Periodic Automorphisms of Surfaces: Invariant Circles and Maximal Orders

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CONTENTS

W. H. Meeks has asked the following question: For what g does every (orientation preserving) periodic automorphism of a closed orientable surface of genus g have an invariant circle? A variant of this question due to R. D. Edwards asks for the existence of invariant *essential* circles. Using a construction of Meeks we show that the answer to his question is negative for all but 43 values of g $\leq$ 10000, all of which lie below g = 105. We then show that the work of S. C. Wang on Edwards' question generalizes to nonorientable surfaces and automorphisms of *odd* order. Motivated by this, we ask for the maximal *odd* order of a periodic automorphism of a given nonorientable surface. We obtain a fairly complete answer to this question and also observe an amusing relation between this order and Fermat primes.

1. INTRODUCTION AND MAIN RESULTS

In this paper F_g denotes a closed orientable surface of genus g and N_g a closed nonorientable surface of the same genus. By a *periodic automorphism of order n* we mean a diffeomorphism $f : \Sigma \to \Sigma$ (where $\Sigma = F_g$ or N_g) such that $f^n = \mathrm{id}_\Sigma$ and $f^k \neq \mathrm{id}_\Sigma$ for $1 \leq k \leq n-1$. The classical case to consider is

(A) The surface Σ is orientable and f is orientation preserving.

For this case we study the following question of [Meeks 1979], which also appears as [Kirby 1997, Problem 2.8]:

(Q1) For what genus g does *every* (orientation preserving) automorphism f of F_g have an invariant circle?

Here *invariant circle* means an embedded circle $C \subset \Sigma$ such that $f(C) = C$.

By the cited work of Meeks the answer is positive for $g \leq 10$, negative for $g = 11$, and according to [Kirby 1997] it is known that the answer is negative for infinitely many g (though, to our knowledge, no

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proof of this fact exists in the literature). Nevertheless, in some sense 'most' periodic automorphisms have an invariant circle (this and the previous statements will all be discussed below), which led Meeks to conjecture that the answer to (Q1) is positive for infinitely many g. However, by using Meeks' own technique we can prove the following theorem, providing some evidence against this conjecture.

Theorem 1.1. *There are precisely* 43 *values of* g *in the range* $g \leq 10000$ *such that every orientation preserving periodic automorphism of* F_g *has an invariant circle, namely,* $g = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 27, 28, 30, 32, 35, 39, 42, 43, 44, 45, 48, 49, 50, 51, 60, 65, 66, 72, 73, 87, 90, 105$.

This theorem and some related results are discussed in Section 3.

The existence of invariant *essential* circles, i.e. embedded circles $C \subset \Sigma$ that are not nullhomotopic, was studied by Wang [1989]. He showed that an orientation preserving periodic automorphism $F_g \to F_g$ ($g \geq 2$) of *prime* order p has an invariant essential circle if and only if $p \leq g+1$, and that $2g+2$ (when g is even) or $2g-2$ (when g is odd and $g > 3$) constitutes a sharp upper bound for the order of automorphisms with invariant essential circles (for $g = 3$ the bound is 6). In Section 4 we generalize the results of Wang to nonorientable surfaces N_g and automorphisms of odd order, and we explain the difficulties in trying to extend these results to the other cases (automorphisms of N_g of even order or orientation reversing automorphisms of F_g). Some of the results of that section can be summarized as follows:

Theorem 1.2. (i) *Any periodic automorphism*

$$f : N_g \to N_g$$

of order n *equal to the power of an odd prime has an invariant essential circle. If* n *contains only two different prime factors, then* f *still has an invariant (but possibly inessential) circle.*

(ii) *There are infinitely many* N_g *which admit a periodic automorphism of odd order without invariant circles.*

In the theory of surface automorphisms there is a further problem with a long and distinguished history, namely, the question:

(Q2) What is the maximal order $o(\Sigma)$ of a periodic automorphism $f : \Sigma \to \Sigma$?

The case for orientation preserving automorphisms of F_g was settled in [Wiman 1895] and [Harvey 1966], and the complete answer was first found in [Steiger 1935] and rediscovered by [Wang 1991] (and others). Their result reads as follows:

Theorem 1.3.

$$o(F_g) = \begin{cases} 4g+2, & g \text{ odd, } g \geq 3; \\ 4g+4, & g \text{ even, } g \geq 2; \end{cases}$$

$$o(N_g) = \begin{cases} 2g, & g \text{ odd, } g \geq 3; \\ 2g-2, & g \text{ even, } g \geq 4. \end{cases}$$

In the remaining cases the order of f can be arbitrarily large.

The discussion in Section 4 suggests that it is worthwile to focus attention on the case

(B) The surface Σ is nonorientable and f has *odd* order.

Motivated by this, we now ask question (Q2) for this case. Write $o^*(g)$ for the maximal odd order of an automorphism $f : N_g \to N_g$, with $g \geq 3$ understood.

For g even, let 2^k be the largest power of 2 that divides $g-2$, and set

$$\alpha(g) = \frac{2^k+1}{2^k}(g-2).$$

For g odd, let k be the *smallest* natural number (not including 0) such that

1. *either* 2^k is the largest power of 2 that divides $g-1$,
2. *or* 2^k divides $g-1$ and 2^{k+1} does not divide $g+2^k-1$ and $\gcd(2^k+1, g+2^k-1) = 1$.

Then set

$$\alpha(g) = \frac{2^k+1}{2^k}(g+\varepsilon 2^k - 1),$$

with $\varepsilon = 0$ if k comes from the first alternative, and $\varepsilon = 1$ if it comes from the second. We abbreviate the right hand side of this equation to $\alpha_\varepsilon(g)$.

Here are the main results proved in Section 5.

Theorem 1.4. (i) *For* g *even,* $o^*(g) = \alpha(g)$.

(ii) *For* g *odd,* $\alpha(g) \leq o^*(g) \leq 3(g+1)/2$, *and* $o^*(g) = \alpha(g)$ *for* $g \not\equiv 257 \bmod 8160$ *or* $g \leq 16575$.

This seems to constitute overwhelming evidence for the conjecture that $o^*(g) = \alpha(g)$ for all g, but the conjecture fails for $g = 16577$. A different formulation, as we shall see, is that the first example where $o^*(g)$ is strictly bigger than $\alpha(g)$ arises for $k = 5$ and $\varepsilon = 0$.

Theorem 1.5. *For g odd, the number $o^*(g)$ satisfies $o^*(g) \geq g$, and equality $o^*(g) = g$ can only hold if g is a Fermat number $g = \mathfrak{F}_j = 2^{2^j} + 1$, $j \in \mathbb{N}_0$. In fact, for the Fermat primes $\mathfrak{F}_j$, $0 \leq j \leq 4$, we have $o^*(\mathfrak{F}_j) = \mathfrak{F}_j$; for the composite Fermat numbers $\mathfrak{F}_j$, $5 \leq j \leq 9$, we have $o^*(\mathfrak{F}_j) > \mathfrak{F}_j$.*

The present paper grew out of Rattaggi's diploma thesis [1998] under Geiges' guidance at the ETH Zürich. In that thesis the reader can find an extensive bibliography pertinent to the questions considered here, more background information, examples, and some further related results.

2. PERIODIC AUTOMORPHISMS AND BRANCHED COVERS

A periodic automorphism $f : \Sigma \to \Sigma$ of order n induces a $\mathbb{Z}_n$-action on Σ such that the quotient map

$$\pi : \Sigma \to \Sigma' := \Sigma/\mathbb{Z}_n$$

is an n-sheeted cyclic branched covering. It is sometimes convenient to assume (and we shall do so implicitly below) that Σ is endowed with a metric for which f is an isometry. Such a metric can be obtained simply by averaging any given metric. One may even assume this to be a constant curvature metric. This is achieved by lifting an elliptic, Euclidean, or hyperbolic structure from $\Sigma/\mathbb{Z}_n$ (which we regard as an orbifold) to Σ; for the existence of such a structure on the orbifold see [Thurston 1985, Chapter 13] and [Scott 1983, Section 2]. Such an invariant constant curvature metric can be used to show that if C is the image of an injective continuous map $S^1 \to \Sigma$ and $f(C) = C$, then there exists a smoothly embedded invariant circle in the same free homotopy class.

We recall some well-known facts about branched coverings; see for instance [Harvey 1966; Berstein and Edmonds 1979; Miranda 1995]. Let l be the number of branch points (in Σ') of the branched covering $\pi : \Sigma \to \Sigma'$ and write $B = \{b_1, \ldots, b_l\}$ for the branching set. Let m_i, $i = 1, \ldots, l$, be the corresponding branching indices, which means that in suitable local complex coordinates around one of the preimages of b_i the map π is given by $z \mapsto z^{m_i}$. Denote the Euler characteristic by χ. Then, assuming that Σ' has no boundary (or reflector curves when regarded as an orbifold), we have the *Riemann–Hurwitz formula*, subsequently referred to as (R-H),

$$\frac{\chi(\Sigma)}{n} = \chi(\Sigma') - \sum_{i=1}^{l} \left(1 - \frac{1}{m_i}\right).$$

The monodromy around the branch points induces a surjective representation

$$\rho : \pi_1(\Sigma' - B) \to \mathbb{Z}_n,$$

where the element x_i of $\pi_1(\Sigma' - B)$ represented by a small loop around b_i maps to an element of order m_i in $\mathbb{Z}_n$. Conversely, from such a representation one can reconstruct the cyclic branched covering.

The existence of a periodic automorphism can be deduced from purely algebraic conditions; see [Bujalance et al. 1990]. We summarize these for the cases (A) and (B) that we consider in the present paper; for the other cases see [Rattaggi 1998].

In case (A) we observe that $\Sigma' = \Sigma/\mathbb{Z}_n$ is orientable and without boundary. Then by Theorems 3.1.2 and 3.1.5 of [Bujalance et al. 1990] we have the following necessary and sufficient conditions for the existence of an orientation preserving periodic automorphism $f : F_g \to F_g$ of order n with quotient $F_{g'} = F_g/\mathbb{Z}_n$:

(A1) m_i divides n for $i = 1, \ldots, l$.
(A2) (R-H): $(2g-2)/n = 2g' - 2 + \sum_{i=1}^{l}(1 - 1/m_i)$.
(A3) $\operatorname{lcm}(m_1, \ldots, \hat{m}_i, \ldots, m_l) = \operatorname{lcm}(m_1, \ldots, m_l)$ for $i = 1, \ldots, l$, where $\hat{m}_i$ indicates the omission of m_i.
(A4) If $g' = 0$, then $n = \operatorname{lcm}(m_1, \ldots, m_l)$.

This case (A) is also discussed in [Harvey 1966, Theorem 4]. The algebraic conditions listed there (for instance, $l \neq 1$ and, if $g' = 0$, then $l \geq 3$) can be deduced from (A1)–(A4). For $g \geq 2$ (i.e., in the hyperbolic case), the monodromy representation ρ has to satisfy only the two conditions mentioned above: it needs to be surjective, and $\rho(x_i)$ must have order m_i (see [Harvey 1966, Theorem 3]).

In case (B), Theorem 3.1.3 and Corollary 3.2.3 of [Bujalance et al. 1990] state that Σ' is nonorientable and without boundary. Furthermore, necessary and sufficient for the existence of a periodic automorphism $f : N_g \to N_g$ of odd order n with quotient $N_{g'} = N_g/\mathbb{Z}_n$ are:

(B1) m_i divides n for $i = 1, \dots, l$.
(B2) (R-H): $(g-2)/n = g' - 2 + \sum_{i=1}^{l}(1 - 1/m_i)$.
(B3) If $g' = 1$, then $n = \operatorname{lcm}(m_1, \dots, m_l)$.

Write $\pi_1(\Sigma' - B)$ in the standard form

$$\langle c_1, \dots, c_{g'}, x_1, \dots, x_l \mid c_1^2 \dots c_{g'}^2\, x_1 \dots x_l = 1\rangle,$$

and denote by Γ^+ the subgroup of elements of

$$\pi_1(\Sigma' - B)$$

containing an even number of the factors $c_i^{\pm 1}$. If $g \geq 3$ (again, the hyperbolic case), then the only condition the monodromy representation ρ has to satisfy (in addition to those discussed above) is that

$$\rho(\Gamma^+) = \mathbb{Z}_n;$$

see [Bujalance 1983, Proposition 3.2].

3. AUTOMORPHISMS WITHOUT INVARIANT CIRCLES

In this section we explain the mathematical principles behind the numerical search which we carried out to prove Theorem 1.1. Thus here we consider orientable surfaces F_g and orientation preserving periodic automorphisms $f : F_g \to F_g$ (of order n). First we summarize the results for $g = 0$ or 1 (compare [Scott 1983; Wang 1989]). If $g = 0$ then f is a rotation by $2\pi/n$ around some axis, and f has an invariant circle. If $g = 1$ then f can either be isotopic to the identity (f is a shift and n arbitrary) or f is one of four maps not isotopic to the identity of order $2, 3, 4, 6$, respectively. In any case, f has an invariant circle.

For the remainder of this section we assume $g \geq 2$.

Meeks [1979] has shown that every $f : F_g \to F_g$ with order n containing at most two different prime factors or with $g \leq 10$ has an invariant circle. Also, on F_{11} there is essentially only one automorphism (of order 30) with no invariant circles ('essentially' meaning up to conjugation and taking a power relatively prime to 30). So automorphisms with no invariant circles appear to be scarce.

Meeks also shows the following necessary and sufficient conditions — together with (A1)–(A4) — for the existence of a periodic automorphism $f : F_g \to F_g$ without invariant circles:

(A5) $g' = 0$, that is, $\Sigma' = S^2$.
(A6) There is a surjective representation

$$\rho : \pi_1(\Sigma' - B) = \langle x_1, \dots, x_l \mid x_1 \dots x_l = 1\rangle \to \mathbb{Z}_n$$

with $\rho(x_i)$ of order m_i and such that $\rho(x_1^{\varepsilon_1} \dots x_l^{\varepsilon_l})$ is *not* a generator of $\mathbb{Z}_n$ for any choice of $\varepsilon_i \in \{0,1\}$, $i = 1, \dots, l$.

That last condition can be explained as follows: Elements of $\pi_1(\Sigma' - B)$ which can be represented by an embedded circle C' are precisely those of the form $x_1^{\varepsilon_1} \dots x_l^{\varepsilon_l}$ by [Meeks and Patrusky 1978, Theorem 1], and the preimage C of C' is an invariant circle exactly when $C \to C'$ is an n-fold cover and $\rho([C'])$ a generator of $\mathbb{Z}_n$.

Conditions (A1)–(A6) clearly allow a numerical search for automorphisms without invariant circles. By the results of Meeks, one can restrict attention to orders n with at least three prime factors (in particular, $n \geq 30$). Furthermore, if one can find an automorphism f satisfying (A1)–(A5) with $n > 2g + 2$ and $m_i \neq n$ for all $i = 1, \dots, l$, then one need not check condition (A6), for the condition $n > 2g + 2$ implies according to [Wang 1989] that f has no invariant *essential* circles, and $m_i \neq n$ implies that f has no fixed points and hence no invariant inessential circles (if f had an invariant inessential circle, this circle would bound a disc which would have to be invariant because of $g > 1$, and to this disc the Brouwer fixed point theorem would apply).

Starting with $l = 3$ (observe that (A5) together with (R-H) implies $l \geq 3$), one can already find infinitely many F_g admitting an automorphism without invariant circles, for example:

$$\begin{aligned} g &= -4 + 15k \quad \text{with } k \in \mathbb{N}, \\ n &= 30k, \quad (r_1, r_2, r_3) = (2, 3, 5), \\ m_i &= n/r_i \quad \text{for } i = 1, 2, 3, \\ \rho(x_1) &= r_1, \quad \rho(x_2) = r_2, \quad \rho(x_3) = -r_3 \bmod n, \end{aligned}$$

or in fact infinitely many F_g with any given number of nonequivalent automorphisms without invariant circles.

Continuing with $l = 4$ and 5, one can find an automorphism of F_g without invariant circles for any $g \leq 10000$ except the ones listed in Theorem 1.1. To finish the proof of Theorem 1.1 one has to check that the corresponding F_g do not admit an automorphism without invariant circles for larger values of l. It follows from the Riemann–Hurwitz formula that this search is finite. Indeed, with $g' = 0$ and $r_i = n/m_i$ we have

$$2g - 2 = n(l-2) - \sum_{i=1}^{l} r_i.$$

With $r_i = n/m_i \leq n/2$ we get

$$2g - 2 \geq n(\tfrac{1}{2}l - 2).$$

Hence, for given g and l there are only finitely many possible values for n and thus only finitely many solutions of (A1)–(A6). Furthermore, (R-H) implies that for $l \geq 18$ there is no automorphism with $g' = 0$, $n \geq 30$, and $g \leq 105$.

Detailed lists of periodic automorphisms without invariant circles and some remarks on the programming of the search described in this section can be found in [Rattaggi 1998].

4. NONORIENTABLE SURFACES

In the present section we want to generalize the results of Meeks and Wang to automorphisms f of nonorientable surfaces $\Sigma = N_g$ of *odd* order n. The reason for this restriction will be explained at the end of this section.

As mentioned in Section 2, the quotient surface

$$\Sigma' = \Sigma/\langle f\rangle$$

is again nonorientable and without boundary.

We first deal with the special cases $\Sigma = N_1 = \mathbb{RP}^2$ and $\Sigma = N_2 =$ Klein bottle. If $g = 1$, then (B2) = (R-H) reads

$$g' - 2 + \sum_{i=1}^{l} \left(1 - \frac{1}{m_i}\right) = \frac{g-2}{n} = \frac{-1}{n} < 0.$$

This implies $g' = 1$ and $l \leq 1$ (since $1 - 1/m_i \geq \frac{1}{2}$). The case $l = 0$ can be excluded because here $n = 1$. Thus $l = 1$ and $m_1 = n$.

If we think of $\mathbb{RP}^2$ as a disc with opposite points on the boundary identified, then these data can be realized by a rotation by $2\pi/n$ around the centre of this disc. This is (up to equivalence) the only automorphism corresponding to the given data, since the (surjective) monodromy representation

$$\rho:\ \pi_1(\Sigma' - B) = \mathbb{Z} \to \mathbb{Z}_n$$

can only be reduction modulo n. The circle at infinity (corresponding to half the circumference of the disc) is an invariant essential circle under the described rotation; any circle around the centre is an invariant inessential circle.

If $g = 2$, then (R-H) becomes

$$g' - 2 + \sum_{i=1}^{l} \left(1 - \frac{1}{m_i}\right) = 0.$$

Hence $g' \leq 2$. If $g' = 1$, then

$$\sum_{i=1}^{l} \left(1 - \frac{1}{m_i}\right) = 1,$$

thus $l = 2$, $m_1 = m_2 = 2$, and, by (B3), we get $n = \operatorname{lcm}(m_1, m_2) = 2$, contradicting our assumption that n be odd. The only remaining case is $g' = 2$ and $l = 0$, that is, Σ' also has to be a Klein bottle and $\Sigma \to \Sigma'$ an n-fold unbranched covering. Such a covering can be given by realizing Σ' as the quotient of $\mathbb{R}^2$ under $\alpha(x,y) = (x+1/n, -y)$ and $\beta(x,y) = (x, y+1)$, and Σ as the quotient of $\mathbb{R}^2$ under α^n and β, with the obvious projection $\Sigma \to \Sigma'$. Since $\mathbb{Z}_n$ is abelian, the monodromy representation

$$\rho:\ \pi_1(\Sigma') = \langle \alpha, \beta \mid \alpha\beta\alpha^{-1}\beta = 1\rangle \to \mathbb{Z}_n$$

factors through

$$H_1(\Sigma') = \langle \alpha, \beta \mid \alpha\beta = \beta\alpha,\ \beta^2 = 1\rangle \cong \mathbb{Z} \oplus \mathbb{Z}_2$$

and hence is unique up to isomorphism (since n is odd). So up to equivalence the automorphism $(x,y) \mapsto (x+1/n, -y)$ of Σ is the only automorphism of N_2 of odd order n. This automorphism has both essential and inessential invariant circles.

From now on we only consider surfaces $\Sigma = N_g$ with $g \geq 3$, and $f : N_g \to N_g$ will always denote an automorphism of odd order n.

The following three lemmas are the analogues of Lemmas 1, 2, 3 of [Wang 1989] (see also Section 1 of [Meeks 1979]) for the case (B) considered here.

Lemma 4.1. *Let C be an invariant circle of f and $r \leq n$ the order of the restriction of f to C. Then*

(i) *$r = n$, and*
(ii) *there are no singular points of f on C.*

Proof. (i) Choose a point $x \in C$ and a short geodesic arc A perpendicular to C at x. The map f^r is an isometry which fixes C pointwise. Hence f^r is either the identity on A, or it reflects A in x. In any case

$f^{2r}(a) = a$ for all $a \in A$. Hence f^{2r} is the identity on N_g, since f^{2r} fixes a point x and an orthogonal frame at x (an isometry of a connected manifold is determined by its value and differential at one point; see [Carmo 1992, Lemma 8.4.2]). Thus $2r = n$ or $2r = 2n$. But n is odd, hence $r = n$.

(ii) We argue by contradiction. Assume $y \in C$ is a singular point of f, that is, $f^q(y) = y$ for some $q < n$. So $f^q|C$ is an isometry of a circle with a fixed point, but not the identity. Therefore $f^q|C$ is orientation reversing and so is $f|C$. By the Lefschetz fixed point theorem, $f|C$ must have a fixed point, which implies that $f|C$ must be of order $r = 2$. This contradicts (i), because of our global assumption that n be odd. □

In the following lemma we adhere to the notation of Section 2. The proof is the same as in the classical case (A).

Lemma 4.2. *If C is an invariant circle of f, then $C' = \pi(C)$ is a circle in $\Sigma' - B$ and $\rho([C'])$ is a generator of $\mathbb{Z}_n$. Conversely, if C' is a circle in $\Sigma' - B$ and $\rho([C'])$ a generator of $\mathbb{Z}_n$, then $C = \pi^{-1}(C')$ is an invariant circle of f.*

Proof. An invariant circle C of f does not contain any singular points by Lemma 4.1. Hence $\langle f \rangle$ acts freely (with order n) on C, so $f|C$ (being an isometry) is a shift by $m \cdot \text{length}\,(C)/n$ with m coprime to n, and $C' = \pi(C)$ is a circle in $\Sigma' - B$ with $C \to C'$ an n-fold unbranched covering. Hence $\rho([C'])$ generates $\mathbb{Z}_n$.

Conversely, if $\rho([C'])$ generates $\mathbb{Z}_n$, then $C = \pi^{-1}(C')$ is connected and thus an invariant circle. □

Lemma 4.3. *Let C be an invariant circle of f. Then C separates N_g if and only if $C' = \pi(C)$ separates $\Sigma' = N_g/\langle f \rangle$.*

Proof. Suppose C separates N_g. Then we can write N_g as a disjoint union $N_g = A_1 \cup A_2 \cup C$ with A_1 and A_2 connected open sets with $\overline{A}_i - A_i = C$ for $i = 1, 2$. These sets are either preserved or interchanged by f, but since f^n is the identity on N_g and n is odd, we must have $f(A_i) = A_i$ for $i = 1, 2$. Arguing by contradiction, we assume that C' does not separate Σ'. Given two nonsingular points $a_i \in A_i$, we can then find a path γ in $\Sigma' - (B \cup C')$ joining $\pi(a_1)$ with $\pi(a_2)$. The lift $\tilde{\gamma}$ of γ with initial point a_1 ends in $\tilde{\gamma}(1) \in \pi^{-1}(\pi(a_2)) \subset A_2$ (because $f(A_2) = A_2$), but $\tilde{\gamma}$ does not pass through C. This contradiction proves that C' separates Σ'.

Conversely, if C' separates Σ', write $\Sigma' = A_1' \cup A_2' \cup C'$. If $C = \pi^{-1}(C')$ did not separate N_g, we could choose $a_i' \in A_i'$ and a path $\tilde{\gamma}$ in $N_g - C$ from an $a_1 \in \pi^{-1}(a_1')$ to an $a_2 \in \pi^{-1}(a_2')$. Then $\pi(\tilde{\gamma})$ would be a path from a_1' to a_2' not passing through C'. □

Proof of of Theorem 1.2. (i) Let n be a power of the prime p. Then an element of $\mathbb{Z}_n$ generates this group if and only if it is not divisible by p.

By Lemmas 4.2 and 4.3 it is sufficient to find an element $[C']$ of $\pi_1(\Sigma' - B)$ represented by a loop C' not separating Σ' such that $\rho([C'])$ is a generator of $\mathbb{Z}_n$.

We write

$$\begin{aligned}\pi_1(\Sigma' - B) \\ &= \langle c_1, \dots, c_{g'}, x_1, \dots, x_l \mid c_1^2 \, \dots \, c_{g'}^2 \, x_1 \, \dots \, x_l = 1 \rangle,\end{aligned}$$

as before. First consider the case $l = 0$. Since ρ is surjective, there exists a c_i such that $\rho(c_i)$ generates $\mathbb{Z}_n$, and we take C' to be a circle representing c_i.

Now assume $l \geq 1$. If there exists a c_i such that $\rho(c_i)$ generates $\mathbb{Z}_n$, we are done as before. Otherwise, all $\rho(c_i)$ are divisible by p, and since ρ is surjective there is an x_j with $\rho(x_j)$ not divisible by p. The class $c_i x_j$ can then be represented by a circle C' with the desired properties.

The proof of the existence of an invariant circle if $n = p^a q^b$ goes along the same lines and is analogous to the proof of Theorem 2 in [Meeks 1979]: Since ρ is surjective, either one of the generators c_i, x_j maps to a generator of $\mathbb{Z}_n$ (in which case we are done), or we find two elements u_1, u_2 among this set of standard generators of $\pi_1(\Sigma' - B)$ such that $\rho(u_1)$ is divisible by p but not by q, and $\rho(u_2)$ is divisible by q but not by p. Then $\rho(u_1 u_2)$ is a generator of $\mathbb{Z}_n$, and $u_1 u_2$ can be represented by an embedded circle.

(ii) We choose $g' = 1$, $l = 2$ and, with $k \in \mathbb{N}$ odd,

$$g = 1785k - 18, \; n = 1785k, \; m_1 = 595k, \; m_2 = 105k.$$

Notice that $1785 = 3 \cdot 5 \cdot 7 \cdot 17$. Writing $\pi_1(\Sigma' - B)$ in terms of standard generators c_1, x_1, x_2 as before, we define

$$\rho : \pi_1(\Sigma' - B) \to \mathbb{Z}_n$$

by

$$\rho(c_1) = n - 10, \; \rho(x_1) = 3, \; \rho(x_2) = 17.$$

Then (B1)–(B3) and the conditions on ρ stipulated in Section 2 are easily verified. In particular,

$$\rho(c_1^2 x_1^7) = 1 \bmod n,$$

so $\rho(\Gamma^+) = \mathbb{Z}_n$.

It remains to check that the automorphism f of N_g defined by these data has no invariant circle. According to [Chillingworth 1972, p. 145, case (iv)], the only elements of $\pi_1(\Sigma' - B)$ which can be represented by an embedded circle are those conjugate to

$$1,\ c_1,\ c_1^2,\ c_1x_1,\ (c_1x_1)^2,\ x_1,\ c_1^2x_1,$$

or their inverses. By Lemma 4.2 we have to show that none of these elements maps to a generator of $\mathbb{Z}_n$ under the monodromy representation ρ. Clearly we may disregard multiples and inverses, and since $\mathbb{Z}_n$ is abelian the same holds for conjugates. We are left with the three generators c_1, x_1, x_2 of $\pi_1(\Sigma'-B)$ and c_1x_1, $c_1^2x_1$, which map under ρ to

$$n-10,\ 3,\ 17,\ n-7,\ n-17,$$

respectively, neither of which is relatively prime to n. $\square$

We conclude this section with a brief comment about the cases not considered in this paper: orientation reversing automorphisms $f : F_g \to F_g$ or automorphisms $f : N_g \to N_g$ of even order. In either case it can happen that the quotient surface $\Sigma' = \Sigma/\langle f\rangle$ has nonempty boundary, in which case Lemma 4.1 and subsequent arguments based on it will fail. If Σ' happens to be closed, then Lemmas 4.1 and 4.2 still hold. But a nonseparating circle $C' \subset \Sigma'$ may now lift to a separating circle $C \subset \Sigma$. However, as long as we are only interested in the circle $C \subset \Sigma$ being essential, the arguments of this section carry through (subject to the assumption that Σ' be closed) with Lemma 4.3 replaced by the following statement.

Lemma 4.4. *Let C be an invariant circle of f not containing a singular point, and $C' = \pi(C)$. If $n[C'] \neq 1 \in \pi_1(\Sigma')$, where n is the order of f, then C is essential.*

Proof. Denote by $\pi_\#$ the homomorphism $\pi_1(\Sigma) \to \pi_1(\Sigma')$ induced by the projection $\pi : \Sigma \to \Sigma'$. Under the assumptions of the lemma we have $\pi_\#([C]) = n[C'] \neq 1 \in \pi_1(\Sigma')$, and thus $[C] \neq 1 \in \pi_1(\Sigma)$, which implies that C is an essential circle. $\square$

5. MAXIMAL ORDERS OF PERIODIC AUTOMORPHISMS

We now turn to the proof of Theorem 1.4. Let $g \geq 3$ be given and write $n_0 = o^*(g)$ for the maximal odd order of periodic automorphisms $f : N_g \to N_g$. Write f_0 for an automorphism which realizes this maximal odd order, g_0' for the genus of the orbit space, and l_0 for the corresponding number of branch points of $N_g \to N_{g_0'} = N_g/\langle f_0\rangle$. We shall see presently that g_0' does not depend on the choice of f_0.

Lemma 5.1. *Let $f : N_g \to N_g$ be an automorphism of odd order n and g' the genus of the quotient surface $N_g/\langle f\rangle$. Then g and g' have the same parity.*

Proof. The Riemann–Hurwitz formula yields

$$\begin{aligned} g &= 2 + ng' - 2n + ln - \sum_{i=1}^{l} \frac{n}{m_i} \\ &\equiv g' \bmod 2, \end{aligned}$$

regardless of the parity of l. $\square$

Lemma 5.2. *For g odd we have $g_0' = 1$ and $l_0 = 2$, for g even we have $g_0' = 2$ and $l_0 = 1$.*

Proof. For g odd there is a periodic automorphism f with $n = g$, $g' = 1$, $l = 2$, and $m_1 = m_2 = n$ (see Figure 1). This implies $n_0 \geq g$. By the preceding lemma we know that g' is odd. Assume $g' \geq 3$. Then (R-H) yields

$$\frac{g-2}{n} = g' - 2 + \sum_{i=1}^{l}\left(1 - \frac{1}{m_i}\right) \geq 1,$$

hence

$$n \leq g - 2 < g \leq n_0.$$

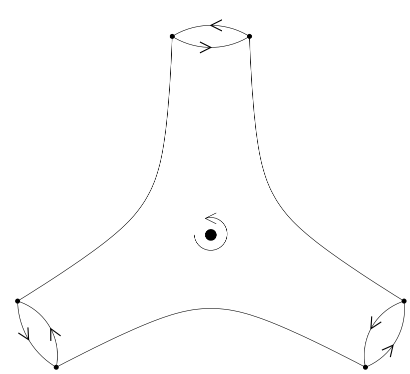

FIGURE 1. The case g odd, $n = g$.

Therefore $g_0' = 1$. Inserting this in (R-H) we find

$$0 < \frac{g-2}{n_0} = -1 + \sum_{i=1}^{l_0}\left(1 - \frac{1}{m_i}\right) < -1 + l_0,$$

hence $l_0 \geq 2$, but also

$$\frac{g-2}{n_0} = -1 + \sum_{i=1}^{l_0}\left(1 - \frac{1}{m_i}\right) \geq -1 + \frac{2}{3}l_0,$$

and thus

$$l_0 \leq \frac{3}{2}\left(\frac{g-2}{n_0} + 1\right) \leq \frac{3}{2}\left(\frac{g-2}{g} + 1\right) = 3 - \frac{3}{g}.$$

We conclude that $l_0 = 2$.

The case g even is treated analogously, starting from the existence of an automorphism f with $n = g-1$, $g' = 2$, $l = 1$, $m_1 = n$ (Figure 2). $\square$

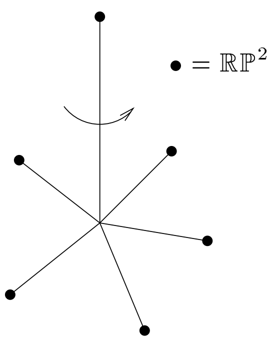

FIGURE 2. The case g even, $n = g - 1$.

Proof of Theorem 1.4(i) (g *even*). The conditions (B1)–(B3) are satisfied with $n = \alpha(g)$ (as defined in Section 1), $g' = 2$, $l = 1$, and $m_1 = 1 + 2^k$. Hence $o^*(g) \geq \alpha(g)$. To show $o^*(g) \leq \alpha(g)$ we can restrict attention to $g' = 2$ and $l = 1$ by the lemma just proved. The Riemann–Hurwitz formula then reads

$$\frac{g-2}{n} = 1 - \frac{1}{m_1},$$

hence $(g-2)m_1 = n(m_1 - 1)$. In the definition of $\alpha(g)$ we wrote 2^k for the largest power of 2 that divides $g - 2$. Since n (and hence m_1) is odd, we can write $m_1 - 1$ in the form $r2^k$ with $r \in \mathbb{N}$. Hence

$$\begin{aligned} n &= \frac{m_1(g-2)}{m_1 - 1} \\ &= \frac{r2^k + 1}{r2^k}(g-2) \\ &\leq \frac{2^k+1}{2^k}(g-2) = \alpha(g). \end{aligned}$$

$\square$

Proof of Theorem 1.4(ii) (g *odd*). By Lemma 5.2 we know that in this case we only have to consider automorphisms with $g' = 1$ and $l = 2$. The statement $o^*(g) = \alpha(g)$ for $g \leq 16575$ we have checked numerically by testing for solutions of (B1)–(B3).

The claim that $o^*(g) \geq \alpha(g)$ for all odd $g \geq 3$ is proved by exhibiting a solution with $n = \alpha(g)$ for those very equations. Recall from Section 1 the definition of k in the formula for $\alpha(g)$. Then the desired solution is given by setting $n = \alpha(g)$, $m_1 = 2^k + 1$, and $m_2 = n$ if $\alpha(g) = \alpha_0(g)$, and $m_2 = n/m_1$ if $\alpha(g) = \alpha_1(g)$.

We now turn to the proof of $o^*(g) \leq 3(g+1)/2$ for all odd $g \geq 3$. This argument will also provide the model for the proof of $o^*(g) = \alpha(g)$ for odd $g \not\equiv 257 \bmod 8160$. Let g be given and, arguing by contradiction, assume that we have a periodic automorphism of N_g of odd order $n > 3(g+1)/2$. Then (R-H) implies

$$1 - \frac{1}{m_1} - \frac{1}{m_2} = \frac{g-2}{n} < \frac{2(g-2)}{3(g+1)} < \frac{2}{3},$$

hence

$$\frac{1}{m_1} + \frac{1}{m_2} > \frac{g+7}{3(g+1)} > \frac{1}{3}.$$

Without loss of generality we may assume $m_1 \leq m_2$, so the only possible cases are

(i) $m_1 = 3$,
(ii) $m_1 = 5, m_2 = 5$,
(iii) $m_1 = 5, m_2 = 7$.

The second and third case can easily be shown to lead, using (B1)–(B3), to small values of g (5 or 25, respectively), for which we already know

$$o^*(g) = \alpha(g) \leq 3(g+1)/2.$$

In the first case, if $\gcd(3, m_2) = 1$, then by (B3) we have $n = \operatorname{lcm}(m_1, m_2) = 3m_2$, and inserting this in (R-H) gives $n = 3(g+1)/2$, contradicting our assumption. Similarly, if

$$\gcd(3, m_2) = 3,$$

then $n = \operatorname{lcm}(m_1, m_2) = m_2$, hence

$$\frac{1}{m_1} + \frac{1}{m_2} = \frac{1}{3} + \frac{1}{n} < \frac{1}{3} + \frac{2}{3(g+1)} = \frac{g+3}{3(g+1)},$$

contradicting the estimate derived above.

We now proceed according to the same scheme. In the formula for $\alpha(g)$ we have $k = 1$ if

1. either 2 divides $g-1$ but 4 does not,
2. or 2 divides $g-1$ and 4 does not divide $g+1$ and $\gcd(3, g+1) = 1$.

The second alternative is equivalent to $g \equiv 1$ or $9 \bmod 12$, and we get $\alpha(g) = 3(g+1)/2$. We have already shown that $\alpha(g) \leq o^*(g) \leq 3(g+1)/2$, hence $o^*(g) = \alpha(g)$ in this case.

The first alternative is equivalent to $g \equiv 3, 7, 11 \bmod 12$, and here $\alpha(g) = 3(g-1)/2$. By an argument similar to the one given above, the assumption $n > 3(g-1)/2$ can be led to a contradiction.

The next case to consider would be $g \equiv 5 \bmod 24$, when $k = 2$ and $\alpha(g) = \alpha_0(g) = 5(g-1)/4$. The assumption $n > 5(g-1)/4$ now leads to the estimate $1/m_1 + 1/m_2 > 1/5$, and we need the numerical result $o^*(g) = \alpha(g)$ for g up to 120 to reduce the possibilities for (m_1, m_2) to a manageable list, which can then be treated individually as above. The first time this line of argument fails to prove $o^*(g) = \alpha(g)$ is when $g \equiv 257 \bmod 8160$. □

For $g \equiv 257 \bmod 8160$ we may distinguish two subcases. The case $g \equiv 8417 \bmod 16320$ corresponds to $k = 5$ and $\alpha = \alpha_0$. The case $g \equiv 257 \bmod 16320$ corresponds to $k \geq 6$. (It is worth observing that the case $\alpha = \alpha_1$ with $k = 5$ or $k = 3$ never occurs; this is a simple exercise.) Here are examples for $\alpha(g) < o^*(g)$ with $k = 5$ and $k = 6$ (in both cases $g' = 1$, $l = 2$):

- $g = 57377$, $n = 60007$, $m_1 = 23$, $m_2 = 2609$, $\alpha(g) = 59169$;
- $g = 16577$, $n = 17197$, $m_1 = 29$, $m_2 = 593$, $\alpha(g) = 16835$.

Proof of Theorem 1.5. Here we consider odd g. Recall that the number k in the definition of $\alpha(g)$ is such that 2^k divides $g-1$. In particular, $g-1 \geq 2^k$. This inequality is equivalent to

$$\frac{2^k+1}{2^k}(g-1) \geq g,$$

with equality if and only if $g = 2^k + 1$, hence

$$o^*(g) \geq \alpha(g) \geq \frac{2^k+1}{2^k}(g-1) \geq g,$$

and $o^*(g) = g$ cannot occur unless $g = 2^k + 1$.

Lemma 5.3. *Assume $g = 2^k + 1$. If there is a natural number y with $1 \leq y \leq k-1$ and $\gcd(2^y+1, 2^{k-y}+1) = 1$, then $o^*(g) > g$.*

Proof. This can be proved simply by exhibiting a solution to (B1)–(B3) with $n > g$. Such a solution is given by

$$g' = 1, \quad l = 2, \quad m_1 = 2^y + 1, \quad m_2 = 2^{k-y} + 1,$$
$$n = \operatorname{lcm}(m_1, m_2) = m_1 m_2.$$

Alternatively, the estimate for $o^*(g)$ given before this lemma shows: a necessary condition for $o^*(2^k + 1) = 2^k + 1$ to hold is that $\alpha(g) = \alpha_0(g) = (2^k + 1)(g-1)/2^k$. But if $y \in \mathbb{N}$ exists as described in the lemma, then

$$\begin{aligned}\gcd(2^y+1, g+2^y-1) &= \gcd(2^y+1, 2^k+2^y) \\ &= \gcd(2^y+1, 2^{k-y}+1) = 1;\end{aligned}$$

hence, with y^* the smallest such y,

$$\alpha(g) = \frac{2^{y^*}+1}{2^{y^*}}(g + 2^{y^*} - 1) > \alpha_0(g). \qquad \square$$

Lemma 5.4. *Let $x, y \in \mathbb{N}$ with y a divisor of x. If x is an even multiple of y, then $2^y + 1$ divides $2^x - 1$. If x is an odd multiple of y, then $2^y + 1$ divides $2^x + 1$.*

Proof. We have

$$2^x - 1 = (2^y+1)(2^{x-y} - 2^{x-2y} + - \cdots + 2^y - 1)$$

in the first case, and

$$2^x + 1 = (2^y+1)(2^{x-y} - 2^{x-2y} + - \cdots - 2^y + 1)$$

in the second. □

We now complete the proof of Theorem 5. So suppose $o^*(g) = g$. We have already seen that this forces g to be of the form $g = 2^k + 1$. If g is not a Fermat number, then k can be written in the form $k = 2^j(2a+1)$ with $j \in \mathbb{N}_0$, $a \in \mathbb{N}$. Set $x = 2^j \cdot 2a$ and $y = 2^j$. Then $2^y + 1$ divides $2^x - 1 = 2^{k-y} - 1$ by the preceding lemma. Since all the prime factors of $2^y + 1$ are odd, this implies

$$\gcd(2^y+1, 2^{k-y}+1) = 1.$$

Thus $o^*(g) > g$ by Lemma 5.3.

The results $o^*(\mathcal{F}_j) = \mathcal{F}_j$ for $0 \leq j \leq 4$ and $o^*(\mathcal{F}_j) > \mathcal{F}_j$ for $5 \leq j \leq 9$ have been checked numerically. For instance, for $g = \mathcal{F}_5$ we can choose

$$\begin{aligned} m_1 &= 99, \\ m_2 &= 394\,435\,773, \\ n &= 4\,338\,793\,503 > \mathcal{F}_5 \end{aligned}$$

(and $g' = 1$, $l = 2$, as in the other examples of this section). □

Observe that Lemma 5.3 cannot be used to show $o^*(\mathcal{F}_j) > \mathcal{F}_j$, for both 2^y+1 and $2^{2^j-y}+1$ (with $1 \le y \le 2^j - 1$) are divisible by $2^{2^c}+1$, where 2^c is the largest power of 2 dividing y (use Lemma 5.4). This also implies that $\alpha(\mathcal{F}_j) = \alpha_0(\mathcal{F}_j) = \mathcal{F}_j$. So $g = \mathcal{F}_j$, $5 \le j \le 9$, are further examples where $o^*(g) > \alpha(g)$.

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