracted to a pair of antipodal vertices. Each of the six colors corresponds to such a pair; the three large basins each contain a vertex. For the conic-fixing $h_{19}$, the basin plot on each conic is conjugate to this one.
FIGURE 11. $\mathbb{RP}^2$ dynamics of $h_{19}$. The vertical and horizontal scales are roughly from $-2$ to $2$. The large radial basins are immediate, i.e., each contains one of the 72-points and come in pairs as do the period-2 attractors. Notice the repelling behavior along the 45-lines and particularly at their intersection in the 36-point $p_{22}$.

FIGURE 12. More $\mathbb{RP}^2$ dynamics of $h_{19}$. The vertical strip on the left is a copy of the same area in Figure 11; the rest of the figure shows trajectories, colored according to their destinations, of the points in the strip.
FIGURE 13. Configuration of 72-lines on a bub-$\mathbb{RP}^2$. Under bub$_{22}$, the five pairs of 72-lines $\{\mathcal{L}_{\alpha_1}, \mathcal{L}_{\alpha_2}\}$, for $a = 1, 3, 4, 5, 6$, map to themselves. Accordingly, each line meets $\mathcal{R}_{22}$ in an $\mathbb{RP}^1$. The picture shows their configuration in the affine plane of Figures 11 and 12. A given pair $\mathcal{L}_{\alpha_1}$ and $\mathcal{L}_{\alpha_2}$ passes through the 72-points $p_{\alpha_1}$ and $p_{\alpha_2}$, respectively; they intersect in the corresponding repelling and fixed 36-point $p_{\alpha}$. $p_{\alpha_1, z}$. Compare this pattern of streaks to that of the 72-lines given in Figure 13. Since $\mathcal{C}_3$ and $\mathcal{C}_4$ are tangent to $\mathcal{R}_{22}$ at $p_{m_{1,2}}$, the icosahedral 19-map opens up a triangular angle of $\pi/3$ to $4\pi/3$. Thus, the behavior at a 72-point consists of “fourth-powering”. Figure 14 displays this local squeezing. (Figures 13–15 were produced using Mathematica.)

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REFERENCES

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