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QUADRATIC GROWTH OF SOLUTIONS OF FULLY NONLINEAR SECOND ORDER EQUATIONS IN $\mathbb{R}^n$

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Abstract. Fully nonlinear (degenerate) elliptic and parabolic equations of second order in $\mathbb{R}^N$ of the forms $u + F(D^2u) = f(x)$ and $u_t + F(D^2u) = 0$, where F is a nonincreasing function from the symmetric $N \times N$ matrices equipped with their usual ordering to $\mathbb{R}$, are considered. Existence and uniqueness theorems are proved in the class of solutions of quadratic growth when the data (f in the elliptic case and the initial data in the parabolic case) have suitable properties. In the parabolic case, a semiflow is obtained and the "inverse problem" of determining properties of similar flows necessary and sufficient to guarantee that they are the time t maps for such an equation is solved.

Introduction. The simplest form the Cauchy problem for a fully nonlinear (possibly degenerate) parabolic equation may have is

$$u_t + F(D^2u) = 0 \quad \text{on } (0,\infty) \times \mathbb{R}^n, \quad u(0,x) = \psi(x) \quad \text{for } x \in \mathbb{R}^n, \tag{CP}$$

in which $\mathcal{S}^n$ is the set of real symmetric $n \times n$ matrices equipped with its usual order and $F : \mathcal{S}^n \to \mathbb{R}$ is continuous and satisfies the "ellipticity" condition

$$F(B) \le F(A) \quad \text{when } A, B \in \mathcal{S}^n \quad \text{and} \quad A \le B. \tag{0.1}$$

We are interested in results concerning (CP) which place no further restrictions on F beyond continuity and (0.1); that is, we seek to determine a natural large space of functions on $\mathbb{R}^n$ such that for each ψ in this space (CP) has a solution u on $[0,\infty) \times \mathbb{R}^n$ satisfying conditions which guarantee its uniqueness. Our resolution of this problem involves the space $QUC(\mathbb{R}^n) = Q(\mathbb{R}^n) + UC(\mathbb{R}^n)$ of sums of pure quadratic functions on $\mathbb{R}^n$ (the elements of $Q(\mathbb{R}^n)$) and uniformly continuous functions on $\mathbb{R}^n$ (the elements of $UC(\mathbb{R}^n)$). Theorem 2.4 below asserts that for each $\psi \in QUC(\mathbb{R}^n)$, (CP) has a unique solution u which grows at most quadratically in x and that

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$u(t,\cdot) \equiv S(t)\psi$ defines a semigroup $S(t)$ on $QUC(\mathbb{R}^n)$. We are also interested in the correspondence between $S(t)$ and F and determine, in Theorem 3.6 below, properties enjoyed by $S(t)$ which guarantee that it arises as the time t map for (CP). The generality of these results is possible only when solutions are interpreted in the "viscosity sense" and all the proofs use of theory of viscosity solutions of fully nonlinear equations (see [14], [12], [13], [2], [3], [5] and the references therein).

Nonlinear semigroup theory, the structure of the theory of viscosity solutions and good pedagogy suggest that before treating (CP) one should discuss the related elliptic "stationary" version

$$u + F(D^2u) = f(x) \quad \text{in } \mathbb{R}^n \tag{S}$$

in the same spirit. Indeed, in Section 2 it is shown that if $f \in QUC(\mathbb{R}^n)$ then (S) has a unique solution u with at most quadratic growth and, moreover, $u \in QUC(\mathbb{R}^n)$.

We remark that quadratically growing data ψ, f which do not belong to the class $QUC(\mathbb{R}^n)$ are also considered in the course of discussion in Sections 1 and 2 below; however, we have been unable to succeed in this without placing an additional restriction on F. It is not known at the present time whether this is an artifact of our proofs or if the results in fact fail for general F's.

As regards earlier related works, Ishii [12] treats generalizations of (S) of the form $G(x,u,Du,D^2u) = 0$ on more general unbounded sets and incorporates Dirichlet boundary conditions in the presence of boundary. His results imply (upon using remarks made in the current work – in particular, the class $QUC(\mathbb{R}^n)$ is not introduced in [12]) that (S) is solvable for $u \in QUC(\mathbb{R}^n)$ when $f \in QUC(\mathbb{R}^n)$; however, the uniqueness of solutions with quadratic growth is not immediate from [12] and we may emphasize a further difference between results of [12] and the current work by remarking that more general equations $u + G(Du, D^2u) = f(x)$ are not necessarily solvable even if f is a pure quadratic (see [7]). Some growth questions are considered for equations of the form $u + H(Du) + F(D^2u) = f(x)$ in [8]. The "inverse problem" for first order equations corresponding to that of identifying properties of S which guarantee that it arises as the time t map for the first order variant of (CP) was solved in [15]. Recently, interesting new applications of viscosity solutions have appeared in [1], [9].

1. The stationary problem.

We begin with the problem (S) and are concerned with the uniqueness and existence of continuous viscosity solutions of (S). We will not recall the definitions of viscosity subsolutions, supersolutions and solutions at this point; we use the conventions stated in [2], the results of which are used in our proofs in any case. (We remark that the results of [2] have been extended and simplified in [3]). At issue is the discovery of appropriate restrictions to place upon F, f and u so that (S) has a unique solution with the desired properties. It is worthwhile to reintroduce the spaces mentioned in the introduction more formally. The spaces below all consist of functions $g : \mathbb{R}^n \to \mathbb{R}$; we denote the Euclidean inner-product of x, $y \in \mathbb{R}^n$ by $\langle x, y\rangle$ and set:

$$UC(\mathbb{R}^n) = \{g : g \text{ is uniformly continuous}\},$$

$$Q(\mathbb{R}^n) = \{g : \exists A \in \mathcal{S}^n \ni g(x) = \langle Ax, x\rangle \quad \text{for } x \in \mathbb{R}^n\}$$

where $\langle x, y\rangle$ denotes the Euclidean inner–product of x, $y \in \mathbb{R}^n$,

$$QL(\mathbb{R}^n) = \{g : \exists \text{ a modulus } \omega \ni$$
$$|g(x) - g(y)| \le \omega((1 + |x|^2 + |y|^2)^{1/2}|x - y|) \text{ for } x,\, y \in \mathbb{R}^n\},$$

where a modulus is a continuous subadditive function on $[0, \infty)$ vanishing at 0,

$$QUC(\mathbb{R}^n) = Q(\mathbb{R}^n) + UC(\mathbb{R}^n)$$

and

$$QG(\mathbb{R}^n) = \left\{g : \exists C \ni |g(x)| \le C(1 + |x|^2) \text{ for } x \in \mathbb{R}^n\right\}.$$

The mneumonics here are evident; in particular, $Q(\mathbb{R}^n)$ is the set of pure quadratic functions, $QL(\mathbb{R}^n)$ is the set of functions possessing "quadratic like" moduli of continuity, $QUC(\mathbb{R}^n)$ is self-explanatory and $QG(\mathbb{R}^n)$ is the set of functions with at most quadratic growth. Observe that $QUC(\mathbb{R}^n) \subset QL(\mathbb{R}^n) \subset QG(\mathbb{R}^n)$ and if $g \in QUC(\mathbb{R}^n)$, then g has a unique decomposition $g = q + h$ with $q \in Q(\mathbb{R}^n)$ and $h \in UC(\mathbb{R}^n)$ – indeed,

$$q(x) = \lim_{\lambda\to\infty} \frac{g(\lambda x)}{\lambda^2}.$$

We will prove:

Theorem 1.1. *Let F be continuous and* (0.1) *hold.*

(i) *If $f \in UC(\mathbb{R}^n)$, then* (S) *has a unique solution $u \in QG(\mathbb{R}^n)$ which in fact satisfies $u \in UC(\mathbb{R}^n)$.*

(ii) *If $f \in QUC(\mathbb{R}^n)$ then* (S) *has a unique solution $u \in QG(\mathbb{R}^n)$ which in fact satisfies $u \in QUC(\mathbb{R}^n)$.*

(iii) *If $f \in QL(\mathbb{R}^n)$ and F has the property that for $K \ge 0$*

$$\limsup_{r\downarrow 0} \{|F(Z + P) - F(Z)| : Z,\, P \in \mathcal{S}^n,\ \|Z\|\,\|P\| \le K,\ \|P\| \le r\} = 0, \tag{1.2}$$

then (S) *has a unique continuous solution $u \in QG(\mathbb{R}^n)$ which in fact satisfies $u \in QL(\mathbb{R}^n)$.*

Remark 1.2. It is easy to give counterexamples to existence for equations $u + F(D^2u) = f(x)$ when f has faster than quadratic growth. For example, if $F : \mathbb{R} \to \mathbb{R}$ is nondecreasing, nonnegative and

$$\lim_{M\to\infty} \frac{F(M)}{M^2} = \infty, \tag{1.3}$$

then $u + F(-u'') = -x^4$ has no subsolutions (we take $n = 1$ here). Indeed, clearly any subsolution satisfies $u \le -x^4$ since $F \ge 0$ and so for each $M > 0$, $u(x) + Mx^2$ has a maximum at some point $\hat{x}$. Then $u(\hat{x}) \ge u(0) - M\hat{x}^2$ and, if u is a subsolution, $u(0) - M\hat{x}^2 + F(2M) \le u(\hat{x}) + F(2M) \le -\hat{x}^4$, which implies that $-\frac{1}{2}M^2 + F(2M) \le -u(0)$ which in turn contradicts (1.3) for large M. Thus, this equation has no subsolutions on $\mathbb{R}$.

It should be remarked that some of the methods we use below are variants of ones developed in [10], [5] and [6] for the first order case; see also [12]. We begin with a lemma.

Lemma 1.3. *Let F be continuous,* (0.1) *hold, a, b, $c \geq 0$ and f satisfy*

$$|f(x) - f(y)| \leq a + b|x-y|^2 + c(|x|^2 + |y|^2) \quad \text{for } x, y \in \mathbb{R}^n. \tag{1.4}$$

If u is an upper semicontinuous subsolution of (S), *v is a lower semicontinuous supersolution of* (S), *$u^+ \in QG(\mathbb{R}^n)$ and $v^- \in QG(\mathbb{R}^n)$, then there is a constant $\hat{a}$ such that*

$$u(x) - v(y) - \left(2b|x-y|^2 + 2c(|x|^2 + |y|^2)\right) \leq \hat{a} \quad \text{for } x, y \in \mathbb{R}^n. \tag{1.5}$$

Proof: By the assumptions on u, v there is a constant $K > 0$ such that

$$u(x) - v(y) \leq K\left(1 + |x|^2 + |y|^2\right) \quad \text{for } x, y \in \mathbb{R}^n. \tag{1.6}$$

For $R > 1$ choose a radial C^2 function $\beta_R : \mathbb{R}^n \to [0, \infty)$ such that

$$\begin{cases} \liminf\limits_{|x| \to \infty} \dfrac{\beta_R(x)}{|x|^2} \geq 2K, \\ \beta_R(x) = 0 \ \text{ for } |x|^2 \leq R^2 \ \text{ and} \\ D^2\beta_R \text{ is bounded on } \mathbb{R}^n \text{ independently of } R. \end{cases} \tag{1.7}$$

Set

$$\Phi(x,y) = u(x) - v(y) - \left(2b|x-y|^2 + 2c(|x|^2 + |y|^2) + \beta_R(x) + \beta_R(y)\right).$$

In view of (1.6) and (1.7), Φ will attain its maximum on $\mathbb{R}^n \times \mathbb{R}^n$ at some point $(\hat{x}, \hat{y})$; we assume, without loss of generality for what follows, that $\Phi(\hat{x}, \hat{y}) \geq 0$ so

$$2b|\hat{x} - \hat{y}|^2 + 2c(|\hat{x}|^2 + |\hat{y}|^2) \leq u(\hat{x}) - v(\hat{y}). \tag{1.8}$$

In the notation of [2], it is immediate from [2, Theorem 1] that the fact that $(\hat{x}, \hat{y})$ is a maximum of $\hat{u}(x) - \hat{v}(y) - 2b|x-y|^2$ where $\hat{u}(x) = u(x) - (2c|x|^2 + \beta_R(x))$, $\hat{v}(y) = v(y) + 2c|y|^2 + \beta_R(y)$ guarantees that there are X, $Y \in \mathcal{S}^n$ such that

$$\begin{cases} X \leq Y, \ \|X\|, \|Y\| \leq 16b, \\ \left(u(\hat{x}), 4b(\hat{x}-\hat{y}) + 4c\hat{x} + D\beta_R(\hat{x}), X + 4cI + D^2\beta_R(\hat{x})\right) \in \overline{\mathcal{D}}^{2,+}u(\hat{x}), \text{ and} \\ \left(v(\hat{y}), 4b(\hat{x}-\hat{y}) - 4c\hat{y} - D\beta_R(\hat{y}), Y - 4cI - D^2\beta_R(\hat{y})\right) \in \overline{\mathcal{D}}^{2,-}v(\hat{y}). \end{cases} \tag{1.9}$$

Since u is a subsolution and v is a supersolution of (S) we then have

$$\begin{cases} u(\hat{x}) \leq f(\hat{x}) - F(X + 4cI + D^2\beta_R(\hat{x})), \\ v(\hat{y}) \geq f(\hat{y}) - F(Y - 4cI - D^2\beta_R(\hat{y})) \end{cases} \tag{1.10}$$

where I is the identity matrix. Using (1.8) and (1.10) we obtain

$$\begin{aligned} u(\hat{x}) - v(\hat{y}) &\leq 2\left(f(\hat{x}) - f(\hat{y}) - (b|\hat{x}-\hat{y}|^2 + c(|\hat{x}|^2 + |\hat{y}|^2))\right) \\ &\quad + 2\left(F(Y - 4cI - D^2\beta_R(\hat{y})) - F(X + 4cI + D^2\beta_R(\hat{x}))\right). \end{aligned} \tag{1.11}$$

In view of (0.1), (1.4) and (1.9), (1.11) yields

$$\begin{cases} \text{(i)} \ u(\hat{x}) - v(\hat{y}) \le 2a + d \text{ where} \\ \text{(ii)} \ d = \sup\limits_{\substack{Y \in \mathcal{S}^n, \, \|Y\| \le 16b \\ \hat{x}, \hat{y} \in \mathbb{R}^n}} 2\left(F(Y - 4cI - D^2\beta_R(\hat{y})) - F(Y + 4cI + D^2\beta_R(\hat{x}))\right). \end{cases} \tag{1.12}$$

Finally, the relation

$$\Phi(x,y) \le \Phi(\hat{x},\hat{y}) \le u(\hat{x}) - v(\hat{y})$$

in conjunction with (1.12), the definition of Φ and (1.7) implies, in the limit $R \to \infty$, that (1.5) holds with $\hat{a} = 2a + d$ (where d is given by (1.12)).

Proof of Theorem 1.1: We begin with (i). If $f \in UC(\mathbb{R}^n)$, then for every $\gamma > 0$ there is an $M_\gamma \ge 0$ such that $|f(x) - f(y)| \le \gamma + M_\gamma |x - y|$ and so, for $\epsilon > 0$

$$|f(x) - f(y)| \le \gamma + M_\gamma |x - y| \le \gamma + \frac{1}{2} M_\gamma \epsilon + \frac{1}{2\epsilon} M_\gamma |x - y|^2. \tag{1.13}$$

Assuming that u, $v \in QG(\mathbb{R}^n)$ are sub and super solutions of (S) as in Lemma 1.3, we learn from the lemma (with $b = 0$, $c = \frac{1}{2\epsilon} M_\gamma$) that $u(x) - v(y) - \frac{1}{\epsilon} M_\gamma |x - y|^2$ is bounded. Using this information, we repeat the proof of the lemma replacing β_R by $\lambda |x|^2$ where $\lambda > 0$ is arbitrary; this leads at once to the estimate

$$u(x) - v(y) - \frac{1}{\epsilon} M_\gamma |x - y|^2 - \lambda(|x|^2 + |y|^2) \le 2\gamma + M_\gamma \epsilon + d(\gamma, \epsilon, \lambda) \tag{1.14}$$

where now (1.12) (ii) has become

$$d(\gamma, \epsilon, \lambda) = \sup_{Z \in \mathcal{S}^n, \, \|Z\| \le \frac{16}{\epsilon} M_\gamma} 2\left(F(Z - 2\lambda I) - F(Z + 2\lambda I)\right). \tag{1.15}$$

Since $\lim_{\lambda \downarrow 0} d(\gamma, \epsilon, \lambda) = 0$, we may put $x = y$ in (1.14) and then take the iterated limit $\lambda \downarrow 0$, $\epsilon \downarrow 0$, $\gamma \downarrow 0$ to conclude that $u \le v$. Moreover, if $u = v$ is a solution, (1.14) implies (upon taking the limit $\lambda \downarrow 0$) that $u \in UC(\mathbb{R}^n)$.

We have now shown that solutions $u \in QG(\mathbb{R}^n)$ are unique and lie in $UC(\mathbb{R}^n)$ via a comparison result assuming only the appropriate growth and semicontinuity properties. In these circumstances, Perron's method (see [11], [12], [13]) provides us with the desired solution provided we can produce a subsolution U and a supersolution $V \ge U$ with quadratic growth. Since $f \in UC(\mathbb{R}^n)$, f grows at most linearly and therefore one easily computes constants K, L such that $U = -(K + L|x|^2)$, $V = -U$ have the desired properties.

We pass to the case (ii), $f \in QUC(\mathbb{R}^n)$. Let $f = q + h$ where $q \in Q(\mathbb{R}^n)$ and $h \in UC(\mathbb{R}^n)$; say $q(x) = \langle Ax, x \rangle$ where $A \in \mathcal{S}^n$. It is just a definition check to see that u is a solution of (S) if and only if $w = u - q$ is a solution of

$$w + G(D^2 w) = h \tag{1.16}$$

where

$$G(B) = F(B + 2A) \quad \text{for } B \in \mathcal{S}^n. \tag{1.17}$$

By (i), (1.16) has a solution $w \in UC(\mathbb{R}^n)$ which is unique in $QG(\mathbb{R}^n)$ and then $u = w + q \in QUC(\mathbb{R}^n)$ is a solution of (S). The uniqueness of u in $QG(\mathbb{R}^n)$ follows from the uniqueness of solutions of (1.16) from (i).

We pass to the case (iii) where $f \in QL(\mathbb{R}^n)$. In this case, for every $\gamma > 0$ there is an M_γ such that $|f(x) - f(y)| \leq \gamma + M_\gamma |x-y|(1+|x|^2+|y|^2)^{1/2}$, so for $\epsilon > 0$

$$|f(x) - f(y)| \leq \gamma + \frac{1}{2}\epsilon M_\gamma + \frac{1}{2\epsilon} M_\gamma |x-y|^2 + \frac{1}{2}\epsilon M_\gamma (|x|^2 + |y|^2). \tag{1.18}$$

Letting u and v be a subsolution and a supersolution as before, Lemma 1.3 implies that $u(x) - v(y) - (\frac{1}{\epsilon} M_\gamma |x-y|^2 + \epsilon M_\gamma(|x|^2+|y|^2))$ is bounded and thus $u(x)-v(y) - (\frac{1}{\epsilon} M_\gamma |x-y|^2 + 2\epsilon M_\gamma(|x|^2+|y|^2))$ attains its maximum on $\mathbb{R}^n \times \mathbb{R}^n$. Proceeding as above, we end up with an estimate

$$u(x) - v(y) - \left(\tfrac{1}{\epsilon} M_\gamma |x-y|^2 + 2\epsilon M_\gamma (|x|^2 + |y|^2)\right) \leq 2\gamma + \epsilon M_\gamma + d(\epsilon, \gamma) \tag{1.19}$$

with

$$d(\epsilon,\gamma) = 2 \sup_{\|Z\| \leq \frac{16}{\epsilon} M_\gamma} \left(F(Z - 4\epsilon M_\gamma I) - F(Z + 4\epsilon M_\gamma I)\right). \tag{1.20}$$

In view of (1.2), $\lim_{\epsilon \downarrow 0} d(\epsilon,\gamma) = 0$ and the comparison $u \leq v$ then follows upon taking the limit $\epsilon \downarrow 0$, $\gamma \downarrow 0$.

If $u = v$ above, the estimate (1.19) implies that $u(x) - u(y) \to 0$ as $(1+|x|^2+|y|^2)^{1/2}|x-y| \to 0$; to see this, fix $\gamma > 0$, put $\epsilon = |x-y|/(1+|x|^2+|y|^2)^{1/2}$ in (1.19), notice that $\epsilon \leq |x-y|(1+|x|^2+|y|^2)^{1/2}$ and conclude that

$$\limsup_{r \downarrow 0} \{|u(x) - u(y)| : |x-y|(1+|x|^2+|y|^2)^{1/2} \leq r\} \leq \gamma,$$

thereby establishing that any solution $u \in QG(\mathbb{R}^n)$ of (S) in fact belongs to $QL(\mathbb{R}^n)$. Existence follows from Perron's method and uniqueness as before.

Remark 1.4. Of course, there is an analogous result for functions with other power like moduli of continuity. Suppose $\alpha < 1$ and

$$|f(x) - f(y)| \leq \omega\left((1+|x|^2+|y|^2)^{\alpha/2}|x-y|\right) \tag{1.21}$$

for some modulus ω; the corresponding condition on F is

$$\limsup_{r \downarrow 0} \left\{|F(Z+P) - F(Z)|; Z, P \in \mathcal{S}^n, \|Z\|^{1/\alpha}\|P\| \leq K, \|P\| \leq r\right\} = 0. \tag{1.22}$$

When (1.21) and (1.22) hold, (S) has a unique continuous solution $u \in QG(\mathbb{R}^n)$. To see this, observe that (1.21) and $\alpha < 1$ imply that for each $\gamma > 0$ there is an M_γ such that for $\delta > 0$

$$\begin{aligned} |f(x) - f(y)| &\leq \gamma + \tfrac{1}{2}\epsilon M_\gamma + \tfrac{1}{2\epsilon} M_\gamma |x-y|^2 + \tfrac{1}{2}\epsilon M_\gamma (|x|^2+|y|^2)^\alpha \\ &\leq \gamma + \tfrac{1}{2}\epsilon M_\gamma + \frac{1-\alpha}{2}(\delta M_\gamma)^{1/(1-\alpha)} + \tfrac{1}{2\epsilon} M_\gamma |x-y|^2 \tfrac{\alpha}{2}\left(\tfrac{\epsilon}{\delta}\right)^{1/\alpha}(|x|^2+|y|^2). \end{aligned} \tag{1.23}$$

When this information is played into the uniqueness proof above it yields, in place of (1.14),

$$u(x) - v(y) - \left(\frac{1}{\epsilon}M_\gamma|x-y|^2 + \alpha(\frac{\epsilon}{\delta})^{1/\alpha}(|x|^2+|y|^2)\right)$$
$$\le 2\gamma + \epsilon M_\gamma + (1-\alpha)(\delta M_\gamma)^{1/(1-\alpha)} + d(\epsilon,\delta,\gamma)$$

with

$$d(\epsilon,\delta,\gamma) = 2 \sup_{\|Z\|\le 16M_\gamma/\epsilon} \left(F(Z - 2\alpha(\frac{\epsilon}{\delta})^{1/\alpha}I) - F(Z + 2\alpha(\frac{\epsilon}{\delta})^{1/\alpha}I)\right)$$

so $d(\epsilon,\delta,\gamma) \to 0$ as $\epsilon \downarrow 0$ for fixed δ, $\gamma > 0$. The comparison follows in the iterated limit $\epsilon \downarrow 0$ then $\delta \downarrow 0$ then $\gamma \downarrow 0$, and we leave the rest to the reader.

Remark 1.5. Let us review the steps of the above proofs in the context of the more general equation

$$u + F(x, D^2u) = 0 \tag{1.24}$$

where $F(x,\cdot)$ satisfies (0.1) for $x \in \mathbb{R}^n$. Assume that u, v are a subsolution and a supersolution of (1.24) such that $u(x) - v(y)$ satisfies (1.6). An analogue of Lemma 1.3 in this situation would provide a bound on suitable expressions $\Phi(x,y) = u(x) - v(y) - (B|x-y|^2 + C(|x|^2+|y|^2))$. Parallel arguments to the proof of Lemma 1.3 in this case lead to an estimate

$$u(x) - v(y) - \big(B|x-y|^2 + C(|x|^2+|y|^2) + \beta_R(x) + \beta_R(y)\big) \le d \tag{1.25}$$

where $(\hat{x},\hat{y})$ is a maximum of $u(x) - v(y) - (B|x-y|^2 + C(|x|^2+|y|^2))$ and

$$\begin{aligned} d = \frac{1}{\alpha}\big(&F(\hat{y}, Y - 2CI - D^2\beta_R(\hat{y})) - F(\hat{x}, X + 2CI + D^2\beta_R(\hat{x})) \\ &- (1-\alpha)\big(B|\hat{x}-\hat{y}|^2 + C(|\hat{x}|^2+|\hat{y}|^2)\big)\big), \end{aligned} \tag{1.26}$$

$\alpha \in (0,1)$ is arbitrary and X, $Y \in \mathcal{S}^n$ satisfy the following relation (see [2, Theorem 1])

$$-16B\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y\end{pmatrix} \le 8B\begin{pmatrix} I & -I \\ -I & I\end{pmatrix}. \tag{1.27}$$

Note that (1.27) implies $X \le Y$ and $\|X\|$, $\|Y\| \le 16B$. Assuming the uniform continuity condition that for each $R > 0$

$$\lim_{r\downarrow 0}\sup\{|F(x,X+P) - F(x,X)| : X,P \in \mathcal{S}^n,\ x\in\mathbb{R}^n,\ \|X\|\le R,\ \|P\|\le r\} = 0 \tag{1.28}$$

we will merely require a bound on

$$F(\hat{y},Y) - F(\hat{x},X) - (1-\alpha)\big(B|\hat{x}-\hat{y}|^2 + C|\hat{x}|^2 + |\hat{y}|^2)\big) \tag{1.29}$$

subject to (1.27). This we postulate with $C = 0$, corresponding to the case $f \in QUC(\mathbb{R}^n)$; that is, we assume there exists $B > 0$, $\alpha \in (0,1)$, $K > 0$ such that

$$F(y,Y) - F(x,X) - (1-\alpha)B|x-y|^2 \le K \quad \text{when (1.27) holds.} \tag{1.30}$$

To continue, we consider a maximum $(\hat{x},\hat{y})$ of $u(x)-v(y)-(B|x-y|^2+\lambda(|x|^2+|y|^2))$ as before and conclude that

$$u(x) - v(y) - (B|x - y|^2 + \lambda(|x|^2 + |y|^2)) \leq d_1 \tag{1.31}$$

where now we choose

$$d_1 = \frac{1}{\alpha}\left(F(\hat{y}, Y - 2\lambda I) - F(\hat{x}, X + 2\lambda I)\right)$$

and (1.27) still holds. Assuming (1.28), we deduce

$$d_1 \leq \frac{1}{\alpha}\left(F(\hat{y}, Y) - F(\hat{x}, X)\right) + \delta(\lambda, B) \tag{1.32}$$

where $\delta(\lambda, B) \to 0$ as $\lambda \downarrow 0$ for fixed $B > 0$. Finally, we assume the standard condition

$$F(y, Y) - F(x, X) \leq g(B|x - y|^2 + |x - y|) \tag{1.33}$$

for some function g satisfying $g(0+) = 0$. To conclude, we need a lemma:

Lemma 1.6. *Let $u, v\colon \mathbb{R}^n \to \mathbb{R}$. For $B > 0$ set*

$$M_B = \sup_{x,y\in\mathbb{R}^n} \{u(x) - v(y) - B|x - y|^2\}. \tag{1.34}$$

If $M_B < \infty$ for some $B > 0$ and x_B, y_B are such that

$$M_B - \left(u(x_B) - v(y_B) - B|x_B - y_B|^2\right) \to 0$$

then

$$B|x_B - y_B|^2 \to 0 \text{ as } B \to \infty. \tag{1.35}$$

Proof: It is clear that M_B decreases as B increases so $\lim_{B\to\infty} M_B$ exists. Moreover, putting

$$M_B - \left(u(x_B) - v(y_B) - B|x_B - y_B|^2\right) = \delta_B$$

we have

$$M_B \leq u(x_B) - v(y_B) - \tfrac{B}{2}|x_B - y_B|^2 - \tfrac{B}{2}|x_B - y_B|^2 + \delta_B \leq M_{B/2} - \tfrac{B}{2}|x_B - y_B|^2 + \delta_B$$

and so

$$\tfrac{B}{2}|x_B - y_B|^2 \leq M_{B/2} - M_B + \delta_B$$

which implies (1.35).

Applying the lemma, we see that $B|\hat{x}-\hat{y}|^2 + |\hat{x}-\hat{y}| \leq \gamma(\lambda, B)$ where $\gamma(\lambda, B) \to 0$ in the iterated limit $\lambda \downarrow 0$, $B \to \infty$. Now (1.31), (1.32), (1.33) imply that

$$u(x) - v(y) - \left(B|x - y|^2 + \lambda(|x|^2 + |y|^2)\right) \leq g(\gamma(\lambda, B)) + \delta(\lambda, B).$$

We conclude upon setting $x = y$ and taking the iterated limit. The proof also shows

2. The Cauchy problem. We consider the Cauchy problem (CP). The function classes appropriate for this problem are as follows (everywhere below $g : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$):

$$C([0,\infty) \times \mathbb{R}^n) = \{g : g \text{ is continuous}\}$$

$$UC_x([0,\infty) \times \mathbb{R}^n) = \big\{g \in C([0,\infty) \times \mathbb{R}^n);\ g(t,x) \text{ is uniformly continuous in } x \text{ uniformly for } t \geq 0\big\}$$

$$Q_x([0,\infty) \times \mathbb{R}^n) = \{g;\ \exists A \in \mathcal{S}^n \ni g(t,x) = \langle Ax, x\rangle \text{ for } x \in \mathbb{R}^n\} .= Q(\mathbb{R}^n),$$

$$QL_x([0,\infty) \times \mathbb{R}^n) = \big\{g \in C([0,\infty) \times \mathbb{R}^n);\ \exists \text{ a modulus } \omega \ni |g(t,x) - g(t,y)| \leq \omega\left((1 + |x| + |y|)|x-y|\right) \text{ for } x,\ y \in \mathbb{R}^n,\ t \geq 0\big\},$$

$$QUC_x([0,\infty) \times \mathbb{R}^n) = Q_x([0,\infty) \times \mathbb{R}^n) + UC_x([0,\infty) \times \mathbb{R}^n)$$

and

$$QG_x([0,\infty)\times\mathbb{R}^n) = \big\{g; \forall T > 0\, \exists C \ni |g(t,x)| \leq C(1+|x|^2) \text{ for } (t,x) \in [0,T] \times \mathbb{R}^n\big\}.$$

We will prove:

Theorem 2.7. *Let F be continuous and* (0.1) *hold.*

(i) *If $\psi \in UC(\mathbb{R}^n)$, then* (CP) *has a unique solution $u \in QG_x([0,\infty) \times \mathbb{R}^n) \cap C([0,\infty) \times \mathbb{R}^n)$; moreover, $u \in UC_x([0,\infty) \times \mathbb{R}^n)$.*

(ii) *If $\psi \in QUC_x([0,\infty)\times\mathbb{R}^n)$, then* (CP) *has a unique solution $u \in QG_x([0,\infty)\times \mathbb{R}^n) \cap C([0,\infty) \times \mathbb{R}^n)$; moreover $u \in QUC_x([0,\infty) \times \mathbb{R}^n)$.*

(iii) *If $\psi \in QL(\mathbb{R}^n)$ and F has the property that for $K \geq 0$*

$$\limsup_{r\downarrow 0} \{|F(Z+P) - F(Z)|;\ Z,\ P \in \mathcal{S}^n,\ \|Z\|\,\|P\| \leq K,\ \|P\| \leq r\} = 0 \tag{2.1}$$

then (CP) *has a unique solution $u \in QG_x([0,\infty) \times \mathbb{R}^n) \cap C([0,\infty) \times \mathbb{R}^n)$; moreover, $u \in QL_x([0,\infty) \times \mathbb{R}^n)$.*

The analogue of Lemma 1.3 is:

Lemma 2.8. *Let F be continuous,* (0.1) *hold, a, b, $c \geq 0$ and ψ satisfy*

$$|\psi(x) - \psi(y)| \leq a + b|x-y|^2 + c(|x|^2 + |y|^2) \quad \text{for } x, y \in \mathbb{R}^n. \tag{2.2}$$

If u is upper semicontinuous on $[0,\infty) \times \mathbb{R}^n$ and a subsolution of the equation of (CP) *on $(0,\infty) \times \mathbb{R}^n$, v is lower semicontinuous on $[0,\infty) \times \mathbb{R}^n$ and a supersolution of the equation of* (CP) *on $(0,\infty) \times \mathbb{R}^n$,*

$$u(0,x) \leq \psi(x) \leq v(0,x) \quad \text{on } \mathbb{R}^n, \tag{2.3}$$

$u^+ \in QG_x([0,\infty) \times \mathbb{R}^n)$, and $v^- \in QG_x([0,\infty) \times \mathbb{R}^n)$, then there is a constant α such that

$$u(t,x) - v(t,y) - \big(b|x-y|^2 + c(|x|^2 + |y|^2)\big) \leq a + \alpha t \quad \text{for } x,\ y \in \mathbb{R}^n. \tag{2.4}$$

Proof: Let $T > 0$. By the assumptions on u, v there is a constant such that

$$u(t,x) - v(t,y) \le C(1 + |x|^2 + |y|^2) \quad \text{for } x, y \in \mathbb{R}^n,\ 0 \le t \le T. \tag{2.5}$$

Let β_R satisfy (1.7), $\alpha > 0$ and set

$$\Phi(t,x,y) = u(t,x) - v(t,y) - \big(b|x-y|^2 + c(|x|^2 + |y|^2) + \beta_R(x) + \beta_R(y)\big) - \alpha t.$$

In view of (2.5) and (1.7), Φ will attain its maximum over $[0,T] \times \mathbb{R}^n \times \mathbb{R}^n$ at some point $(\hat{t}, \hat{x}, \hat{y})$. If $\hat{t} = 0$, we have

$$\Phi(\hat{t}, \hat{x}, \hat{y}) = \Phi(0, \hat{x}, \hat{y}) \le \psi(\hat{x}) - \psi(\hat{y}) - \big(b|\hat{x} - \hat{y}|^2 + c(|\hat{x}|^2 + |\hat{y}|^2)\big) \le a \tag{2.6}$$

since (2.2) and (2.3) hold.

If $\hat{t} > 0$, since u is a subsolution and v is a supersolution, we conclude from [3, Theorem 5] and standard remarks on parabolic equations (or [4]) that there are X, $Y \in \mathcal{S}^n$ such that

$$X \le Y, \ \ \|X\|, \ \|Y\| \le 8b \tag{2.7}$$

and

$$\alpha \le F(Y - 2cI - D^2\beta_R(\hat{y})) - F(X + 2cI + D^2\beta_R(\hat{x})); \tag{2.8}$$

clearly we may choose $\alpha = \alpha(b,c)$ so that this is impossible. Hence, with this choice of α we must have (2.6). Noting that $T > 0$ was arbitrary, we conclude that $\Phi(t,x,y) \le \Phi(\hat{t}, \hat{x}, \hat{y}) \le a$ and then, letting $R \to \infty$, that

$$u(t,x) - v(t,y) - \big(b|x-y|^2 + c(|x|^2 + |y|^2)\big) \le a + \alpha(b,c)t. \tag{2.9}$$

Observe that, in view of (0.1) and $X \le Y$, we may choose

$$\alpha(b,c) = 2 \sup_{\substack{X \in \mathcal{S}^n,\, |X| \le 8b \\ R \ge 1,\, \hat{x}, \hat{y} \in \mathbb{R}^n}} \big(F(X - 4cI - D^2\beta_R(\hat{y})) - F(X + 4cI + D^2\beta_R(\hat{x})\big)\,. \tag{2.10}$$

Proof of Theorem 2.7: We begin with (i). If $\psi \in UC(\mathbb{R}^n)$, then for every $\gamma > 0$ we have inequalities of the form (1.13) for ψ in place of f. Assuming that u, $v \in QG(\mathbb{R}^n)$ are sub and supersolutions of (CP) as in Lemma 2.8, we learn from the lemma (with $b = 0$, $c = \frac{1}{2\epsilon}M_\gamma$) that $u(t,x) - v(t,y) - \frac{1}{2\epsilon}M_\gamma|x-y|^2$ is bounded for bounded t. Using this information, we repeat the proof of the lemma replacing β_R by $\lambda|x|^2$ where $\lambda > 0$ is arbitrary; this leads at once to the estimate

$$u(t,x) - v(t,y) - \frac{1}{2\epsilon}M_\gamma|x-y|^2 - \lambda(|x|^2 + |y|^2) \le \gamma + \frac{1}{2}M_\gamma\epsilon + \alpha(\gamma,\epsilon,\lambda)t \tag{2.11}$$

where now

$$\alpha(\gamma,\epsilon,\lambda) = \sup_{Z \in \mathcal{S}^n,\, \|Z\| \le \frac{4}{\epsilon}M_\gamma} 2(F(Z - 2\lambda I) - F(Z + 2\lambda I)). \tag{2.12}$$

Since $\lim_{\lambda \downarrow 0} \alpha(\gamma,\epsilon,\lambda) = 0$, we may put $x = y$ in (2.11) and then take the iterated limit $\lambda \downarrow 0$, $\epsilon \downarrow 0$, $\gamma \downarrow 0$ to conclude that $u \le v$.

Moreover, if $u = v$ is a solution, (2.11) implies (upon taking the limit $\lambda \downarrow 0$) that u is uniformly continuous in x uniformly in $t \geq 0$. To see that if $u \in QG_x([0,\infty) \times \mathbb{R}^n)$ is a solution, then $u \in UC_x([0,\infty) \times \mathbb{R}^n)$, we thus only need to show that u is continuous in t. For this it will suffice to show that u is continuous at $t = 0$ with estimates that depend only on the modulus of continuity of ψ (for we know that u is uniformly continuous in x uniformly in t, so such estimates may be reapplied for $t > 0$). If $z \in \mathbb{R}^n$ we have

$$\psi(x) \leq \psi(z) + \gamma + \frac{1}{2}\epsilon M_\gamma + \frac{1}{2\epsilon} M_\gamma |x - z|^2 \quad \text{on } \mathbb{R}^n \tag{2.13}$$

and then one checks then that

$$v(t,x) = \psi(z) + \gamma + \frac{1}{2}\epsilon M_\gamma + \frac{1}{2\epsilon} M_\gamma |x - z|^2 - tF(\frac{1}{\epsilon} M_\gamma I) \tag{2.14}$$

is a solution of the equation of (CP) and $v(0,x) \geq \psi(x)$. Hence, by the comparison already established,

$$\begin{aligned} u(t,x) - \psi(z) &= u(t,x) - v(t,x) + \gamma + \frac{1}{2}\epsilon M_\gamma + \frac{1}{2\epsilon} M_\gamma |x - z|^2 - tF(\frac{1}{\epsilon} M_\gamma I) \\ &\leq \gamma + \frac{1}{2}\epsilon M_\gamma + \frac{1}{2\epsilon} M_\gamma |x - z|^2 - tF(\frac{1}{\epsilon} M_\gamma I); \end{aligned} \tag{2.14}$$

estimating in the other direction in the same way, we conclude that

$$|u(t,x) - \psi(z)| \leq \gamma + \frac{1}{2}\epsilon M_\gamma + \frac{1}{2\epsilon} M_\gamma |x - z|^2 + t|F(\frac{1}{\epsilon} M_\gamma I)|. \tag{2.16}$$

The continuity is now evident.

In order to establish the existence, we proceed in a similar way assuming that

$$\psi \in C^2, \quad D\psi \text{ and } D^2\psi \text{ are bounded.} \tag{2.17}$$

It suffices to treat this case because if $\psi \in UC(\mathbb{R}^n)$ there is a sequence $\psi_j \in C^2$ with $D\psi_j$, $D^2\psi_j$ bounded and $\sup_{\mathbb{R}^n} |\psi(x) - \psi_j(x)| \to 0$ as $j \to \infty$. It is immediate that the solution u_j of (CP) with ψ replaced by ψ_j converges uniformly to the desired solution of (CP). If (2.17) holds, we may simplify the writing of a supersolution $\bar{v}$ from (2.14) to

$$\bar{v}(t,x) = \psi(z) + \langle D\psi(z), x - z\rangle + \frac{c}{2}|x - z|^2 - tF(cI) \tag{2.18}$$

where $c \geq \|D^2\psi(x)\|$ for $x \in \mathbb{R}^n$; there is also the corresponding subsolution

$$\underline{v}(t,x) = \psi(z) + \langle D\psi(z), x - z\rangle - \frac{c}{2}|x - z|^2 - tF(-cI). \tag{2.19}$$

Regarding $\bar{v}$, $\underline{v}$ as parameterized by $z \in \mathbb{R}^n$, it is evident that

$$\overline{V} = \inf_{z \in \mathbb{R}^n} \bar{v}, \quad \underline{V} = \sup_{z \in \mathbb{R}^n} \underline{v}$$

provide locally Lipschitz continuous super and subsolutions of (2.3) which satisfy $\overline{V} \geq \underline{V}$ and assume the initial data ψ continuously. Existence now follows from Perron's method.

We pass to the case $\psi \in QUC(\mathbb{R}^n)$. Let $\psi = q + h$ where $q \in Q_x([0,\infty) \times \mathbb{R}^n)$ and $h \in UC_x([0,\infty) \times \mathbb{R}^n)$; say $q(t,x) = \langle Ax, x\rangle$ where $A \in \mathcal{S}^n$. It is just a definition check to see that u is a solution of (CP) if and only if $w = u - q$ is a solution of

$$w_t + G(D^2 w) = 0 \;\text{ on } (0,\infty) \times \mathbb{R}^n \quad w(0,x) = h(x) \;\text{ on } \mathbb{R}^n \tag{2.20}$$

where

$$G(B) = F(B + 2A) \quad \text{for } B \in \mathcal{S}^n. \tag{2.21}$$

By (i), (2.20) has a solution $w \in UC_x([0,\infty) \times \mathbb{R}^n)$ which is unique in $QG_x([0,\infty) \times \mathbb{R}^n)$ and then $u = w + q \in QUC(\mathbb{R}^n)$ is a solution of (CP). The uniqueness of u in $QG_x([0,\infty) \times \mathbb{R}^n)$ follows from the uniqueness of solutions of (2.20) from (i).

We pass to the case $\psi \in QL(\mathbb{R}^n)$ assuming that (2.1) holds. Now we have

$$|\psi(x) - \psi(y)| \leq \gamma + \frac{1}{2}\epsilon M_\gamma + \frac{1}{2\epsilon} M_\gamma |x-y|^2 + \frac{1}{2}\epsilon M_\gamma (|x|^2 + |y|^2). \tag{2.22}$$

The analogue of the estimate (2.11) , (2.12) in this case is

$$u(t,x) - v(t,y) - \frac{1}{2\epsilon} M_\gamma |x-y|^2 - M_\gamma \epsilon (|x|^2 + |y|^2) \leq \gamma + \frac{1}{2} M_\gamma \epsilon + \alpha(\gamma,\epsilon) t \tag{2.23}$$

$$\alpha(\gamma,\epsilon) = \sup_{Z \in \mathcal{S}^n,\, \|Z\| \leq \frac{4}{\epsilon} M_\gamma} 2\left(F(Z - 2M_\gamma \epsilon I) - F(Z + 2M_\gamma \epsilon I)\right). \tag{2.24}$$

We now have $\lim_{\epsilon \downarrow 0} \alpha(\gamma,\epsilon) = 0$ because of (2.1) and putting $x = y$ and passing to the limits $\epsilon \downarrow 0$ then $\gamma \downarrow 0$ in (2.23) we find $u \leq v$.

Turning to the question of existence, we assume that $u = v$ is a solution and will first show that for T, $\delta > 0$ there is a $\kappa > 0$ depending only on F, T and the x-modulus of $\psi \in Q(\mathbb{R}^n)$ such that

$$|u(t,x) - u(t,y)| \leq \delta \;\text{ for } 0 \leq t \leq T \text{ and } |x-y|(1 + |x|^2 + |y|^2)^{1/2} \leq \kappa. \tag{2.25}$$

Putting $\epsilon = |x-y|/(1 + |x|^2 + |y|^2)^{1/2}$ and $r = |x-y|(1 + |x|^2 + |y|^2)^{1/2}$ in (2.24) we have $\epsilon \leq r$ and so, using (2.24),

$$|u(t,x) - u(t,y)| \leq \gamma + M_\gamma r + \sup_{0 < \epsilon \leq r} \alpha(\gamma,\epsilon) T. \tag{2.26}$$

In this inequality we first choose $\gamma < \delta/2$ and then determine κ by the condition $M_\gamma r + \alpha(\gamma, r) < \delta/2$ for $r < \kappa$.

Next we want to make a similar estimate, depending only on the x modulus of ψ, on the continuity of u in t. The supersolution analogous to v in (2.14) in this case is

$$v(t,x) = \psi(z) + \gamma + \frac{1}{2}\epsilon M_\gamma (1 + x^2 + z^2) + \frac{1}{2\epsilon} M_\gamma |x-z|^2 - tF\left(\frac{1}{\epsilon} M_\gamma I + \epsilon M_\gamma I\right) \tag{2.27}$$

which leads to the estimate

$$u(t,x) \leq \psi(x) + \gamma + \frac{1}{2}\epsilon M_\gamma (1+2|x|^2)^{1/2} - tF(\frac{1}{\epsilon}M_\gamma I + \epsilon M_\gamma I). \tag{2.28}$$

It is then clear that $u(t,x)$ is continuous in t at $t=0$ uniformly for bounded x in a way that only depends on F and the modulus of ψ in $QL(\mathbb{R}^n)$. To complete the existence proof, we choose $\psi \in QL(\mathbb{R}^n)$ truncate it to $\psi_R = \max\{\min\{\psi, R\}, -R\}$ and notice that $\psi_R \in UC(\mathbb{R}^n)$ and has the same $QL(\mathbb{R}^n)$ modulus as ψ independent of $R > 0$. From the above, the solutions u_R of (CP) with ψ replaced by ψ_R provided by part (i) are equicontinuous on bounded sets of $[0,\infty) \times \mathbb{R}^n$, so we obtain a solution of (0.1) by taking a (subsequential) limit of the u_R as $R \to \infty$.

3. The inverse problem. The solution of (0.1) provided by Theorem 2.7 for $\psi \in QUC(\mathbb{R}^n)$ determines a semigroup $S(t)$, $t \geq 0$ on $UC(\mathbb{R}^n)$ and $QUC(\mathbb{R}^n)$. If we equip $QUC(\mathbb{R}^n)$ with the norm–like functional

$$\|\psi\|_\infty = \sup_{x \in \mathbb{R}^n} |\psi(x)|$$

(which takes values in $[0,\infty]$), what was shown above implies that $S(t)$ is order–preserving and nonexpansive (i.e., $\|S(t)\psi - S(t)\varphi\|_\infty \leq \|\psi - \varphi\|_\infty$) on $QUC(\mathbb{R}^n)$. Moreover, $S(t)$ is strongly continuous; that is, $\lim_{t\downarrow 0} \|S(t)\psi - \psi\|_\infty = 0$ for $\psi \in QUC(\mathbb{R}^n)$. While checking these remarks the reader will want to observe that if $\psi = q + h$ and $\hat{\psi} = \hat{q} + \hat{h}$ with q, $\hat{q} \in Q(\mathbb{R}^n)$ and h, $\hat{h} \in UC(\mathbb{R}^n)$, then $q \neq \hat{q}$ implies $\|\psi - \hat{\psi}\|_\infty = \infty$.

Moreover, we can compute some orbits explicitly: if $(a,p,A) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$, then there is a function g such that

$$S(t)(a + \langle p,x\rangle + \langle Ax,x\rangle) = a + \langle p,x\rangle + \langle Ax,x\rangle - g(t); \tag{3.1}$$

indeed,

$$g(t) = tF(2A). \tag{3.2}$$

We also have

$$S(t)(a + \langle p,x\rangle + \psi) = a + \langle p,x\rangle + S(t)\psi \quad \text{for } (a,p,\psi) \in \mathbb{R} \times \mathbb{R}^n \times QUC(\mathbb{R}^n) \tag{3.3}$$

and

$$S(t)\psi(\cdot + y) = (S(t)\psi)(\cdot + y) \quad \text{for } (y,\psi) \in \mathbb{R}^n \times QUC(\mathbb{R}^n), \tag{3.4}$$

as is straightforward to check ((3.3) corresponds to $F(D^2u)$ being independent of u, Du while (3.4) corresponds to independence of x). The reader will have observed several of the notations we need to use here: $S(t)\psi$ is an element of $QUC(\mathbb{R}^n)$ for $t \geq 0$ and its value at x is denoted by $(S(t)\psi)(x)$; when convenient, we will also write expressions like $S(t)(\langle Ax,x\rangle)$ which means $S(t)\psi$ where $\psi(x) = \langle Ax,x\rangle$; likewise, $S(t)\psi(\cdot + y)$ means $S(t)\varphi$ where $\varphi(x) = \psi(x+y)$, etc.

Suppose now that we begin with an order – preserving strongly continuous semigroup $S(t)$ on $QUC(\mathbb{R}^n)$ with the properties (3.1) (in the sense that given (a,p,A), there exists $g\colon [0,\infty) \to \mathbb{R}$ such that (3.1) holds), (3.3) and (3.4). Is there an F such that $S(t)$ comes from solving (CP)? The appropriate definition of F is clear under these assumptions; we first note that the semigroup property and (3.1) guarantee that $g(t) = t\alpha$ is a linear function of t and then define F via (3.2):

$$F(2A) = \frac{1}{t}\left(\langle Ax, x\rangle - S(t)(\langle Ax, x\rangle)\right). \tag{3.5}$$

There is no obvious reason for this function F to be continuous with the assumptions we have made so far, although it is automatically nonincreasing as a map from $\mathcal{S}^n$ to $\mathbb{R}$ (i.e., (0.1) holds). A property of S which guarantees the continuity and which, in view of (3.1), is satisfied by the semigroup constructed by solving (CP) is the following:

$$\begin{cases} \text{If } \psi_j,\ \psi \in Q(\mathbb{R}^n)\ \ j = 1, \cdots, \text{ and } \psi_j \to \psi \text{ uniformly on bounded sets,} \\ \text{then } S(t)\psi_j \to S(t)\psi \text{ uniformly on bounded sets for } t \geq 0. \end{cases} \tag{3.6}$$

We will add one further hypothesis corresponding to the local nature of $F(D^2u)$:

$$\begin{cases} \text{If } \psi,\ \varphi \in QUC(\mathbb{R}^n) \text{ have bounded uniformly continuous derivatives of} \\ \text{orders 2 through 4 and } \psi(x) - \varphi(x) = O(|x|^3) \text{ as } |x| \to 0, \text{ then} \\ (S(t)\psi(0) - S(t)\varphi(0)) = o(t) \text{ as } t \downarrow 0. \end{cases} \tag{3.7}$$

Theorem 3.9. *Let $S(t)$ be an order – preserving strongly continuous semigroup $QUC(\mathbb{R}^n)$ with the properties* (3.1), (3.3), (3.4), (3.6), (3.7). *Let F be given by* (3.5). *Then* (0.1) *holds, F is continuous and for $\psi \in QUC(\mathbb{R}^n)$, $u(t,x) = (S(t)\psi)(x)$ is the solution of* (CP) *provided by Theorem* 2.7.

Proof: What we must show is that for $\psi \in QUC(\mathbb{R}^n)$, $u(t,x) = (S(t)\psi)(x)$ is a viscosity solution of the equation of (CP). Using a type of argument introduced in [15] (see also [16]), we just show it is a subsolution, and for this it is enough to verify that if $g\colon [0,\infty) \to \mathbb{R}$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $(\hat{t}, \hat{x}) \in (0,\infty) \times \mathbb{R}^n$, $\delta > 0$ and

$$u(t,x) - g(t) - \varphi(x) \leq u(\hat{t},\hat{x}) - g(\hat{t}) - \varphi(\hat{x}) \ \text{ for } \hat{t} - \delta \leq t \leq \hat{t} \text{ and } |x - \hat{x}| \leq \delta, \tag{3.8}$$

then

$$g'(\hat{t}) + F(D^2\varphi(\hat{x})) \leq 0. \tag{3.9}$$

It will be simpler to proceed assuming that (3.8) holds globally (i.e., for $(t,x) \in [0,\hat{t}] \times \mathbb{R}^n$) and we may achieve this in a straightforward way by modifying φ and g and we may even assume that φ is a suitable quadratic for large $|x|$. Then we have, using (3.8) and the order preserving nature of S,

$$\begin{aligned} S(h)(u(\hat{t}-h,\cdot)) = u(\hat{t},\cdot) &\leq S(h)(u(\hat{t},\hat{x}) + g(\hat{t}-h) - g(\hat{t}) - \varphi(\hat{x}) + \varphi(\cdot)) \\ &= u(\hat{t},\hat{x}) + g(\hat{t}-h) - g(\hat{t}) - \varphi(\hat{x}) + S(h)\varphi. \end{aligned} \tag{3.10}$$

Next we use (3.7) and translation invariance (3.4) to assert

$$\begin{aligned} (S(h)\varphi)(\hat{x}) &= S(h)\left(\varphi(\hat{x}) + \langle D\varphi(\hat{x}), x - \hat{x}\rangle + \tfrac{1}{2}\left\langle D^2\varphi(\hat{x})(x-\hat{x}), x - \hat{x}\right\rangle\right)(\hat{x}) + o(h) \\ &= \varphi(\hat{x}) - hF(D^2\varphi(\hat{x})) + o(h), \end{aligned} \tag{3.11}$$

and use (3.11) in (3.10) evaluated at $\hat{x}$ to conclude that

$$0 \leq g(\hat{t}-h) - g(\hat{t}) - hF(D^2\varphi(\hat{x})) + o(h)$$

and (3.10) follows upon dividing by h and letting $h \downarrow 0$.

To complete the story of this section, we must convince the reader that the semigroup $S(t)$ arising from the solution of (CP) has the local property (3.7). This we do by proving:

Proposition 3.10. *Let F be continuous and satisfy* (0.1). *Let $\psi \in QUC(\mathbb{R}^n)$ have bounded uniformly continuous derivatives of orders* 2 *through* 4 *and u be the solution of* (CP). *Then*

$$\lim_{t\downarrow 0} \frac{u(t,x)-\psi(x)}{t} = -F(D^2\varphi(x)) \text{ holds uniformly on bounded sets.} \tag{3.12}$$

Sketch of proof: Choose a sequence of maps $F_j\colon \mathcal{S}^n \to \mathbb{R}$ such that $F_j \in C^2(\mathcal{S}^n)$ and $|F_j(D^2\psi(x)) - F(D^2\psi(x))| \leq \frac{1}{j}$ for $x \in \mathbb{R}^n$. Put $u_j(t,x) = \psi(x) - tF_j(D^2\psi(x))$ and observe that

$$\begin{cases} u_{jt} + F(D^2u_j) = F(D^2(\psi - tF_j(D^2\psi(x)))) - F_j(D^2\psi(x)) = g_j(t,x), \\ u_j(0,x) = \psi(x) \end{cases}$$

where $\limsup_{t\downarrow 0} |g_j(t,x)| \leq \frac{1}{j}$ holds uniformly in x. Observing that then $u_j - \int_0^t \|g_j(s,\cdot)\|_\infty ds$ is a subsolution of the problem solved by u, etc., one sees that

$$\limsup_{t\downarrow 0} \left\| \frac{u_j(t,\cdot)-u(t,\cdot)}{t} \right\| \leq \limsup_{t\downarrow 0} \frac{1}{t}\int_0^t \|g_j(s,\cdot)\|_\infty \, ds \leq \frac{1}{j}$$

and so

$$\left| \frac{u(t,x)-\psi(x)}{t} + F(D^2\psi(x)) \right| \leq \left| F(D^2\psi(x)) - F_j(D^2\psi(x)) \right| + \frac{|u(t,x)-u_j(t,x)|}{t}$$

implies, letting $t \downarrow 0$,

$$\limsup_{t\downarrow 0} \left\| \frac{u(t,\cdot)-\psi(\cdot)}{t} + F(D^2\psi(\cdot)) \right\|_\infty \leq \frac{2}{j};$$

the proposition is established upon letting $j \to \infty$.

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