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PERIODIC SOLUTIONS OF DISSIPATIVE DYNAMICAL SYSTEMS WITH SINGULAR POTENTIALS

P. HABETS
UCL, Institut Mathématique, B-1348 Louvain-la-Neuve, Belgium

L. SANCHEZ
INIC/CMAF, Avenida Professor Gama Pinto, 2, P. 1699 Lisboa, Portugal

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Abstract. We give sufficient conditions for the existence of solutions of the periodic boundary value problem

$$\ddot{u} + \frac{d}{dt}[\nabla f(u)] + \nabla g(u) = h(t), \qquad u(0) = u(T), \quad \dot{u}(0) = \dot{u}(T),$$

where $u \in \mathbb{R}^n$. The potential g is supposed to be singular at the origin and to satisfy a condition similar to the "strong force" assumption of W. Gordon. An extension is worked out for systems with several singularities.

1. Introduction. We consider the periodic boundary value problem

$$\ddot{u} + \frac{d}{dt}[\nabla f(u)] + \nabla g(u) = h(t), \qquad u(0) = u(T), \quad \dot{u}(0) = \dot{u}(T), \tag{1.1}$$

under the general assumption that $g(u)$ becomes infinite as $u \to 0$ and look for solutions $u(t)$ of (1.1) that do not cross the singularity $u = 0$.

In the last two years there has appeared a rich literature on this problem in the absence of dissipation; i.e., when $f \equiv 0$. Such contributions go back to the work of Gordon [5] who introduced the "strong force" assumption which characterizes the behaviour of $g(u)$ near the singularity. In one way or another, this condition has been further explored, either for potentials of attractive type (see, e.g., [1], [2], [3], [6] and their references) or for potentials of repulsive type [4]. The method used in these papers is of variational character and the strong force condition is used to get some compactness for the functional associated with the equation.

Our approach to problem (1.1) is mainly based on degree theoretical methods, and covers both attractive and repulsive potentials together with complete dissipation. We also investigate a conservative case with a repulsive potential. To obtain existence of solutions, we embed (1.1) in a one-parameter family of homotopic equations with solutions for which we can obtain a priori estimates. It follows that a

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coincidence degree can be computed when g is replaced by some reference potential g_0 with a particularly simple structure. An important device to get our estimates, namely to bound solutions away from the singularity, consists in the use of a condition quite similar to the "strong force" introduced by Gordon [5] (see assumption (C-2)). Our results are in the spirit of Lazer and Solimini [8] who investigated the case $n = 1$ and $f \equiv 0$. Further results in the scalar case can be found in Habets and Sanchez [7].

The paper is organized as follows. In Section 2 we state our assumptions and main results. Proofs are collected in Section 3. Lemmas 1 to 6 display the role of our assumptions in obtaining the a priori estimates. In Lemma 7 a coincidence degree associated with a reference potential is computed. In the last section we give a theorem with two singularities of attractive type. This exemplifies how our approach can be used to deal with several singularities.

2. Assumptions and main results. Consider the periodic boundary value problem

$$\ddot{u} + \frac{d}{dt}(\nabla f)(u) + \nabla g(u) = h(t), \qquad u(0) = u(T), \quad \dot{u}(0) = \dot{u}(T), \tag{2.1}$$

where $h \in L^1(0,T)$, $f \in C^2(\mathbb{R}^n)$, $g \in C^1(\mathbb{R}^n \backslash \{0\})$ and $n > 1$. For convenience we state here the assumptions we shall use on h, f and g.

(A-1) There exists $a_1 > 0$ such that for any u

$$\frac{\partial^2 f}{\partial u^2}(u) \geq a_1 I \quad \text{or} \quad -\frac{\partial^2 f}{\partial u^2}(u) \geq a_1 I,$$

where $\partial^2 f/\partial u^2$ is the hessian matrix of f and I the identity matrix.

(A-2) There exists an increasing function $a_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $u \in \mathbb{R}^n \backslash \{0\}$

$$u^T \nabla g(u) \leq a_2(|u|).$$

(B-1) There exists $R > 0$ such that for any $|u| \geq R$

$$|\nabla g(u)| < |\overline{h}|,$$

where

$$\overline{h} = \frac{1}{T}\int_0^T h.$$

(B-2) There exists a function $b_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\lim_{s \to +\infty} b_2(s)/s = +\infty$$

and for all $u \in \mathbb{R}^n \backslash \{0\}$

$$|u^T \nabla g(u)| \geq b_2(|u|).$$

(C-1) $\lim_{u \to 0} |u^T \nabla g(u)| = +\infty$.

(C-2) **Strong force condition.** There exist a neighborhood $\mathcal{N}$ of $0 \in \mathbb{R}^n$, a function $\varphi \in C^1(\mathcal{N}\backslash\{0\}, \mathbb{R})$ and $c > 0$ such that:

(i) $\lim_{u \to 0} \varphi(u) = +\infty$;

(ii) $\forall u \in \mathcal{N}\backslash\{0\}$, $|\nabla \varphi(u)|^2 \leq c(|u^T \nabla g(u)| + 1)$.

Remark. Condition (C-2) is similar to the strong force condition introduced by Gordon [5], where $|u^T\nabla g(u)|$ is replaced by $|g(u)|$. It is easy to see that a radial potential $g(u) = |u|^{-\alpha}$ satisfies (C-2) if and only if $\alpha \geq 2$.

The main results of this paper are the following theorems.

Theorem 1. *Let $f \in C^2(\mathbb{R}^n)$, $g \in C^1(\mathbb{R}^n \backslash \{0\})$ and $h \in L^2(0,T)$. Assume* (A-1), (B-1), (C-1) *and* (C-2) *are satisfied. Then the problem* (2.1) *has at least one solution.*

Theorem 2. *Suppose $f \in C^2(\mathbb{R}^n)$ satisfies* (A-1), *$h \in L^2(0,T)$ and $g \in C^1(\mathbb{R}^n \backslash \{0\})$ satisfies* (B-2), (C-1), (C-2) *and*

$$\lim_{|u|\to\infty} u^T\nabla g(u) \cdot \lim_{u\to 0} u^T \nabla g(u) = -\infty.$$

Then if n is odd, the problem (2.1) *has at least one solution.*

Our third result refers to the case $f(u) \equiv 0$; i.e., to the periodic boundary value problem

$$\ddot{u} + \nabla g(u) = h(t), \qquad u(0) = u(T), \quad \dot{u}(0) = \dot{u}(T). \tag{2.2}$$

Theorem 3. *Let $g \in C^1(\mathbb{R}^n \backslash \{0\})$ and $h \in L^1(0,T)$. Assume* (A-2), (B-1), (C-1) *and* (C-2) *are satisfied. Then the problem* (2.2) *has at least one solution.*

3. Proofs.

In order to study (2.1) or (2.2) we shall consider the homotopies

$$\ddot{u} + \frac{d}{dt}\big(\nabla f(u)\big) + \nabla g_\lambda(u) = h_\lambda(t), \quad u(0) = u(T), \quad \dot{u}(0) = \dot{u}(T), \tag{3.1}$$

or

$$\ddot{u} + (1-\lambda)\dot{u} + \nabla g_\lambda(u) = h_\lambda(t), \quad u(0) = u(T), \quad \dot{u}(0) = \dot{u}(T), \tag{3.2}$$

where $\lambda \in [0,1]$, $g_\lambda(u) = \lambda g_1(u) + (1-\lambda)g_0(u)$, $g_1 \equiv g$, g_0 is some convenient potential and $h_\lambda(t) = \lambda h(t) + (1-\lambda)\bar{h}$. To apply degree theory, we have first to compute a priori bounds for the solutions of (3.1) and (3.2).

In what follows we use the symbol $\|u\|_p$ to denote the usual norm of a function $u \in L^p(0,T)$, $1 \leq p \leq \infty$.

Lemma 1. *Suppose $f \in C^2(\mathbb{R}^n)$, $g_i \in C^1(\mathbb{R}^n \backslash \{0\})$, $i = 0,1$, and $h \in L^2(0,T)$. If* (A-1) *is satisfied, there exists $N > 0$ such that for any $\lambda \in [0,1]$ and $u(t)$ solution of* (3.1) *one has*

$$\|\dot{u}\|_2 < N.$$

Proof: Multiplying (3.1) by $\dot{u}^T$ and integrating gives

$$a_1\|\dot{u}\|_2^2 \leq \pm \int_0^T \dot{u}^T \frac{\partial^2 f}{\partial u^2}(u)\dot{u} = \pm\lambda \int_0^T \dot{u}^T h \leq \|h\|_2\|\dot{u}\|_2.$$

Lemma 2. *Suppose $g_i \in C(\mathbb{R}^n\backslash\{0\})$, $i = 0,1$, satisfy* (A-2) *and $h \in L^1(0,T)$. Then for any $M > 0$, there exists $N > 0$ such that for any $\lambda \in [0,1]$ and $u(t)$ solution of* (3.2) *that satisfies $\|u\|_\infty \le M$, we have*

$$\|\dot{u}\|_2 < N.$$

Proof: Consider the inner product of (3.2) with $-u^T(t)$ and compute

$$\|\dot{u}\|_2^2 = \int_0^T u^T \nabla g_\lambda(u) - \int_0^T u^T h_\lambda \le a_2(M)T + M\|h\|_1.$$

Lemma 3. *Suppose $f \in C^2(\mathbb{R}^n)$, $g_i \in C^1(\mathbb{R}^n\backslash\{0\})$, $i = 0,1$, and $h \in L^1(0,T)$. If* (B-1) *is satisfied for g_0 and g_1, then for any N, there exists $M > 0$ such that for any $\lambda \in [0,1]$ and $u(t)$ solution of* (3.1) *that satisfies $\|\dot{u}\|_2 \le N$, we have*

$$\|u\|_\infty < M.$$

Proof: Assume $|u(t)| \ge R$ for all t. Integrating (3.1) this implies the contradiction

$$T|\bar{h}| = \Big|\int_0^T h_\lambda\Big| = \Big|\int_0^T \nabla g_\lambda(u)\Big| \le \lambda \int_0^T |\nabla g_0(u)| + (1-\lambda)\int_0^T |\nabla g_1(u)| < T|\bar{h}|.$$

Hence there exists $\bar{t} \in [0,T]$ such that $|u(\bar{t})| < R$ and

$$|u(t)| \le |u(\bar{t})| + \Big|\int_{\bar{t}}^t \dot{u}\Big| \le R + T^{1/2}N.$$

Lemma 4. *Suppose $g_i \in C^1(\mathbb{R}^n\backslash\{0\})$, $i = 0,1$, and $h \in L^1(0,T)$. If* (B-1) *is satisfied for g_0 and g_1, there exists $M > 0$ such that for any $\lambda \in [0,1]$ and $u(t)$ solution of* (3.2), *we have*

$$\|u\|_\infty < M.$$

Proof: Define $r(t) = |u(t)|$ and let t_0 be such that

$$r(t_0) = \max r(t) > R + 2\|h\|_1 T.$$

Extending r by periodicity we consider

$$t_1 = \sup\{t \in [t_0, t_0+T] \,|\, \forall s \in [t_0,t),\ r(s) > R\}.$$

If $t_1 < t_0 + T$, one computes

$$\ddot{r} = \frac{u^T}{r}(h_\lambda - \nabla g_\lambda) - (1-\lambda)\dot{r} + \frac{|\dot{u}|^2 - \dot{r}^2}{r} \ge -(|h_\lambda| + |\nabla g_\lambda|) - (1-\lambda)\dot{r}.$$

Integrating on $[t_0,t)$, one gets for $t \in (t_0,t_1]$

$$\dot{r}(t) \ge -\int_{t_0}^t (|h_\lambda| + |\nabla g_\lambda|) + (1-\lambda)\big(r(t_0) - r(t)\big) > -2\|h\|_1$$

and

$$r(t_1) > r(t_0) - 2\|h\|_1 T > R.$$

By continuity, there is a $t_1' > t_1$ such that $r(t) > R$ on $[t_0, t_1']$, which contradicts the definition of t_1. Hence $t_1 = t_0 + T$ and $\forall t \in [0,T]$, $r(t) > R$.

As in Lemma 3, we prove on the other hand, that there exists $\bar{t} \in [0,T]$ such that $|u(\bar{t})| \le R$, which is a contradiction.

Lemma 5. *Suppose $f \in C^2(\mathbb{R}^n)$, $g_i \in C^1(\mathbb{R}^n \backslash \{0\})$, $i = 0, 1$, satisfy* (B-2) *and in addition there exists $R > 0$ such that if $|u| \geq R$*

$$\left(u^T \nabla g_0(u)\right)\left(u^T \nabla g_1(u)\right) > 0. \tag{3.3}$$

Let $h \in L^1(0,T)$. Then for any $N > 0$, there exists $M > 0$ such that for any $\lambda \in [0,1]$ and $u(t)$ solution of (3.1) *that satisfies $\|\dot{u}\|_2 \leq N$, we have*

$$\|u\|_\infty < M.$$

Proof: Fix a constant $d > 0$ such that, for any $u \in H^1(0,T)$,

$$\|u\|_\infty \leq d(\|u\|_1 + \|u'\|_2)$$

and choose K so large that

$$K > (N^2 + \|h\|_1 dN)(\|h\|_1 dT)^{-1}, \qquad K > R$$

and $b_2(s) > 2\|h\|_1 ds$ whenever $s \geq K$. By condition (3.3) we may assume that (B-2) holds for g_λ uniformly in $\lambda \in [0,1]$. Assume that $|u(t)| \geq K$ for all $t \in [0,T]$. Then integrating the product of (3.1) with u one obtains

$$\int_0^T b_2(|u|) - N^2 \leq \Big| - \|\dot{u}\|_2^2 + \int_0^T u^T \nabla g_\lambda(u) \Big| \leq \Big| \int_0^T u^T h_\lambda \Big| \leq \|u\|_\infty \|h\|_1. \tag{3.4}$$

It follows that

$$2\|h\|_1 d\|u\|_1 - N^2 \leq \|h\|_1 d(\|u\|_1 + N)$$

and hence

$$\|h\|_1 dTK \leq \|h\|_1 d\|u\|_1 \leq N^2 + \|h\|_1 dN$$

contradicting our choice of K. Therefore there exists $\bar{t} \in [0,T]$ such that $|u(\bar{t})| < K$ so that for any $t \in [0,T]$

$$|u(t)| < K + \int_0^T |\dot{u}| \leq K + NT^{1/2} = M.$$

Lemma 6. *Let $f \in C^2(\mathbb{R}^n)$, $h \in L^1(0,T)$ and assume the functions $g_i \in C^1(\mathbb{R}^n \backslash \{0\})$, $i = 0, 1$ satisfy* (C-1), (C-2) *and*

$$\forall u \in \mathcal{N}, \quad \left(u^T \nabla g_0(u)\right)\left(u^T \nabla g_1(u)\right) > 0. \tag{3.5}$$

Then for any $N > 0$ and $M > 0$, we can find $m > 0$ such that for any $\lambda \in [0,1]$ and $u(t)$, solution of (3.1) *or* (3.2) *that satisfies*

$$\|\dot{u}\|_2 < N \quad \textit{and} \quad \|u\|_\infty < M$$

we have

$$\forall t \in [0,T], \quad |u(t)| > m.$$

Proof: Define $\beta > 0$ small enough so that $B(0,\beta) = \{u \in \mathbb{R}^n \mid |u| < \beta\} \subset \mathcal{N}$ and for any $|u| \leq \beta$

$$|u^T \nabla g_i(u)| > (M\|h\|_1 + N^2)/T = K_1.$$

It also follows that

$$|u^T \nabla g_\lambda(u)| > K_1. \tag{3.6}$$

Let us prove first there exists some $\bar{t} \in [0,T]$ such that $|u(\bar{t})| \geq \beta$. If not, the inner product of (3.1) or (3.2) with u gives

$$\Big|\int_0^T u^T \nabla g_\lambda(u)\Big| = \Big|\int_0^T h_\lambda^T u + \|\dot{u}\|_2^2\Big| \leq M\|h\|_1 + N^2 = K_1 T$$

which contradicts (3.6).

Next we shall prove that the function

$$\varphi_\lambda(u(t)) = \lambda\varphi_1(u(t)) + (1-\lambda)\varphi_0(u(t))$$

is bounded. Consider an interval $[t_1, t_2]$ such that

$$|u(t_1)| = \beta \quad \text{and} \quad \forall t \in (t_1, t_2),\ u(t) < \beta.$$

We can write

$$\varphi_\lambda(u(t)) = \varphi_\lambda(u(t_1)) + \int_{t_1}^t \dot{u}^T \nabla\varphi_\lambda(u) \leq K_2 + \Big(\int_{t_1}^t |\nabla\varphi_\lambda|^2\Big)^{1/2} \|\dot{u}\|_2, \tag{3.7}$$

where K_2 is such that for $i = 0, 1$ and $|u| = \beta$, $\varphi_i(u) \leq K_2$. From a convexity argument, (3.5) and (C-2), we deduce

$$\begin{aligned}\int_{t_1}^t |\nabla\varphi_\lambda|^2 &\leq \lambda\int_{t_1}^t |\nabla\varphi_1|^2 + (1-\lambda)\int_{t_1}^t |\nabla\varphi_0|^2 \\ &\leq c\Big[\lambda\int_{t_1}^t (|u^T\nabla g_1(u)| + 1) + (1-\lambda)\int_{t_1}^t (|u^T\nabla g_0(u)| + 1)\Big] \\ &\leq c\Big[T + \int_0^T |u^T\nabla g_\lambda(u)|\Big].\end{aligned} \tag{3.8}$$

On the other hand, it follows from (C-1) and (3.6) that for some $K_3 > 0$

$$0 \geq \big(u^T\nabla g_\lambda(u)\big)^- = \min\big(0, u^T\nabla g_\lambda(u)\big) \geq -K_3$$

or

$$0 \leq \big(u^T\nabla g_\lambda(u)\big)^+ = \max\big(0, u^T\nabla g_\lambda(u)\big) \leq K_3.$$

Also the inner product of (3.1) or (3.2) with u gives

$$\Big|\int_0^T u^T\nabla g_\lambda(u)\Big| \leq \Big|\int_0^T u^T h_\lambda\Big| + \|\dot{u}\|_2^2 \leq M\|h\|_1 + N^2.$$

Hence we have for some $K_4 > 0$

$$\int_0^T |u^T \nabla g_\lambda(u)| = \int_0^T u^T \nabla g_\lambda(u) - 2\int_0^T \left(u^T \nabla g_\lambda(u)\right)^- \leq K_4$$

or

$$\int_0^T |u^T \nabla g_\lambda(u)| = 2\int_0^T \left(u^T \nabla g_\lambda(u)\right)^+ - \int_0^T u^T \nabla g_\lambda(u) \leq K_4.$$

It follows then from (3.7) and (3.8) that for some $K_5 > 0$

$$\varphi_\lambda(u(t)) \leq K_5.$$

This proves that $u(t)$ cannot enter a ball $|u| \leq m$, where $m > 0$ is small enough so that for $i = 0, 1$ and $|u| \leq m$, $\varphi_i(u) > K_5$.

Remark. It is easy to see from the proof that if $\|\dot{u}\|_\infty$ is a priori bounded we only need to find an a priori bound on $\|\nabla \varphi\|_1$. This can be obtained from a condition such as

$$|\nabla \varphi(u)| \leq c\big(|u^T \nabla g(u)| + 1\big)$$

which is a weak force condition. This idea works out fine if $n = 1$ (see Habets and Sanchez [7]).

The Lemmas 1 to 6 provide the a priori bounds necessary to apply coincidence degree theory [9]. The next step will be to transform (3.1) and (3.2) into operator equations. First we introduce the linear operator

$$L : \mathrm{Dom}\, L \subset W^{2,1}(0,T) \to L^1(0,T), \quad u \mapsto Lu$$

defined by

$$\mathrm{Dom}\, L = \{u \in W^{2,1}(0,T) \mid u(0) = u(T),\ \dot{u}(0) = \dot{u}(T)\}, \quad Lu = \ddot{u}$$

and the Nemitski operator

$$N : [0,1] \times \overline{\Omega} \to L^1(0,T), \quad (\lambda, u) \mapsto N(\lambda, u),$$

where

$$\Omega = \{u \in H^1(0,T) \mid \forall t, m < |u(t)| < M,\ \|\dot{u}\|_2 < N\},$$

$M > m > 0$, $N > 0$ are constants and

$$N(\lambda, u) = h_\lambda - \frac{d}{dt}\big(\nabla f(u)\big) - \nabla g_\lambda(u)$$

or

$$N(\lambda, u) = h_\lambda - (1-\lambda)\dot{u} - \nabla g_\lambda(u).$$

With these notations, the equations (3.1) and (3.2) read

$$Lu = N(\lambda, u),$$

where the map N is L-compact. For $\lambda = 0$, this last equation is equivalent to

$$\ddot{u} + \frac{d}{dt}\big(\nabla \hat{f}(u)\big) + \nabla g_0(u) = \overline{h},$$
$$u(0) = u(T), \quad \dot{u}(0) = \dot{u}(T),$$

with $\hat{f}(u) = f(u)$ or $\hat{f}(u) = |u|^2/2$.

The next Lemma computes the coincidence degree

$$d_L\big(L - N(0,\cdot), \Omega\big).$$

Lemma 7. *Let $\hat{f} \in C^2(\mathbb{R}^n)$ satisfy* (A-1). *Assume $g_0 \in C^2(\mathbb{R}^n \backslash \{0\})$ is such that $g_0 - \bar{h}^T u$ has only a finite number of nondegenerate critical points $u_1, \ldots u_p$ in $D = \{u \in \mathbb{R}^n : m < |u| < M\}$ and no critical points on ∂D. Let μ_i denote the sum of the multiplicities of the positive eigenvalues of $(\partial^2 g/\partial u^2)(u_i)$. Then we have*

$$\big|d_L\big(L - N(0,\cdot), \Omega\big)\big| = \Big|\sum_{i=1}^{p} (-1)^{\mu_i}\Big|.$$

Proof: Using the formalism introduced in [9], we define the operators

$$P : W^{2,1}(0,T) \to \text{Ker } L, \quad u \mapsto Pu = \frac{1}{T}\int_0^T u,$$
$$Q : L^1(0,T) \to \text{co Im } L, \quad u \mapsto Qu = \frac{1}{T}\int_0^T u,$$

we introduce the isomorphism J : co Im $L \to$ Ker L, $u \mapsto u$ and denote by K : Im $L \to$ Dom L the generalized inverse of L. The coincidence degree is then defined up to its sign as

$$d_L\big(L - N(0,\cdot), \Omega\big) = d(M_1, \Omega, 0)$$

with

$$M_1 u = u - Pu - JQN(0,u) - K(I - Q)N(0,u).$$

Next we consider the homotopy

$$M_\lambda u = u - Pu - JQN(0,u) - \lambda K(I - Q)N(0,u). \tag{3.9}$$

For $\lambda \neq 0$, this equation is equivalent to

$$\ddot{u} + \lambda\frac{d}{dt}\big(\nabla \hat{f}(u)\big) + \lambda \nabla g_0(u) = \lambda \overline{h}, \qquad u(0) = u(T), \quad \dot{u}(0) = \dot{u}(T). \tag{3.10}$$

Multiplying (3.10) by $\dot{u}$ and integrating gives $\|\dot{u}\|_2 = 0$. Hence (3.10) has only constant solutions which are the critical points of $g_0(u) - u^T\overline{h}$. Similarly, when $\lambda = 0$, (3.9) reduces to

$$u = Pu + JQN(0,u)$$

and u is a critical point of $g_0(u) - u^T\overline{h}$.

It follows now from classical results in degree theory that

$$d(M_1,\Omega,0) = d(M_0,\Omega,0) = d_B\big(JQN(0,\cdot)|_{\text{Ker } L}, \Omega \cap \text{Ker } L, 0\big)$$

and from the definition of the Brouwer degree that

$$d_B\big(JQN(0,\cdot)|_{\text{Ker } L}, \Omega \cap \text{Ker } L, 0\big) = \sum (-1)^{\mu_i}.$$

Examples. 1. Take $g_0(u) = r^{-2}$, where $r = |u|$. The equation $\nabla g_0(u) = \overline{h}$ has exactly one solution for each $\overline{h} \neq 0$, which is a negative multiple of $\overline{h}$. A simple computation yields

$$\frac{\partial^2 g_0}{\partial u^2}(u) = -\frac{2}{r^4}\Big(I - 4\frac{uu^T}{r^2}\Big)$$

so that in an orthonormal basis $v_1, \dots, v_n$, where $v_1 = u/|u|$, this matrix is

$$\frac{2}{r^4}\begin{bmatrix} 3 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & -1 \end{bmatrix}. \tag{3.11}$$

2. If $g_0(u) = -r^{-2}$, a potential corresponding to the attractive source, we obtain $(\partial^2 g_0/\partial u^2)(u)$ by reversing the signs in (3.11).

3. Consider now the potential

$$g_0(u) = -\Big(\frac{1}{r^2} + r^2 + e_1^T u\Big),$$

where e_1 is a unit vector. The critical points of g_0 are $\overline{u} = -ae_1$, where $a > 1$ and $\overline{\overline{u}} = be_1$, where $0 < b < 1$. An easy calculation shows that in a convenient basis, $(\partial^2 g_0/\partial u^2)(u)$ is represented by the matrix

$$-2 \text{ diag}\Big(1 + \frac{3}{r^4}, 1 - \frac{1}{r^4}, \dots, 1 - \frac{1}{r^4}\Big)$$

so that at $\overline{u}$ there are no positive eigenvalues and at $\overline{\overline{u}}$ there are exactly $n-1$ positive ones.

Proof of Theorem 1: Let $g_1 = g$ and consider the homotopy (3.1) where we choose $g_0(u) = r^{-2}$ if $u^T\nabla g(u) < 0$ in a neighborhood of the origin, and $g_0(u) = -r^{-2}$ if $u^T\nabla g(u) > 0$ in such a neighborhood. It is easy to see that in both cases g_0 satisfies (B-1), (C-1), (C-2) and (3.5). Therefore we can apply Lemmas 1, 3 and 6 to prove that no solution of (3.1) is on the boundary of

$$\Omega = \{u \in H^1(0,T) : m < |u(t)| < M, \ \|\dot{u}\|_2 < N\}.$$

From Lemma 7 we compute

$$\big|d_L\big(L - N(\lambda,\cdot),\Omega\big)\big| = \big|d_L\big(L - N(0,\cdot),\Omega\big)\big| = 1$$

and the thesis follows from standard degree arguments.

Proof of Theorem 2: We consider the homotopy (3.1) together with $g_1 = g$ and $g_0(u) = \mp\big((1/r^2) + r^2 + e_1^T u\big) + \bar{h}^T u$. This potential satisfies (B-2), (C-1), (C-2), (3.3) and (3.5) so that we can apply Lemmas 1, 5 and 6. From Lemma 7, we compute

$$\big|d_L\big(L - N(0,\cdot), \Omega\big)\big| = 1 + (-1)^{n-1}$$

and the result follows.

Proof of Theorem 3: Here we consider the homotopy (3.2) with $g_1 = g$ and $g_0(u) = r^{-2}$. This potential satisfies (A-2), (B-1), (C-1), (C-2) and (3.5) so that we can apply Lemmas 4, 2 and 6. At last we compute from Lemma 7

$$\big|d_L\big(L - N(0,\cdot), \Omega\big)\big| = 1.$$

4. Systems with several singularities. Using the above techniques, we can investigate systems with several singularities. The following theorem exemplifies this remark.

Theorem 4. *Suppose $f \in C^2(\mathbb{R}^n)$ satisfies* (A-1). *Let $a \in \mathbb{R}^n$ and suppose $g \in C^1(\mathbb{R}^n \backslash \{-a, a\})$ satisfies:*

(B′-2) there exists a function $b_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\lim_{s \to +\infty} b_2(s)/s = +\infty$$

and for all $u \in \mathbb{R}^n \backslash \{-a, a\}$

$$u^T \nabla g(u) \geq b_2(|u|);$$

(C′-1) $\lim_{u \to \pm a} (u \mp a)^T \nabla g(u) = +\infty$;

(C′-2) there exists a $\beta > 0$, a function $\varphi \in C^1\big(B(\pm a, \beta) \backslash \{\pm a\}, \mathbb{R}\big)$ and $c > 0$ such that

(i) $\lim_{u \to \pm a} \varphi(u) = +\infty$,

(ii) $\forall u \in B(\pm a, \beta) \backslash \{\pm a\}$, $|\nabla \varphi(u)|^2 \leq c(|(u \mp a)^T \nabla g(u)| + 1)$.

At last, suppose $h \in L^2(0, T)$.

Then the problem (2.1) *has at least one solution.*

Proof: Consider the homotopy (3.1) where

$$g_0(u) = -\frac{1}{|u-a|^2} - \frac{1}{|u+a|^2} + u^2 + u^T(2a + \overline{h}).$$

Adapting Lemmas 1, 5 and 6, one proves that for appropriate choices of m, M and N, there is no solution of (3.3) on the boundary of

$$\Omega = \{u \in H^1(0,T) \mid \|u\|_\infty < M;\ \forall t,\ |u(t) \pm a| > m;\ \|\dot{u}\|_2 < N\}.$$

Also we prove as in Lemma 7 that

$$\big|d_L\big(L - N(0,\cdot), \Omega\big)\big| = 1.$$

To this end, it is sufficient to see that $g_0(u) - u^T h$ has only one critical point. The rest of the proof follows as above.

Remark. Notice that Theorem 4 is a counterpart of Theorem 2. This last result holds only for n odd. Here the introduction of a second singularity, more generally of an even number of them, allows us to deal with all values of n.

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