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SCHAUDER ESTIMATES AND EXISTENCE THEORY FOR ENTIRE SOLUTIONS OF LINEAR PARABOLIC EQUATIONS

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Abstract. Solutions in $\mathbb{R}^{n+1}$ of the linear, parabolic, nonhomogeneous partial differential equation

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}\frac{\partial^2\phi}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial \phi}{\partial x_i} + c\phi - \frac{\partial \phi}{\partial t} = f,$$

and the related homogeneous equation are investigated. By entire solutions is meant solutions of these equations defined in all of $\mathbb{R}^{n+1}$. Schauder-type a priori estimates are developed for entire solutions with prescribed behavior at infinity. These estimates lead to an existence theory for entire solutions with certain behavior required at infinity. Uniqueness of these solutions follows from the maximum principle.

1. Introduction. We investigate here solutions in $\mathbb{R}^{n+1}$ of the linear, parabolic, nonhomogenous, variable coefficient, partial differential equation

$$\begin{aligned}\mathcal{L}\phi :&= a \cdot \mathcal{D}^2\phi + b \cdot \mathcal{D}\phi + c\phi - \partial\phi/\partial t \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}\frac{\partial^2\phi}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial \phi}{\partial x_i} + c\phi - \frac{\partial \phi}{\partial t} = f \end{aligned} \tag{NH}$$

and the related homogeneous equation,

$$\mathcal{L}\phi = 0. \tag{H}$$

Following the prevailing terminology in the literature, we refer to solutions of these equations defined in all of $\mathbb{R}^{n+1}$ as entire solutions. We shall develop Schauder-type a priori estimates for the entire solutions with prescribed behavior at infinity. These estimates, of perhaps independent interest themselves, lead to an existence theory for entire solutions with certain behavior required at infinity. Uniqueness of these solutions follows from the maximum principle.

The coefficients a, b, c of $\mathcal{L}$ are assumed to be Hölder-continuous in $\mathbb{R}^{n+1} \cup \{\infty\}$: and as $x \to \infty$, the matrix a approaches the $n \times n$ identity matrix I, the vector b approaches the $1 \times n$ zero vector and the scalar c approaches zero. Thus $\mathcal{L}$ approaches the heat operator

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near infinity. We also assume that $\mathcal{L}$ is uniformly parabolic in $\mathbb{R}^{n+1}$, and $c \leq 0$. With these assumptions, we shall establish a one-to-one correspondence between the entire solutions of (NH) that are $O(|x|^2 + |t|)^{m/2}$ and the polynomial solutions to the heat equation of total degree no greater than m in x.

Several authors, for example [1, 2, 6, 8-11, 13-15] have investigated existence and/or uniqueness questions for entire solutions of linear elliptic equations. Results analogous to the ones presented here have been obtained for elliptic equations by H. Begehr and G.N. Hile in [2].

The Schauder estimates for entire solutions are obtained from analogous estimates for bounded domains as found in the book of Friedman [5].

2. Norms and a Schauder estimate for entire solutions. For vectors v, w in $\mathbb{R}^{n+1}$, we let $v \cdot w$ denote the dot product of v and w, and $|v|$ the Euclidean norm of v. For $n \times n$ matrices $A = (a_{i,j})$, $B = (b_{i,j})$ in $\mathbb{R}^{n\times n}$, we define

$$A \cdot B = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} b_{i,j}, \text{ and } |A| = (A \cdot A)^{1/2}.$$

(The inequality $|A \cdot B| \leq |A|\,|B|$ holds.) We let $\bar{x} = (x,t) = (x_1, x_2, \cdots, x_n, t)$, $\bar{y} = (y,s) = (y_1, y_2, \cdots, y_n, s)$, and $\bar{z} = (z,r) = (z_1, z_2, \cdots, z_n, r)$ denote variable points in $\mathbb{R}^{n+1}$. The symbol $\bar{0}$ will denote the origin in $\mathbb{R}^{n+1}$, and Ω will denote a bounded $(n+1)$-dimensional domain in $\mathbb{R}^{n+1}$ bounded by $F \times \{t = T_0\}$ and $F_T \times \{t = T\}$ where F_T and F are domains in $\mathbb{R}^n, T > T_0$, and by a manifold (not necessarily connected) S lying in $T_0 \leq t \leq T$. If $\bar{x} = (x,t)$ and $\bar{y} = (y,s)$, then we shall define

$$d(\bar{x}, \bar{y}) = \sqrt{(|x-y|^2 + |t-s|)}$$

where $|\cdot|$ is the Euclidean norm. The function $d(\bar{x}, \bar{y})$ is a metric on $\mathbb{R}^{n+1}$. Hölder-continuity will be defined with respect to this metric. For $\phi = \phi(\bar{x})$ a real valued function defined in Ω, we let $\mathcal{D}\phi$ and $\mathcal{D}^2\phi$ represent the $1 \times n$ vector of first derivatives of ϕ and the $n \times n$ matrix of second derivatives of ϕ, respectively. Thus,

$$\mathcal{D}\phi = (\phi_{x_1}, \phi_{x_2}, \cdots, \phi_{x_n}), \quad \mathcal{D}^2\phi = (\phi_{x_i x_j})_{n\times n}.$$

Let $a = (a_{i,j})_{n\times n}$, $b = (b_1, b_2, \cdots, b_n)$, c be functions defined in Ω with values in $\mathbb{R}^{n\times n}$, $\mathbb{R}^n$, $\mathbb{R}$, respectively. The matrix a is always assumed to be symmetric. We consider the second order linear partial differential operator $\mathcal{L}$, defined by

$$\begin{aligned}\mathcal{L}\phi :&= a \cdot \mathcal{D}^2\phi + b \cdot \mathcal{D}\phi + c\phi - \partial\phi/\partial t \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j} \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial \phi}{\partial x_i} + c\phi - \frac{\partial \phi}{\partial t}.\end{aligned}$$

Associated with the operator $\mathcal{L}$ are the homogeneous equation

$$\mathcal{L}\phi = 0 \tag{H}$$

and the nonhomogeneous equation,

$$\mathcal{L}\phi = f, \tag{NH}$$

where the right-hand side f is assumed to be a real valued function in Ω. We let D_x^m denote any partial derivative of order m with respect to $x_1, x_2, \cdots, x_n$. We will write D_x for D_x^1 and D_t will denote $\partial/\partial t$. For functions $u : \Omega \to \mathbb{R}$, $|u|_0^\Omega$ will denote $\sup\{|u(\bar{x})| : \bar{x} \in \Omega\}$. We shall denote the S_τ the set $S \cap \{(x,t) : t \leq \tau\}$. For any $\bar{x} = (x,t)$ in Ω, let $d_{\bar{x}} = \inf\{d(\bar{x}, \bar{y}) : \bar{y} \in F \cup S_t\}$. For any points $\bar{x}$, $\bar{y}$ in Ω, let $d_{\bar{x},\bar{y}} = \min\{d_{\bar{x}}, d_{\bar{y}}\}$. For a real number α, $0 < \alpha \leq 1$, integer m, and function $u : \Omega \to \mathbb{R}$, let

$$H_\alpha^\Omega(u) = \sup\left\{(d_{\bar{x},\bar{y}})^\alpha \frac{|u(\bar{x}) - u(\bar{y})|}{d(\bar{x},\bar{y})^\alpha} : \bar{x}, \bar{y} \in \Omega\right\},$$

$$|d^m u|_0^\Omega = \sup\left\{(d_{\bar{x}})^m |u(\bar{x})| : \bar{x} \in \Omega\right\}, \quad |du|_0^\Omega = |d^1 u|_0^\Omega,$$

$$H_\alpha^\Omega(d^m u) = \sup\left\{(d_{\bar{x},\bar{y}})^{m+\alpha} \frac{|u(\bar{x}) - u(\bar{y})|}{d(\bar{x},\bar{y})^\alpha} : \bar{x}, \bar{y} \in \Omega\right\},$$

$$|d^m u|_\alpha^\Omega = |d^m u|_0^\Omega + H_\alpha^\Omega(d^m u), \quad |u|_\alpha^\Omega = |d^0 u|_\alpha^\Omega, \quad |du|_\alpha^\Omega = |d^1 u|_\alpha^\Omega,$$

and

$$|u|_{2+\alpha}^\Omega = |u|_\alpha^\Omega + \sum |dD_x u|_\alpha^\Omega + \sum |d^2 D_x^2 u|_\alpha^\Omega + |d^2 D_t u|_\alpha^\Omega,$$

where summations are over all possible derivatives.

By a classical solution of $\mathcal{L}\phi = f$, we shall mean a solution ϕ with ϕ, $D_x\phi$, $D_t\phi$, $D_x^2\phi$ continuous. Friedman has shown (Theorem 11 on page 74 of [5]) that if $a_{i,j}$, b_i, c and f are Hölder-continuous with exponent α in Ω and ϕ is a solution of $\mathcal{L}\phi = f$ then ϕ, $D_x\phi$, $D_x^2\phi$, $D_t\phi$ are all locally Hölder-continuous with exponent α in Ω. The following lemma appears as Theorem 5 on page 64 of [5].

Lemma 1. *Consider the equation $\mathcal{L}\phi = f$ in $\Omega \cup F_T$, and assume*

A) *the entries of the matrix a are locally Hölder-continuous with exponent $\alpha, 0 < \alpha < 1$, in Ω and there is $K_1 > 0$ such that*

$$|a_{i,j}|_\alpha^\Omega \leq K_1, \quad |db_i|_\alpha^\Omega \leq K_1, \quad |d^2 c|_\alpha^\Omega \leq K_1,$$

B) *there is $K_2 > 0$ such that for all $\bar{x}$ in Ω and real $(1 \times n)$ vector ξ,*

$$\sum_{i,j=1}^n a_{i,j}(\bar{x})\xi_i\xi_j \geq K_2|\xi|^2,$$

C) *the function f is locally Hölder-continuous with exponent α in Ω and $|d^2 f|_\alpha^\Omega < \infty$.*

Suppose further that ϕ is a classical solution of $\mathcal{L}\phi = f$ and that ϕ is bounded on Ω. Then there exists $K = K(K_1, K_2)$ such that

$$|\phi|_{2+\alpha}^\Omega \leq K(|\phi|_0^\Omega + |d^2 f|_\alpha^\Omega).$$

We will derive an analogue of Lemma 1 for the case $\Omega = \mathbb{R}^n$. For this purpose it is useful to define the following norms where again $\sigma \in \mathbb{R}$, $0 < \alpha \leq 1$, and now u is a function *defined* on all of $\mathbb{R}^{n+1}$. Let

$$\| u \|_\sigma = \sup\left\{(1 + |t| + |x|^2)^{-\sigma/2} |u(x,t)| : \bar{x} \in \mathbb{R}^{n+1}\right\},$$

$$\| u \|_{(\sigma,\alpha)} = \sup \left\{ \begin{array}{l} (1+|t|+|x|^2)^{(-\sigma+\alpha)/2} \dfrac{|u(\bar{x})-u(\bar{y})|}{d(\bar{x},\bar{y})^\alpha} : \\ \bar{x},\bar{y} \in \mathbb{R}^{n+1}, \quad 0 < d(\bar{x},\bar{y}) \leq \dfrac{1+|t|+|x|^2)^{1/2}}{2} \end{array} \right\},$$

$$\| u \|_{\sigma,\alpha} = \| u \|_\sigma + \| u \|_{(\sigma,\alpha)}, \quad \text{and}$$

$$\| \|u\| \|_{\sigma,\alpha} = \| u \|_{\sigma,1} + \sum \| D_x u \|_{\sigma-1,\alpha} + \sum \| D_x^2 u \|_{\sigma-2,\alpha} + \| D_t u \|_{\sigma-2,\alpha},$$

where the summations are over all partial derivatives. For brevity we shall denote $(1+|t|+|x|^2)^{1/2}$ by $[\bar{x}]$.

Let $B_\sigma = \{u : u \text{ is continuous and } \| u \|_\sigma < \infty\}$ and $B_{\sigma,\alpha} = \{u : \| u \|_{\sigma,\alpha} < \infty\}$. The norms $\| u \|_\sigma$ and $\| u \|_{(\sigma,\alpha)}$ may be viewed as analogous to the norms $|d^m u|_0^\Omega$ and $H_\alpha^\Omega(d^m u)$ where the distance $d_{\bar{x}}$, which vanishes as $\bar{x}$ approaches $\partial\Omega$, has been replaced by the quantity $(1+|t|+|x|^2)^{-1/2}$, which likewise vanishes as $\bar{x}$ approaches the boundary point at infinity of $\mathbb{R}^{n+1}$. The condition $0 < d(\bar{x},\bar{y}) \leq (1+|t|+|x|^2)^{1/2}/2$ is a technical convenience, assuring that $\bar{y}$ approaches infinity along with $\bar{x}$, and more or less in the same direction.

We gather some miscellaneous observations regarding the spaces B_σ and $B_{\sigma,\alpha}$ into the following lemma:

Lemma 2. *Let σ, $\tau \in \mathbb{R}$, $0 < \alpha$, $\beta \leq 1$, and let u and v denote functions defined on $\mathbb{R}^n$ (either both scalar valued, both vector valued or both matrix valued).*

a) *If $\sigma \leq \tau$ then $B_\sigma \subset B_\tau$ and $\| u \|_\tau \leq \| u \|_\sigma$.*

b) *If $\sigma \leq \tau$ and $0 < \beta \leq \alpha \leq 1$, then $B_{\sigma,\alpha} \subset B_{\tau,\beta}$ and $\| u \|_{\tau,\beta} \leq \| u \|_{\sigma,\alpha}$.*

c) *If $u \in B_\sigma$ and $v \in B_\tau$, then $u \cdot v \in B_{\sigma+\tau}$ and $\| u \cdot v \|_{\sigma+\tau} \leq \| u \|_\sigma \| v \|_\tau$.*

d) *If $u \in B_{\sigma,\alpha}$ and $v \in B_{\tau,\alpha}$ then $u \cdot v \in B_{\sigma+\tau,\alpha}$ and $\| u \cdot v \|_{\sigma+\tau,\alpha} \leq \| u \|_{\sigma,\alpha} \| v \|_{\tau,\alpha}$.*

e) *If ϕ is a real valued function in $C^1(\mathbb{R}^{n+1})$ with each $D_x\phi \in B_{\sigma-1}$ and $D_t\phi$ in $B_{\sigma-2}$, then $\| \phi \|_{(\sigma,1)}$ is finite and*

$$\| \phi \|_{(\sigma,1)} \leq M(\sigma) \Big(\sum \| D_x \phi \|_{\sigma-1} + \| D_t \phi \|_{\sigma-2} \Big).$$

f) *Let $\{u_m\}_{m=1}^\infty$ be a sequence of functions in $B_{\sigma,\alpha}$, with all the u_m's having values in the same space $\mathbb{R}$, $\mathbb{R}^n$, or $\mathbb{R}^{n\times n}$, and suppose that we have a uniform bound $\| u_m \|_{\sigma,\alpha} \leq M$ for all m, where $M \geq 0$. Then there is a subsequence $\{u_{m_k}\}$ and a function u in $B_{\sigma,\alpha}$, with $\| u \|_{\sigma,\alpha} \leq M$, such that $u_{m_k} \to u$ in the norm of any space B_τ or $B_{\tau,\beta}$ with $\tau > \sigma, 0 < \beta < \alpha$.*

g) *Let $f \in B_{\sigma,\alpha}$ where $\alpha > 0$. Then f is locally Hölder-continuous with exponent α.*

h) *The spaces B_σ and $B_{\sigma,\alpha}$ are Banach spaces.*

Proof: a) Since $[\bar{x}] = (1+|t|+|x|^2)^{1/2} \geq 1$, then

$$\|u\|_\sigma \geq \sup \left\{ [\bar{x}]^{-\tau} |u(x,t)| : \bar{x} \in \mathbb{R}^{n+1} \right\} = \|u\|_\tau.$$

b) Since $\big([\bar{x}]/d(\bar{x},\bar{y})\big) \geq 2$, and $[\bar{x}] \geq 1$, then

$$\|u\|_{(\sigma,\alpha)} \geq \sup \left\{ [\bar{x}]^{-\tau} \Big(\frac{[\bar{x}]}{d(\bar{x},\bar{y})}\Big)^\beta |u(\bar{x}) - u(\bar{y})| : \bar{x},\bar{y} \in \mathbb{R}^{n+1},\ 0 < d(\bar{x},\bar{y}) \leq [\bar{x}]/2 \right\}$$
$$= \|u\|_{(\tau,\beta)}.$$

c)

$$\|u\cdot v\|_{\sigma+\tau} = \sup\{[\bar{x}]^{-\sigma-\tau}|u\cdot v(\bar{x})| : \bar{x}\in\mathbb{R}^{n+1}\}$$
$$\le \sup\{[\bar{x}]^{-\sigma}|u(\bar{x})| : \bar{x}\in\mathbb{R}^{n+1}\}\sup\{[\bar{y}]^{-\tau}|v(\bar{y})| : \bar{y}\in\mathbb{R}^{n+1}\} = \|u\|_\sigma\|v\|_\tau.$$

d)

$$\|u\cdot v\|_{(\sigma+\tau,\alpha)} =$$
$$\sup\{[\bar{x}]^{-\sigma-\tau+\alpha}\frac{|u(\bar{x})\cdot v(\bar{x})-u(\bar{y})\cdot v(\bar{y})|}{d(\bar{x},\bar{y})^\alpha} : \bar{x},\bar{y}\in\mathbb{R}^{n+1},\ 0<d(\bar{x},\bar{y})\le[\bar{x}]/2\}.$$

Now

$$[\bar{x}]^{-\sigma-\tau+\alpha}\frac{|u(\bar{x})\cdot v(\bar{x})-u(\bar{y})\cdot v(\bar{y})|}{d(\bar{x},\bar{y})^\alpha}$$
$$\le [\bar{x}]^{-\sigma-\tau+\alpha}\Big(|u(\bar{x})|\frac{|v(\bar{x})-v(\bar{y})|}{d(\bar{x},\bar{y})^\alpha}+|v(\bar{y})|\frac{|u(\bar{x})-u(\bar{y})|}{d(\bar{x},\bar{y})^\alpha}\Big)$$
$$\le \|u\|_\sigma\|v\|_{(\tau,\alpha)}+\|u\|_{(\sigma,\alpha)}|v(\bar{y})|[\bar{x}]^{-\tau}$$
$$\le \|u\|_\sigma\|v\|_{(\tau,\alpha)}+\|u\|_{(\sigma,\alpha)}\big([\bar{x}]^{-\tau}|v(\bar{x})|+[x]^{-\tau}|v(\bar{x})-v(\bar{y})|([\bar{x}]/d(\bar{x},\bar{y}))^\alpha\big).$$

The last inequality uses the fact that $([\bar{x}]/d(\bar{x},\bar{y}))^\alpha \ge 2^\alpha \ge 1$. After taking sups, we get

$$\|u\cdot v\|_{(\sigma+\tau,\alpha)} \le \|u\|_\sigma\|v\|_{(\tau,\alpha)}+\|u\|_{(\sigma,\alpha)}\|v\|_\tau+\|u\|_{(\sigma,\alpha)}\|v\|_{(\tau,\alpha)}.$$

We add this to inequality c) to get

$$\|u\cdot v\|_{(\sigma+\tau,\alpha)}+\|u\cdot v\|_{\sigma+\tau} \le \|u\|_{\sigma,\alpha}\|v\|_{\tau,\alpha}.$$

e) By the mean value theorem

$$\phi(\bar{x})-\phi(\bar{y}) = (\partial\phi/\partial x_1,\cdots,\partial\phi/\partial x_n,\partial\phi/\partial t)(\bar{z})\cdot(\bar{x}-\bar{y}),$$

where $\bar{z}$ is some point on the line connecting $\bar{x}$ and $\bar{y}$ in $\mathbb{R}^{n+1}$. Thus

$$|\phi(\bar{x})-\phi(\bar{y})| \le \sum(|\partial\phi/\partial x_i(\bar{z})||x_i-y_i|)+|D_t\phi(\bar{z})||t-s|.$$

Multiplying both sides of the above inequality by $[\bar{x}]^{-\sigma+\alpha}/d(\bar{x},\bar{y})^\alpha$, $0<\alpha\le 1$, yields

$$[\bar{x}]^{-\sigma+\alpha}|\phi(\bar{x})-\phi(\bar{y})|/d(\bar{x},\bar{y})^\alpha \le [\bar{x}]^{-\sigma+\alpha}\sum(|\partial\phi/\partial x_i(\bar{z})||x_i-y_i|/d(\bar{x},\bar{y})^\alpha)$$
$$+[\bar{x}]^{-\sigma+\alpha}|D_t\phi(\bar{z})||t-s|/d(\bar{x},\bar{y})^\alpha$$
$$\le [\bar{x}]^{-\sigma+\alpha}\sum(\|\partial\phi/\partial x_i\|_{\sigma-1}[\bar{z}]^{\sigma-1}/d(\bar{x},\bar{y})^{\alpha-1})+[\bar{x}]^{-\sigma+\alpha}\|D_t\phi\|_{\sigma-2}[\bar{z}]^{\sigma-2}/d(\bar{x},\bar{y})^{\sigma-2}. \tag{1}$$

We claim that if $d(\bar{x},\bar{y})\le[\bar{x}]/2$, then $[\bar{x}]/4\le[\bar{z}]\le 2[\bar{x}]$. Geometric considerations show that for any fixed $\rho>0$ and $\bar{x}\in\mathbb{R}^{n+1}$, $\{\bar{y} : d(\bar{x},\bar{y})\le\rho\}$ is convex; so in particular $\{\bar{y} : d(\bar{x},\bar{y})\le[\bar{x}]/2\}$ is convex and, therefore, $\bar{z}$ satisfies $d(\bar{x},\bar{z})\le[\bar{x}]/2$. Now

$$[\bar{z}]^2 = 1+d(\bar{0},\bar{z})^2 \le 1+(d(\bar{x},\bar{0})+d(\bar{x},\bar{z}))^2 \le 1+([\bar{x}]+[\bar{x}]/2)^2 = 1+9[\bar{x}]^2/4 \le 4[\bar{x}]^2,$$

since $1 \leq [\bar{x}]$. This shows

$$[\bar{z}] \leq 2[\bar{x}]. \tag{2}$$

For any real number $r \geq 0$ we have by elementary algebra the inequality

$$1 \leq \frac{1+\sqrt{r}}{\sqrt{1+r}} \leq \sqrt{2};$$

therefore, for any $\bar{x} \in \mathbb{R}^{n+1}$ we have that

$$1 \leq \frac{1+d(\bar{x},\bar{0})}{[\bar{x}]} \leq \sqrt{2}, \tag{3}$$

so

$$d(\bar{x},\bar{z}) \leq \frac{[\bar{x}]}{2} \leq \frac{1+d(\bar{x},\bar{0})}{2}.$$

We have

$$1+d(\bar{x},\bar{0}) - \frac{1+d(\bar{x},\bar{0})}{2} \leq 1+d(\bar{z},\bar{0}),$$

so by (3)

$$\frac{1+d(\bar{x},\bar{0})}{2} \leq 1+d(\bar{z},\bar{0}) \leq \sqrt{2}[\bar{z}] \text{ and } \frac{[\bar{x}]}{2} \leq \frac{1+d(\bar{x},\bar{0})}{2} \leq \sqrt{2}[\bar{z}],$$

so by (2)

$$\frac{[\bar{x}]}{4} < \frac{[\bar{x}]}{2\sqrt{2}} \leq [\bar{z}] \leq 2[\bar{x}]. \tag{4}$$

This establishes the claim.

Equation (4) allows us in (1) to replace $[\bar{z}]$ by $[\bar{x}]$ multiplied by a suitable constant, so we have

$$\begin{aligned}
&[\bar{x}]^{-\sigma+\alpha}|\phi(\bar{x})-\phi(\bar{y})|/d(\bar{x},\bar{y})^{\alpha} \\
&\leq M(\sigma)\Big(\big(\frac{d(\bar{x},\bar{y})}{[\bar{x}]}\big)^{1-\alpha}\sum\|\partial\phi/\partial x_i\|_{\sigma-1} + \big(\frac{d(\bar{x},\bar{y})}{[\bar{x}]}\big)^{2-\alpha}\|D_t\phi\|_{\sigma-2}\Big) \\
&\leq M(\sigma)\Big(\sum\|\partial\phi/\partial x_i\|^{\sigma-1} + \|D_t\phi\|_{\sigma-2}\Big).
\end{aligned}$$

Thus for all $\bar{y}$ in $\{\bar{y} : d(\bar{x},\bar{y}) \leq [\bar{x}]/2\}$ we have

$$[\bar{x}]^{-\sigma+\alpha}|\phi(\bar{x})-\phi(\bar{y})|/d(\bar{x},\bar{y})^{\alpha} \leq M(\sigma)\Big(\sum\|\partial\phi/\partial x_i\|_{\sigma-1} + \|D_t\phi\|_{\sigma-2}\Big).$$

Taking sups yields

$$\|\phi\|_{(\sigma,\alpha)} \leq M(\sigma)\Big(\sum\|\partial\phi/\partial x_i\|_{\sigma-1} + \|D_t\phi\|_{\sigma-2}\Big)$$

for all $0 < \alpha \leq 1$. In particular,

$$\|\phi\|_{(\sigma,1)} \leq M(\sigma)\Big(\sum\|\partial\phi/\partial x_i\|_{\sigma-1} + \|D_t\phi\|_{\sigma-2}\Big).$$

f) We have $\|u_m\|_{\sigma,\alpha} \le M$ where

$$\begin{aligned}\|u_m\|_{\sigma,\alpha} &= \sup\left\{[\bar{x}]^{-\sigma}|u_m(\bar{x})| : \bar{x} \in \mathbb{R}^{n+1}\right\} \\ &+ \sup\left\{[\bar{x}]^{-\sigma+\alpha}|u_m(\bar{x}) - u_m(\bar{y})|/d(\bar{x}, y^-)^\alpha : \bar{x},\, \bar{y} \in \mathbb{R}^{n+1},\, 0 < d(\bar{x},\bar{y}) \le [\bar{x}]/2\right\},\end{aligned} \tag{5}$$

so $\{u_m\}$ is a uniformly bounded equicontinuous (with respect to the Euclidean metric) family on any compact subset of $\mathbb{R}^{n+1}$. By the Arzela-Ascoli theorem, there is a subsequence, which we relabel $\{u_m\}$, coverging uniformly with respect to the Euclidean norm on compact subsets of $\mathbb{R}^{n+1}$ to some continuous function u. Taking limits in (5), we see that the inequality holds for u; i.e., $u \in B_{\sigma,\alpha}$ and $\|u\|_{\sigma,\alpha} \le M$.

To examine convergence in $B_{\tau,\beta}$ we need to investigate

$$I_m(\bar{x}) := [\bar{x}]^{-\tau}|u_m(\bar{x}) - u(\bar{x})|$$

and

$$J_m(\bar{x},\bar{y}) := [\bar{x}]^{-\tau+\beta}|(u_m(\bar{x}) - u(\bar{x})) - (u_m(\bar{y}) - u(\bar{y}))|/d(\bar{x},\bar{y})^\beta,$$

where $0 < d(\bar{x},\bar{y}) \le [\bar{x}]/2$. Let R be a large positive number.

Case 1: $[\bar{x}] \ge R$. If $[\bar{x}] \ge R$, then

$$I_m(\bar{x}) \le [\bar{x}]^{\sigma-\tau}[\bar{x}]^{-\sigma}(|u_m(\bar{x})| + |u(\bar{x})|) \le R^{\sigma-\tau}(\|u_m\|_\sigma) + \|u\|_\sigma) \le 2MR^{\sigma-\tau}.$$

Similarly

$$J_m(\bar{x},\bar{y}) \le R^{\sigma-\tau}(\|u_m\|_{(\sigma,\beta)} + \|u\|_{(\sigma,\beta)}) \le 2MR^{\sigma-\tau}.$$

Case 2: $1 \le [\bar{x}] \le R$. Since $\{\bar{x} : 1 \le [\bar{x}] \le R\}$ is a compact set,

$$I_m(\bar{x}) \le \max\left\{[\bar{x}]^{-\tau} : 1 \le [\bar{x}] \le R\right\}|u_m(\bar{x}) - u(\bar{x})| \le N(\tau)|u_m(\bar{x}) - u(\bar{x})|, \tag{6}$$

where $N(\tau) = 1$, when $\tau \ge 0$ and $N(\tau) = R^{-\tau}$ when $\tau < 0$. We break Case 2 into two subcases:

Subcase 2i): $1 \le [\bar{x}] \le R$, $0 \le d(\bar{x},\bar{y}) \le \delta$. Using $\tau > \sigma$, $0 < \beta < \alpha$, and $[\bar{x}] \ge 1$, we get

$$J_m(\bar{x},\bar{y}) \le (\|u_m\|_{(\sigma,\alpha)} + \|u\|_{(\sigma,\alpha)})\delta^{\alpha-\beta} \le 2M\delta^{\alpha-\beta}.$$

Subcase 2ii): $1 \le [\bar{x}] \le R$, $d(\bar{x},\bar{y}) \ge \delta$.

$$J_m(\bar{x},\bar{y}) \le \max\left\{1, R^{-\tau+\beta}\right\}\delta^{-\beta}\big(|u_m(\bar{x}) - u(\bar{x}))| + |(u_m(\bar{y}) - u(\bar{y}))|\big). \tag{7}$$

Given $\epsilon > 0$ choose R large enough to make $2MR^{\sigma-\tau} < \epsilon$ and then choose δ small enough to make $2M\delta^{\alpha-\beta} < \epsilon$. Finally choose, by uniform convergence of u_m on the compact set $\{\bar{x} : 1 \le [\bar{x}] \le R\}$, m large enough to make the expressions (6) and (7) less than ϵ. This makes $I_m(\bar{x})$ and $J_m(\bar{x})$ less than ϵ for all $\bar{x}$, $\bar{y}$ so $\|u_m - u\|_{\tau,\beta} \to 0$ as $m \to \infty$.

g) Let Ω be a bounded region in $\mathbb{R}^{n+1}$ such that for any $\bar{x}$ and $\bar{y}$ in Ω, we have $0 < d(\bar{x},\bar{y}) \le [\bar{x}]/2$. For all $\bar{x}, \bar{y}$ in Ω we have that $(|f(\bar{x}) - f(\bar{y})|/d(\bar{x},\bar{y})^\alpha) \le M\|f\|_{(\sigma,\alpha)}$, where $M = M(\Omega,\sigma,\alpha)$ is an upper bound of $[\bar{x}]^{\sigma-\alpha}$ on Ω. Thus f is locally Hölder-continuous with exponent α.

h) Let $\{u_\ell\}$ be a Cauchy sequence in B_σ. Then, $\{u_\ell/[\bar{x}]^\sigma\}$ is uniformly Cauchy in the $\|\cdot\|_\infty$ norm and so there is a continuous function v defined on $\mathbb{R}^{n+1}$, with the same range $\mathbb{R}^n$, $\mathbb{R}^{n+1}$, or $\mathbb{R}^{n\times n}$ as the u_ℓ, such that $u_\ell/[\bar{x}]^\sigma \to v$ uniformly; i.e., $\|u_\ell/[\bar{x}]^\sigma - v\|_\infty \longrightarrow 0$,

$\|u_\ell/[\bar{x}]^\sigma - v[\bar{x}]^\sigma/[\bar{x}]^\sigma\|_\infty \longrightarrow 0$, $\|u_\ell - v[\bar{x}]^\sigma\|_\sigma \longrightarrow 0$. Let $w(\bar{x}) = v(\bar{x})[\bar{x}]^\sigma$. Then $u_\ell \to w$ in the norm of B_σ. Since $\|w\|_\sigma \leq \|u_\ell - w\|_\sigma + \|u_\ell\|_\sigma$ is finite, $w \in B_\sigma$. This shows that B_σ is a Banach space. Next we show that $B_{\sigma,\alpha}$ is a Banach space. Let $\{u_\ell\}$ be a Cauchy sequence in $B_{\sigma,\alpha}$. For all $\epsilon > 0$ there is an integer N such that for $m, n > N$,

$$\sup\left\{[\bar{x}]^{-\sigma}|u_m(x,t) - u_n(x,t)| : \bar{x} \in \mathbb{R}^{n+1}\right\} +$$
$$\sup\left\{[\bar{x}]^{-\sigma+\alpha}\frac{|[u_m(\bar{x}) - u_n(\bar{x})] - [u_m(\bar{y}) - u_n(\bar{y})]|}{d(x,y)^\alpha} : \bar{x},\, \bar{y} \in \mathbb{R}^{n+1},\ 0 < d(\bar{x},\bar{y}) \leq \frac{[\bar{x}]}{2}\right\}$$
$$\leq \epsilon.$$

We already know that there is a continuous function w such that $u_\ell(\bar{x}) \longrightarrow w(\bar{x})$ in the norm of B_σ. If we define

$$U_\ell(\bar{x},\bar{y}) = (u_\ell(\bar{x}) - u_\ell(\bar{y}))/([\bar{x}]^{(\sigma-\alpha)}d(\bar{x},\bar{y})^\alpha)$$

we have that $\{U_\ell\}$ is uniformly Cauchy in $\{(\bar{x},\bar{y}) \in \mathbb{R}^{n+1} : 0 < d(\bar{x},\bar{y}) \leq [\bar{x}]/2\}$ and there is some function W defined on $\{(\bar{x},\bar{y}) : 0 < d(\bar{x},\bar{y}) \leq [\bar{x}]/2\}$ such that $U_\ell \longrightarrow W$ uniformly in $\{\bar{x},\bar{y}) : 0 < d(\bar{x},\bar{y}) \leq [\bar{x}]/2\}$. We see that

$$W(\bar{x},\bar{y}) = (w(\bar{x}) - w(\bar{y}))/([\bar{x}]^{(\sigma-\alpha)}d(\bar{x},\bar{y})^\alpha),$$

since $u_\ell(\bar{x}) \longrightarrow w(\bar{x})$ pointwise and $U_\ell(\bar{x},\bar{y})$ converges pointwise to

$$(w(\bar{x}) - w(\bar{y}))/([\bar{x}]^{(\sigma-\alpha)}d(\bar{x},\bar{y})^\alpha) = W(\bar{x},\bar{y}).$$

To see that $w \in B_{\sigma,\alpha}$, we observe that $\|u_\ell - w\|_{\sigma,\alpha} \longrightarrow 0$ so

$$\|w\|_{\sigma,\alpha} \leq \|u_\ell - w\|_{\sigma,\alpha} + \|u_\ell\|_{\sigma,\alpha} < \infty.$$

Theorem 3. (Schauder estimate). *Let ϕ be a classical solution of the nonhomogeneous equation $\mathcal{L}\phi = f$, with $\|\phi\|_\sigma$ finite for some $\sigma \in \mathbb{R}$. Suppose that for some α, $0 < \alpha < 1, \|f\|_{\sigma-2,\alpha}$ is finite and that for some nonnegative constant M_1,*

$$\|a\|_{0,\alpha}\,, \quad \|b\|_{-1,\alpha}\,, \quad \|c\|_{-2,\alpha} \leq M_1\,. \tag{8}$$

Assume also that the matrix a is symmetric and that there is a constant $M_2 > 0$ such that for all $\bar{x} = (x,t)$ in $\mathbb{R}^{n+1}$ and real $(1 \times n)$ vector ξ,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}(\bar{x})\xi_i\xi_j \geq M_2|\xi|^2.$$

Then there is a constant $M = M(M_1, M_2, n, \alpha, \sigma)$ such that

$$\||\phi|\|_{\sigma,\alpha} \leq M(\|\phi\|_\sigma + \|f\|_{\sigma-2,\alpha}).$$

Proof: Consider the set

$$S = \{\bar{x} = (x,t) : |x| \leq 5 \text{ and } |t| \leq 5\}.$$

We obtain an estimate for ϕ on S. Let

$$\Omega = \{\bar{x} = (x,t) : |x| < 9 \text{ and } |t| < 21\}.$$

We check that the hypotheses for Lemma 1 are satisfied in Ω and begin by showing that $|a_{i,j}|_\alpha^\Omega \le N(M_1,\alpha)$. Observe that $|a_{i,j}|_0^\Omega = \sup\{|a_{i,j}(\bar{x})| : \bar{x} \in \Omega\} \le \|a_{i,j}\|_0$. Since for all $\bar{x}$, $\bar{y}$ in Ω the values of $d_{\bar{x}}$, $d_{\bar{y}}$ and $d_{\bar{x},\bar{y}}$ are no greater than 9, we have

$$\begin{aligned} H_\alpha^\Omega(a_{i,j}) &\le 9^\alpha\Big(\sup\Big\{\frac{|a_{i,j}(\bar{x}) - a_{i,j}(\bar{x}) - a_{i,j}(\bar{y})|}{d(\bar{x},\bar{y})^\alpha} : \bar{x},\bar{y} \in \Omega \text{ and } 0 < d(\bar{x},\bar{y}) \le [\bar{x}]/2\Big\} \\ &\quad + \sup\Big\{\frac{|a_{i,j}(\bar{x}) - a_{i,j}(\bar{y})|}{d\bar{x},\bar{y})^\alpha} : \bar{x},\bar{y} \in \Omega \text{ and } d(\bar{x},\bar{y}) \ge [\bar{x}]/2\Big\}\Big) \\ &\le N(\alpha)\big(\|a_{i,j}\|_{(0,\alpha)} + \|a_{i,j}\|_0\big) \le N(M_1,\alpha). \end{aligned}$$

We will now show that $|db_i|_\alpha^\Omega \le N(M_1,\alpha)$. We have

$$|db_i|_0^\Omega \le 9\sup\{|b_i(\bar{x})| : \bar{x} \in \Omega\} \le 9\|b\|_{-1}.$$

$$\begin{aligned} H_\alpha^\Omega(db_i) &\le 9^{1+\alpha}\Big(\sup\Big\{\frac{|b_i(\bar{x}) - b_i(\bar{y})|}{d(\bar{x},\bar{y})^\alpha} : \bar{x},\bar{y} \in \Omega,\ \ 0 < d(\bar{x},\bar{y}) \le [\bar{x}]/2\Big\} \\ &\quad + \sup\Big\{\frac{|b_i(\bar{x}) - b_i(\bar{y})|}{d(\bar{x},\bar{y})^\alpha} : \bar{x},\bar{y} \in \Omega,\ d(\bar{x},\bar{y}) \ge [\bar{x}]/2\Big\}\Big) \\ &\le M(\alpha)(\|b\|_{(-1,\alpha)} + \|b\|_{-1}) = M(\alpha)\|b\|_{-1,\alpha}, \end{aligned}$$

and by hypothesis

$$M(\alpha)\|b\|_{-1,\alpha} \le M(\alpha)M_1 = N(M_1,\alpha).$$

In a similar fashion, we can show that $|d^2c|_\alpha^\Omega \le N(M_1,\alpha)$, $|d^2 f|_\alpha^\Omega \le M(\sigma,\alpha)\|f\|_{\sigma-2,\alpha} < \infty$, and that $|\phi|_0^\Omega \le M(\sigma)\|\phi\|_\sigma$.

By Lemma 1 and the preceding estimates, there is $M = M(M_1, M_2, n, \alpha, \sigma)$ such that

$$|\phi|_{2+\alpha}^\Omega \le M(|\phi|_0^\Omega + |d^2 f|_\alpha^\Omega) \le M(\|\phi\|_\sigma + \|f\|_{\sigma-2,\alpha}). \tag{9}$$

Now if $|x| < 5$, $|t| < 5$ and $0 < d(\bar{x},\bar{y}) < [\bar{x}]/2$ then we have

$$1 \le d_{\bar{x}},\, d_{\bar{y}},\, d_{\bar{x},\bar{y}} \le 9, \ \text{ and } \ 1 \le [\bar{x}] \le \sqrt{(1+5+25)} < 6,$$

so we may write (9) as

$$\begin{aligned} &[\bar{x}]^{-\sigma}|\phi(\bar{x})| + [\bar{x}]^{\alpha-\sigma}\frac{|\phi(\bar{x}) - \phi(\bar{y})}{d(\bar{x},\bar{y})^\alpha} \\ &+ \sum\Big([\bar{x}]^{1-\sigma}|D_x\phi(\bar{x})| + [\bar{x}]^{1+\alpha-\sigma}\frac{|D_x\phi(\bar{x}) - D_x\phi(\bar{y})|}{d(x,y)^\alpha}\Big) \\ &+ \sum\Big([\bar{x}]^{2-\sigma}|D_x^2\phi(\bar{x})| + [\bar{x}]^{2+\alpha-\sigma}\frac{|D_x^2\phi(\bar{x}) - D_x^2\phi(\bar{y})|}{d(x,y)^\alpha}\Big) \\ &+ [\bar{x}]^{2-\sigma}|D_t\phi(\bar{x})| + [\bar{x}]^{2+\alpha-\sigma}\frac{|D_t\phi(\bar{x}) - D_t\phi(\bar{y})|}{d(x,y)^\alpha} \\ &\le M(\|\phi\|_\sigma + \|f\|_{\sigma-2,\alpha}). \end{aligned} \tag{10}$$

We will presently show the inequality (10) to hold in all of $\mathbb{R}^{n+1}$ and then taking sups in (10) we will obtain

$$\begin{aligned} &\|\phi\|_\sigma + \|\phi\|_{(\sigma,\alpha)} + \sum \left(\|D_x\phi\|_{\sigma-1} + \|D_x\phi\|_{(\sigma-1,\alpha)}\right) \\ &+ \sum \left(\|D_x^2\phi\|_{\sigma-2} + \|D_x^2\phi\|_{(\sigma-2,\alpha)}\right) + \|D_t\phi\|_{\sigma-2} + \|D_t\phi\|_{(\sigma-2,\alpha)} \\ \leq\ & M(\|\phi\|_\sigma + \|f\|_{\sigma-2,\alpha}). \end{aligned} \tag{11}$$

We now derive the inequality (10) for a point $\bar{x}$ from $\{(x,t) : |x| \geq 5 \text{ or } |t| \geq 5\}$. For this point $\bar{x}$ we shall define $\rho = \rho(\bar{x}) = [\bar{x}]/6$. Let

$$\Omega(\bar{x}) = \left\{\bar{z} = (z,r) \in \mathbb{R}^{n+1} : \rho < \max\{|z|, \sqrt{|r|}\} < 10\rho\right\}.$$

We show that $\bar{x} \in \Omega(\bar{x})$. If $\bar{x} \notin \Omega(\bar{x})$ then either $|x| \leq \rho$ and $|t| \leq \rho^2$, or $|x| \geq 10\rho$ or $|t| \geq 100\rho^2$. Suppose that $|x| \leq \rho$ and $|t| \leq \rho^2$. Then

$$|x^2| + |t| < 2\rho^2 < \frac{2(1 + |t| + |x|^2)}{36},$$

and hence $17|x|^2 + 17|t| < 1$. This is impossible since either $|x| \geq 5$ or $|t| \geq 5$. Now suppose that $|x| \geq 10\rho$. Then

$$|x|^2 \geq \frac{100(1 + |t| + |x|^2)}{36} > 2|x|^2,$$

which is a contradiction. If $|t| \geq 100\rho^2$, then

$$|t| \geq \frac{100(1 + |t| + |x|^2)}{36} > 2|t|.$$

Again this is a contradiction and so $\bar{x} \in \Omega(\bar{x})$.

We now check that the hypotheses for Lemma 1 are satisfied in $\Omega(\bar{x})$. We first note that for $\bar{z}, \bar{y}$ in $\Omega(\bar{x})$ we have, $d_{\bar{z}},\ d_{\bar{y}},\ d_{\bar{x},\bar{y}} \leq 10\rho(\bar{x})$. Also for all $\bar{z} = (x,r) \in \Omega$,

$$1/36 \leq \rho^2 \leq (1 + |r| + |z|^2) \leq 1 + 100\rho^2 + 100\rho^2 \leq 225\rho^2,$$

so $1/6 < \rho \leq [\bar{z}] \leq 15\rho$. We show that $|a_{i,j}|_\alpha^\Omega \leq M(M_1, \alpha)$. We have

$$|a_{i,j}|_\alpha^\Omega = |a_{i,j}|_0^\Omega + H_\alpha^\Omega(a_{i,j}),$$

$$|a_{i,j}|_0^\Omega = \sup\{|a_{i,j}(\bar{z})| : \bar{z} \in \Omega\} \leq \|a_{i,j}\|_0 \leq M_1,$$

$$\begin{aligned} H_\alpha^\Omega(a_{i,j}) &\leq (10\rho)^\alpha \Big(\sup\{\|a_{i,j}\|_{(0,\alpha)}[\bar{z}]^{-\alpha} : \bar{z} \in \Omega\} \\ &+ \sup\{|a_{i,j}(\bar{z}) - a_{i,j}(\bar{y})|/([\bar{z}]/2)^\alpha : \bar{z}, \bar{y} \in \Omega \text{ and } d(\bar{z},\bar{y}) \geq [\bar{z}]/2\}\Big) \\ &\leq N(\alpha)\rho^\alpha \Big(\|a_{i,j}\|_{0,\alpha)}\rho^{-\alpha} + \sup\{|a_{i,j}(\bar{z}) - a_{i,j}(\bar{y})|/(\rho/2)^\alpha : \bar{z}, \bar{y} \in \Omega \text{ and } d(\bar{z},\bar{y}) \geq [\bar{z}]/2\}\Big), \end{aligned}$$

since for $\bar{z} \in \Omega$ we have $\rho \leq [\bar{z}] \leq 15\rho$. Hence

$$H_\alpha^\Omega(a_{i,j}) \leq N(\alpha)\|a_{i,j}\|_{0,\alpha} \leq N(\alpha)M_1 = M(M_1, \alpha).$$

We now show that we have $|db_i|_\alpha^\Omega \le M$.

$$|db_i|_0^\Omega \le 10\rho \sup\left\{\frac{[\bar z]|b_i(\bar z)|}{\rho} : \bar z \in \Omega\right\} \le M\|b\|_{-1},$$

$$H_\alpha^\Omega(db_i) \le M(\alpha)\|b\|_{-1,\alpha} \le M(\alpha)M_1 = M(M_1,\alpha).$$

Similarly we can show that $|d^2c|_\alpha^\Omega \le M$.

We have now shown that part A of the hypotheses of Lemma 1 is satisfied. We proceed to show that part C is satisfied. It follows from Lemma 2g that f is Hölder-continuous with expoent α. We show that $|d^2 f|_\alpha^\Omega \le N(\sigma,\alpha)\rho^\sigma \|f\|_{\sigma-2,\alpha} < \infty$. We estimate

$$|d^2 f|_\alpha^\Omega = |d^2 f|_0^\Omega + H_\alpha^\Omega(d^2 f).$$

$$\begin{aligned}|d^2 f|_0^\Omega &\le (10\rho)^2 \sup\{|f(\bar z)| : \bar z \in \Omega\} \le (10\rho)^2 \sup\{\|f\|_{\sigma-2}[\bar z]^{\sigma-2} : \bar z \in \Omega\}\\ &\le \begin{cases} (10\rho)^2(15\rho)^{\sigma-2}\|f\|_{\sigma-2} & \sigma \ge 2 \\ (10\rho)^2\rho^{\sigma-2}\|f\|_{\sigma-2}, & \sigma < 2\end{cases} \le M(\sigma)\rho^\sigma\|f\|_{\sigma-2},\end{aligned}$$

$$\begin{aligned}H_\alpha^\Omega(d^2 f) &\le M(\alpha)\rho^{\alpha+2}\begin{cases} \|f\|_{(\sigma-2,\alpha)}(15\rho)^{\sigma-2-\alpha} + \|f\|_{\alpha-2}(15)^{\sigma-2-\alpha}, & \sigma \ge 2+\alpha \\ \|f\|_{(\sigma-2,\alpha)}\rho^{\sigma-2-\alpha} + \|f\|_{\sigma-2}\rho^{\sigma-2-\alpha}, & \sigma < 2+\alpha\end{cases}\\ &\le N(\sigma,\alpha)\rho^\sigma\|f\|_{\sigma-2,\alpha}.\end{aligned}$$

Finally we show that $|\phi|_0^\Omega < N(\sigma)\rho^\sigma\|\phi\|_\sigma < \infty$. We have by hypothesis

$$\|\phi\|_\sigma = \sup\{[\bar z]^{-\sigma}|\phi(\bar z)| : \bar z \in \Omega\} < \infty.$$

For $\bar z \in \Omega$, we have $\rho \le [\bar z] \le 15\rho$, so

$$|\phi|_0^\Omega = \sup\{|\phi(\bar z)| : \bar z \in \Omega\} \le \sup\{[\bar z]^\sigma\|\phi\|_\sigma : \bar z \in \Omega\} \le \begin{cases}(15\rho)^\sigma\|\phi\|_\sigma, & \text{if } \sigma \ge 0 \\ \rho^\sigma\|\phi\|_\sigma, & \text{if } \sigma \le 0\end{cases}$$

so

$$|\phi|_0^\Omega \le N(\sigma)\rho^\sigma\|\phi\|_\sigma < \infty.$$

We apply Lemma 1 to $\Omega(\bar x)$ and get

$$|\phi|_{2+\alpha}^\Omega \le M(M_1, M_2, n, \alpha, \sigma)(|\phi|_0^\Omega + |d^2 f|_\alpha^\Omega);$$

thus

$$|\phi|_\alpha^\Omega + \sum |dD_x\phi|_\alpha^\Omega + \sum |d^2 D_x^2\phi|_\alpha^\Omega + |d^2 D_t\phi|_\alpha^\Omega \le M\rho^\sigma(\|\phi\|_\sigma + \|f\|_{\sigma-2,\alpha}),$$

where summations are over all possible derivatives. Since $[\bar x] = 6\rho$, it follows that $\rho \ge 1/6$. We claim that if $d(\bar x, \bar y) \le [\bar x]/2 = 3\rho$, then $\bar y \in \Omega(\bar x)$. We note that $6\rho^2 \ge 1$ as follows from $6\rho^2 = [(1+|t|+|x|^2)/6] \ge 1$, since either $|x| \ge 5$ or $|t| \ge 5$. We now show that $\bar y \in \Omega$. From $[\bar x] = 6\rho$, we get that

$$1 + d(\bar x, \bar 0)^2 = 1 + |t| + |x|^2 = 36\rho^2,$$

so

$$36\rho^2 - 1 = d(\bar{x},\bar{0})^2 < 36\rho^2, \quad 36\rho^2 - 6\rho^2 \le d(\bar{x},\bar{0})^2 < 36\rho^2$$

and hence

$$\sqrt{30}\rho \le d(\bar{x},\bar{0}) < 6\rho.$$

We now have the conditions $0 \le d(\bar{x},\bar{y}) \le 3\rho$ and $\sqrt{30}\rho \le d(\bar{x},\bar{0}) < 6\rho$, which give us by the triangle inequality that

$$(\sqrt{30} - 3)\rho \le d(\bar{y},\bar{0}) < 9\rho, \qquad 2\rho \le d(\bar{y},\bar{0}) < 9\rho$$

so

$$4\rho^2 \le |y|^2 + |s| < 81\rho^2.$$

Thus $|y| < 9\rho, |s| < 81\rho^2$, and either $|y| \ge 2\rho$ or $|s| \ge 4\rho^2$, giving $\bar{y} \in \Omega(\bar{x})$. Also, $\rho \le d_{\bar{x}}$, $d_{\bar{y}}$, $d_{\bar{x},\bar{y}} \le 15\rho$, which gives $[\bar{x}]/6 \le d_{\bar{x}}$, $d_{\bar{y}}$, $d_{\bar{x},\bar{y}} \le 15[\bar{x}]/6$. As (10) follows from (9), we now have (11), which gives

$$\|\phi\|_\sigma + \sum \|D_x\phi\|_{\sigma-1,\alpha} + \sum \|D_x^2\phi\|_{\sigma-2,\alpha} + \|D_t\phi\|_{\sigma-2,\alpha} \le M(\|\phi\|_\sigma + \|f\|_{\sigma-2,\alpha}). \tag{12}$$

By Lemma 2e,

$$\|\phi\|_{(\sigma,1)} \le M(\sigma)\Big(\sum \|D_x\phi\|_{\sigma-1} + \|D_t\phi\|_{\sigma-2}\Big)$$

so we can add $\|\phi\|_{(\sigma,1)}$ to the left-hand side of (12) to get

$$\|\phi\|_{\sigma,1} + \sum \|D_x\phi\|_{\sigma-1,\alpha} + \sum \|D_x^2\phi\|_{\sigma-2,\alpha} + \|D_t\phi\|_{\sigma-2,\alpha} \le M(\|\phi\|_\sigma + \|f\|_{\sigma-2,\alpha}),$$

i.e.,

$$\|\|\phi\|\|_{\sigma,1} \le M(\|\phi\|_\sigma + \|f\|_{\sigma-2,\alpha}).$$

This proves Theorem 3.

3. The nonhomogeneous heat equation. Before further discussing entire solutions of the general parabolic equation, it is useful both as motivation and as a mathematical aid to study entire solutions of the equation

$$\sum_{i=1}^{n} \frac{\partial^2\phi}{\partial x_i^2} - \frac{\partial\phi}{\partial t} + f = 0.$$

In $\mathbb{R}^{n+1}$, let k denote the heat kernel

$$k(x,t) = \begin{cases} e^{-|x|^2/(4t)}/(4\pi t)^{n/2}, & t > 0 \\ 0, & t \le 0, \end{cases}$$

and for a real valued function f on $\mathbb{R}^{n+1}$ let

$$\mathcal{K}f(\bar{x}) = \int_{\mathbb{R}^{n+1}} k(\bar{x} - \bar{y}) f(\bar{y})\, dy\, ds.$$

Recall that

$$\int_{\mathbb{R}^n} k((x,t)-(y,s))\,dy = \begin{cases} 1, & t > s \\ 0, & t \le s. \end{cases}$$

Lemma 4. *Let f be a real valued function in $\mathbb{R}^{n+1}$ in the space $B_{-\tau}$ where $2 < \tau < n+2$. Then for $\bar{x} \in \mathbb{R}^{n+1}$ the integral $\mathcal{K}f(\bar{x})$ exists and*

$$|\mathcal{K}f(\bar{x})| \le [\bar{x}]^{2-\tau} M(n,\tau)\|f\|_{-\tau}.$$

Proof: We first show that

$$|\mathcal{K}f(\bar{x})| \le M(\tau)\|f\|_{-\tau} \ \text{ if } \ d(\bar{x},\bar{0}) \le 1. \tag{13}$$

We have

$$|\mathcal{K}f(\bar{x})| \le \|f\|_{-\tau} \int_{\mathbb{R}^{n+1}} k(\bar{x}-\bar{y})(1+|s|+|y|^2)^{-\tau/2}\,dy\,ds,$$

and

$$\int_{\mathbb{R}^{n+1}} k(\bar{x},\bar{y})(1+|s|+|y|^2)^{-\tau/2}\,dy\,ds \le \int_{t-2\ge 2} |s|^{-\tau/2}\,ds + \int_{|t-s|\le 2} ds. \tag{14}$$

Since $|t| \le d(\bar{x},\bar{0})^2 \le 1$, it follows that

$$(14) \ \le \int_{s\le -1} |s|^{-\tau/2}\,ds + \int_{|t-s|\le 2} ds = M(\tau).$$

This establishes (13).

We now show that

$$|\mathcal{K}f(\lambda x, \lambda^2 t)| \le \lambda^{2-\tau}\|f\|_{-\tau} M(n,\tau) \tag{15}$$

if $d(\bar{x},\bar{0}) = 1$, $\lambda > 0$. We have

$$|\mathcal{K}f(\lambda x,\lambda^2 t)| \le \int_{\substack{\mathbb{R}^{n+1}\\ \lambda^2 t > s}} \frac{e^{-|\lambda x - y|^2/(4(\lambda^2 t - s))}}{[4\pi(\lambda^2 t - s)]^{n/2}} |f(y,s)|\,dy\,ds.$$

Let $y = \lambda z$ and $s = \lambda^2 r$. Then

$$|\mathcal{K}f(\lambda x,\lambda^2 t)| \le \lambda^{2-\tau}\|f\|_{-\tau} \int_{\mathbb{R}^{n+1}} k(\bar{x}-\bar{z})(|z|^2+|r|)^{-\tau/2}\,dz\,dr.$$

We derive an upper bound for

$$\int_{\mathbb{R}^{n+1}} k(\bar{x}-\bar{z})(|z|^2+|r|)^{-\tau/2}\,dz\,dr.$$

Let

$$\Omega_1 = \{(z,r) \in \mathbb{R}^{n+1} : |z|^2+|r| \ge 1/2\}, \ \text{ and } \ \Omega_2 = \{(z,r) \in \mathbb{R}^{n+1} : |z|^2+|r| \le 1/2\}.$$

In Ω_1, $1+|z|^2+|r| \le 3(|z|^2+|r|)$, so

$$\int_{\Omega_1} k(\bar{x}-\bar{z})(|z|^2+|r|)^{-\tau/2}\,dz\,dr \le 3^{\tau/2}\int_{\mathbb{R}^{n+1}} k(\bar{x}-\bar{z})(1+|z|^2+|r|)^{-\tau/2}\,dz\,dr \le M(\tau)$$

as was shown in the derivation of (13).

In Ω_2, $|z|^2 + |r| = d(\bar{0}, \bar{z})^2 \leq 1/2$ and $|x|^2 + |t| = 1$, so

$$d(\bar{x}, \bar{0}) \leq d(\bar{x}, \bar{z}) + d(\bar{0}, \bar{z}) \leq d(\bar{x}, \bar{0}) + d(\bar{0}, \bar{z}) + d(\bar{0}, \bar{z}),$$

hence

$$1 \leq d(\bar{x}, \bar{z}) + 1/\sqrt{2} \leq 1 + \sqrt{2},$$

$$1/20 < (1 - 1/\sqrt{2})^2 \leq d(\bar{x}, \bar{z})^2 = |x - z|^2 + |t - r| \leq (1 + \sqrt{2})^2 \leq 6.$$

Thus either $|x - z|^2 \geq 1/40$ or $|t - r| \geq 1/40$. If $|t - r| \geq 1/40$, then

$$k(\bar{x} - \bar{y}) \leq \frac{1}{|t - r|^{n/2}} \leq 40^{n/2} = M(n).$$

Suppose that $|x - z|^2 \geq 1/40$ and let $m = [n/2] + 1$, where $[n/2]$ is the greatest integer no larger than $n/2$. Then

$$k(\bar{x} - \bar{y}) \leq \frac{m!}{|t - r|^{n/2}} \frac{4|t - r|^m}{|x - z|^{2m}} \leq M(n).$$

Therefore,

$$\int_{\Omega_2} k(\bar{x} - \bar{z})(|z|^2 + |r|)^{-\tau/2}\, dz\, dr \leq M(n) \int_{\rho=0}^{1} \int_{r=0}^{1} (\rho^2 + r)^{-\tau/2} \rho^{n-1}\, dr\, d\rho \leq M(n, \tau).$$

This establishes (15).

Suppose now that $\bar{y} \in \mathbb{R}^{n+1}$ and $d(\bar{0}, \bar{y}) = \lambda \geq 1$. Then we can write (y, s) as $(\lambda x, \lambda^2 t)$ where $x = y/\lambda$ and $t = s/\lambda^2$. Observe that $d(\bar{x}, \bar{0}) = 1$. We have from (15) that

$$[\bar{y}]^{\tau-2}|\mathcal{K}f(\bar{y})| \leq [\bar{y}]^{\tau-2}\lambda^{2-\tau}\|f\|_{-\tau}M(n, \tau) \leq 2^{(\tau-2)/2}\|f\|_{-\tau}M(n, \tau).$$

If $d(\bar{0}, \bar{y}) \leq 1$ then, by (13)

$$(1 + |s| + |y|^2)^{(\tau-2)/2}|\mathcal{K}f(\bar{y})| \leq 2^{(\tau-2)/2}\|f\|_{-\tau}M(\tau).$$

Thus the inequality of Lemma 4 is verified in all cases.

Lemma 5. *Let f be real valued on $\mathbb{R}^{n+1}$ and in some space $B_{-\tau,\alpha}$, $0 < \alpha < 1$, and $\tau > 2$. Then $\mathcal{K}f(\bar{x})$ has a continuous derivative with respect to t and continuous second derivatives with respect to the x_i's at each point of $\mathbb{R}^{n+1}$, and $\mathcal{K}f$ satisfies*

$$\sum_{i=1}^{n} \frac{\partial^2 \phi}{\partial x_i^2} - \frac{\partial \phi}{\partial t} + f = 0.$$

Moreover, $\mathcal{K}f$ vanishes at infinity.

Proof: By making τ smaller if necessary, we have by Lemma 4 that $\mathcal{K}f$ exists and vanishes at infinity. Let R be some large number and let

$$S_R = \{(y, s) : |y| \leq R \text{ and } |s| \leq R\}.$$

We investigate the region

$$\Omega = \{\bar{x} : |x| < R/3 \text{ and } |t| < R/3\}.$$

Let $\bar{x} \in \Omega$. We write

$$\mathcal{K}f(\bar{x}) = \int_{S_R} k(\bar{x} - \bar{y}) f(\bar{y})\, dy\, ds + \int_{\mathbf{R}^{n+1}\setminus S_R} k(\bar{x} - \bar{y}) f(\bar{y})\, dy\, ds.$$

It follows from results of Dressel ([3] and [4]) and Friedman (Theorem 11 on page 74 of [5]), that if f is Hölder continuous with exponent α, the first integral has continuous second derivatives with respect to the x_i's and a continuous derivative with respect to t and it satisfies

$$\sum_{i=1}^{n} \frac{\partial^2 \phi}{\partial x_i^2} - \frac{\partial \phi}{\partial t} + f = 0$$

for $\bar{x} \in \Omega$. For $\bar{x} \in \Omega$, the second integral gives upon differentiation under the integral sign any number of times with respect to the x_i's and/or with respect to t a linear combination of expressions of the form

$$\int_{\mathbf{R}^{n+1}\setminus S_r} k(\bar{x} - \bar{y}) f(\bar{y}) \Big(\prod_{i=1}^{n} (\bar{x} - \bar{y})^{\alpha(i)}\Big) / (t - s)^{\sum \alpha(i)}\, dy\, ds,$$

where $\alpha(i)$ are nonnegative integers. To justify differentiation under the integral sign, we shall show that if α is a nonnegative integer, integrals of the form

$$I(\bar{x}) = \int_{\mathbf{R}^{n+1}\setminus S_R} k(\bar{x} - \bar{y}) |f(\bar{y})| |x - y|^{\alpha} / (t - s)^{\alpha}\, dy\, ds$$

converge uniformly in $\bar{x}$ for $\bar{x} \in \Omega$, i.e., there is a function $F(\Omega, \bar{y}) = F(R, \bar{y})$ such that

$$|F(R, \bar{y})| \geq k(\bar{x} - \bar{y}) |f(\bar{y})| |x - y|^{\alpha} / |t - s|^{\alpha}$$

for all $\bar{x} \in \Omega(R)$ and

$$\int_{\mathbf{R}^{n+1}\setminus S_R} |F(R, \bar{y})|\, dy\, ds < \infty.$$

We shall assume that $\tau > 2 + 2\epsilon$, where $\epsilon > 0$. Let $\beta = n/2 + \alpha + \epsilon$. We shall write

$$I(\bar{x}) = J_1(\bar{x}) + J_2(\bar{x}) + J_3(\bar{x})$$

where

$$J_1(\bar{x}) = \int_{\substack{|s| \leq R \\ |y| \geq R}} k(\bar{x} - \bar{y}) |f(\bar{y})| |x - y|^{\alpha} / |t - s|^{\alpha}\, dy\, ds,$$

$$J_2(\bar{x}) = \int_{\substack{|s| \geq R \\ |y| \geq R}} k(\bar{x} - \bar{y}) |f(\bar{y})| |x - y|^{\alpha} / |t - s|^{\alpha}\, dy\, ds,$$

and

$$J_3(\bar{x}) = \int_{\substack{|s| \geq R \\ |y| \leq R}} k(\bar{x} - \bar{y}) |f(\bar{y})| |x - y|^{\alpha} / |t - s|^{\alpha}\, dy\, ds.$$

We first consider $J_1(\bar{x})$. If $|s| \le R$ and $|y| \ge R$ and $\bar{x} \in \Omega$, then

$$|t-s| \le |s| + |t| \le R + R/3 = 4R/3, \quad |x-y| \ge |y| - |x| \ge |y| - R/3 \ge 2|y|/3.$$

Noting that

$$k(\bar{x}-\bar{y}) \le \frac{M}{|t-s|^{n/2}} \frac{(4|t-s|)^{\beta}}{|x-y|^{2\beta}}$$

for some constant M, we have

$$\frac{|x-y|^{\alpha}}{|t-s|^{\alpha}} k(\bar{x}-\bar{y})|f(\bar{y})| \le \frac{|x-y|^{\alpha}}{|t-s|^{\alpha}} k(\bar{x}-\bar{y})\|f\|_{-\tau}(1+|s|+|y|^2)^{-\tau/2}$$

$$\le M(f,\tau)\frac{|x-y|^{\alpha}}{|t-s|^{\alpha+n/2}} \frac{(4|t-s|)^{\beta}}{|x-y|^{2\beta}} \le M(f,\tau,\alpha,\beta,R)|y|^{\alpha-2\beta} =: F(R,\bar{y}),$$

and since $\alpha - 2\beta + n < 0$, then

$$\int_{|s|\le R}\int_{|y|\ge R} (|y|)^{\alpha-2\beta}\, dy\, ds < \infty.$$

Next we consider $J_2(\bar{x})$. If $|s| \ge R$ and $|y| \ge R$ and $\bar{x} \in \Omega$, then

$$|t-s| \ge |s| - |t| \ge R - R/3 = 2R/3 \text{ and } |x-y| \ge |y| - |x| \ge |y| - R/3 \ge 2|y|/3.$$

Further since $|t-s| \le |t| + |s|$ and $|t| \le R/3 \le |t-s|/2$, it follows that $|s| \ge |t-s|/2$. Thus

$$\frac{|x-y|^{\alpha}}{|t-s|^{\alpha}} k(\bar{x}-\bar{y})|f(\bar{y})| \le \frac{|x-y|^{\alpha}}{|t-s|^{\alpha}} k(\bar{x}-\bar{y})\|f\|_{-\tau}(1+|s|+|y|^2)^{-\tau/2}$$

$$\le M\frac{|x-y|^{\alpha}}{|t-s|^{\alpha+n/2}} \frac{(4|t-s|)^{\beta}}{|x-y|^{2\beta}} \frac{|s|^{(2\epsilon-\tau)/2}}{|t-s|^{\epsilon}} \le M(|y|)^{\alpha-2\beta}|s|^{(2\epsilon-\tau)/2} := F(R,\bar{y}),$$

and since $\alpha - 2\beta + n < 0$, $\tau > 2 + 2\epsilon$, then

$$\int_{|s|\ge R} |s|^{(2\epsilon-\tau)/2} \int_{|y|\ge R} (|y|)^{\alpha-2\beta}\, dy\, ds < \infty$$

Finally we consider $J_3(\bar{x})$. If $|s| \ge R$ and $|y| \le R$ and $\bar{x} \in \Omega$, then

$$|t-s| \ge |s| - |t| \ge R - R/3 = 2R/3 \text{ and } |x-y| \le |y| + |x| \le |y| + R/3 \le 4R/3,$$

so

$$\frac{|x-y|^{\alpha}}{|t-s|^{\alpha}} k(\bar{x}-\bar{y})|f(\bar{y})| \le M\frac{|x-y|^{\alpha}}{|t-s|^{\alpha+n/2}}|s|^{-\tau/2}$$

$$\le M(2R/3)^{-\alpha-n/2}R^{\alpha}|s|^{-\tau/2} =: F(R,\bar{y}),$$

since $\tau > 2$, then

$$\int_{|s|\ge R} |s|^{-\tau/2} \int_{|y|\le R} dy\, ds < \infty$$

Since differentiation under the integral sign is allowed for $x \in \Omega(R)$, the integral

$$\int_{\mathbf{R}^{n+1}\setminus S_R} k(\bar{x}-\bar{y}) f(\bar{y})\, dy\, ds$$

satisfies

$$\sum_{i=1}^{n} \frac{\partial^2 \phi}{\partial x_i^2} - \frac{\partial \phi}{\partial t} = 0 \text{ in } \Omega.$$

Thus since R can be arbitrarily large, $\mathcal{K}f$ satisfies

$$\sum_{i=1}^{n} \frac{\partial^2 \phi}{\partial x_i^2} - \frac{\partial \phi}{\partial t} + f = 0 \text{ in } \mathbf{R}^{n+1}.$$

Theorem 6. *Let f be real valued on $\mathbb{R}^{n+1}$ and in some space $B_{-\tau,\alpha}$, $\tau > 2$, $0 < \alpha < 1$. Then there exists exactly one entire classical solution ϕ of the equation*

$$\sum_{i=1}^{n} \frac{\partial^2 \phi}{\partial x_i^2} - \frac{\partial \phi}{\partial t} + f = 0$$

vanishing at infinity, namely the solution $\phi = \mathcal{K}f$. Moreover, if $2 < \tau < n+2$, we have for this solution the bound

$$\||\phi\||_{2-\tau,\alpha} \leq M(n,\alpha,\tau)\|f\|_{-\tau,\alpha}.$$

Proof: Uniqueness of the solution follows from the fact that the difference of two solutions would satisfy the heat equation and vanish an infinity, thereby being identically zero by the maximum principle of [12]. Lemma 5 states that $\phi = \mathcal{K}f$ is a solution vanishing at infinity. To derive the bound on ϕ, we apply Theorem 3 to the case of the heat operator to obtain

$$\||\phi\||_{2-\tau,\alpha} \leq M(\|\phi\|_{2-\tau} + \|f\|_{-\tau,\alpha}).$$

However, Lemma 4 implies

$$\begin{aligned}\|\phi\|_{2-\tau} &= \|\mathcal{K}f\|_{2-\tau} = \sup\left\{[\bar{x}]^{-(2-\tau)}|\mathcal{K}f(\bar{x})| : \bar{x} \in \mathbb{R}^{n+1}\right\} \\ &= M(n,\tau)\sup\left\{[\bar{x}]^{-(2-\tau)}\|f\|_{-\tau}[\bar{x}]^{2-\tau} : \bar{x} \in \mathbb{R}^{n+1}\right\} = M(n,\tau)\|f\|_{-\tau}.\end{aligned}$$

Hence

$$\||\phi\||_{2-\tau,\alpha} \leq M(n,\tau)\|f\|_{-\tau,\alpha}.$$

4. An a priori bound for the general parabolic equation. We consider the equation

$$\mathcal{L}\phi := a \cdot \mathcal{D}^2\phi + b \cdot \mathcal{D}\phi + c\phi - \partial\phi/\partial t = f \tag{NH}$$

and derive an a priori bound for entire solutions analogous to Theorem 6. The conditions on the coefficients will guarantee that the operator $\mathcal{L}$ approaches the heat operator near infinity at a certain rate. Let I denote the $n \times n$ identity matrix. We require:

(C_1) There exist constants δ, α, M_1 with $\delta > 0$, $0 < \alpha < 1$, $M_1 \geq 0$ such that

$$\|a - I\|_{-\delta,\alpha},\ \|b\|_{-1-\delta,\alpha},\ \|c\|_{-2-\delta,\alpha} < M_1.$$

(C_2) The matrix a is symmetric. For all $(x,t) \in \mathbb{R}^{n+1}$ and real $1 \times n$ vectors ξ,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}(\bar{x})\xi_i\xi_j \geq M_2|\xi|^2 \text{ for some } M_2 > 0.$$

(C_3) $\qquad c \leq 0.$

Theorem 7. *Suppose conditions (C_1), (C_2), (C_3) hold on the coefficients of $\mathcal{L}$ and that $f \in B_{-\tau,\alpha}$ for some τ, $2 < \tau < n+2$, and α the same as in (C_1). If ϕ is an entire classical solution of (NH) and $\phi \in B_{2-\tau}$, then*

$$|||\phi|||_{2-\tau,\alpha} \leq M(n, \alpha, \delta, \tau, M_1, M_2)\|f\|_{-\tau,\alpha}. \tag{16}$$

Proof: We have

$$\begin{aligned}\|a\|_{0,\alpha} &\leq \|a - I\|_{0,\alpha} + \|I\|_{0,\alpha} \leq \|a - I\|_{-\delta,\alpha} + \|I\|_{0,\alpha} \text{ by Lemma 2(b),} \\ &= \|a - I\|_{-\delta,\alpha} + \|I\|_0 \leq M_1 + \sqrt{n}.\end{aligned} \tag{17}$$

We also have

$$\|b\|_{-1,\alpha} \leq \|b\|_{-1-\delta,\alpha} \leq M_1 \tag{18}$$

and

$$\|c\|_{-2,\alpha} \leq \|c\|_{-2-\delta,\alpha} \leq M_1. \tag{19}$$

Hence Theorem 3 can be applied with $\sigma = 2 - \tau$ to get the estimate

$$|||\phi|||_{2-\tau,\alpha} \leq M(M_1, M_2, n, \alpha, \tau)(\|\phi\|_{2-\tau} + \|f\|_{-\tau,\alpha}). \tag{20}$$

It is, therefore, sufficient to derive instead of (16) the estimate

$$\|\phi\|_{2-\tau} \leq M(M_1, M_2, n, \alpha, \delta, \tau)\|f\|_{-\tau,\alpha}. \tag{21}$$

Suppose that (21) is false. Then there exist sequences $\{a_m\}$, $\{b_m\}$, $\{c_m\}$, $\{f_m\}$, $\{\phi_m\}$, $m = 1, 2, 3, \cdots$ such that all a_m, b_m, c_m satisfy the conditions (C_1), (C_2), (C_3) with the same constants M_1, M_2, α, δ such that f_m is in $B_{-\tau,\alpha}$, and also each ϕ_m is in $B_{2-\tau}$ and

$$a_m \cdot \mathcal{D}^2\phi_m + b_m \cdot \mathcal{D}\phi_m + c_m\phi_m - \partial\phi_m/\partial t = f_m \tag{22}$$

and with

$$\|\phi_m\|_{2-\tau} = 1, \qquad \|f_m\|_{-\tau,\alpha} \leq 1/m. \tag{23}$$

The inequalities (17), (18), (19) are satisfied by a_m, b_m, c_m, respectively, and (20) holds for all pairs (ϕ_m, f_m). We also have that $\{a_m - I\}$, $\{b_m\}$, $\{c_m\}$, $\{f_m\}$, $\{\phi_m\}$, $\{D_x\phi_m\}$, $\{D_x^2\phi_m\}$ and $\{D_t\phi_m\}$ are all uniformly bounded with respect to the appropriate norms in the spaces $B_{-\delta,\alpha}$, $B_{-1-\delta,\alpha}$, $B_{2-\delta,\alpha}$, $B_{-\tau,\alpha}$, $B_{2-\tau,\alpha}$, $B_{1-\tau,\alpha}$, $B_{-\tau,\alpha}$ and $B_{-\tau,\alpha}$. By Lemma 2(f) there exist functions $a - I$, b, c, f, ϕ contained in the same spaces as the corresponding sequences, with ϕ, $D_x\phi$, $D_x^2\phi$, $D_t\phi$ continuous and with ϕ in $B_{2-\tau}$ [we have uniform convergence]. After passing to subsequences [preserving (23)], we have with respect to the norms involved

$$(a_m - I) \longrightarrow (a - I) \text{ in } B_0, \text{ and hence } a_m \longrightarrow a \text{ in } B_0,$$

$$b_m \longrightarrow b \text{ in } B_{-1},\ c_m \longrightarrow c \text{ in } B_{-2},$$

$$f_m \longrightarrow f \text{ in } B_{-2},\ \phi_m \longrightarrow \phi \text{ in } B_{2-\tau+\delta},$$

$$D\phi_m \longrightarrow D\phi \text{ in } B_{1-\tau+\delta},\ D_x^2\phi_m \longrightarrow D_x^2\phi \text{ in } B_{-\tau+\delta},$$

and

$$D_t\phi_m \longrightarrow D_t\phi \text{ in } B_{-\tau+\delta}. \tag{24}$$

So by (23)

$$\|f\|_{-2} = \lim \|f_m\|_{-2} \le \overline{\lim} \|f_m\|_{-\tau} \le \lim(1/m) = 0.$$

Therefore, passing to the limit in (22), we find that ϕ is an entire classical solution of the homogeneous equation

$$a \cdot \mathcal{D}^2\phi + b \cdot \mathcal{D}\phi + c\phi - \partial\phi/\partial t = 0. \tag{25}$$

Clearly the conditions (C_2) and (C_3) on a and c are fulfilled, so the maximum and minimum principles [12] may be applied to the solutions of (25). Since $\phi \in B_{2-\tau}$, ϕ vanishes at infinity and so by applying the maximum and minimum principles to arbitrarily large spheres in $\mathbf{R}^{n+1}$, we conclude that $\phi \equiv 0$. Hence from (24),

$$\phi_m \longrightarrow 0 \text{ in } B_{2-\tau+\delta}, \quad D\phi_m \longrightarrow 0 \text{ in } B_{1-\tau+\delta}, \quad D_x^2\phi_m \longrightarrow 0 \text{ in } B_{-\tau+\delta}. \tag{26}$$

We rewrite (22) as

$$I \cdot \mathcal{D}^2\phi_m - \partial\phi_m/\partial t + g_m = 0 \tag{27}$$

where

$$g_m = (a_m - I) \cdot \mathcal{D}^2\phi_m + b_m \cdot \mathcal{D}\phi_m + c_m\phi_m - f_m \tag{28}$$

and observe that

$$\begin{aligned} &\|g_m\|_{-\tau} \\ &\le \|(a_m - I)\|_{-\delta}\|\mathcal{D}^2\phi_m\|_{-\tau+\delta} + \|b_m\|_{-1-\delta}\|\mathcal{D}\phi_m\|_{1-\tau+\delta} + \|c_m\|_{-2-\delta}\|\phi_m\|_{2-\tau+\delta} + \|f_m\|_{-\tau} \\ &\le M_1\big(\|\mathcal{D}^2\phi_m\|_{-\tau+\delta} + \|\mathcal{D}\phi_m\|_{1-\tau+\delta} + \|\phi_m\|_{2-\tau+\delta}\big) + 1/m. \end{aligned}$$

By (26), the right-hand side goes to zero as $m \to \infty$ so $\|g_m\|_{-\tau} \to 0$ as $m \to \infty$. Note that

$$(a_m - I) \cdot \mathcal{D}^2\phi_m \in B_{-\tau-\delta,\alpha}, \quad b_m \cdot D\phi_m \in B_{-\tau-\delta,\alpha}, \quad c_m\phi_m \in B_{-\tau-\delta,\alpha}$$

so from (28), $g_m \in B_{-\tau,\alpha}$. Note further that if we let Δ represent $\sum_{i=1}^n \partial^2/\partial x_i^2$, then (27) implies

$$\Delta\phi_m - \partial\phi_m/\partial t + g_m = 0.$$

Since $\phi_m \in B_{2-\tau}$, ϕ_m also vanishes at infinity. Hence, Theorem 6 may be applied and we have $\phi_m = \mathcal{K}g_m$. However from Lemma 4,

$$\|\phi_m\|_{2-\tau} \le M(n,\tau)\|g_m\|_{-\tau} \to 0$$

which contradicts (23).

5. Existence of solutions. We now demonstrate the existence of entire solutions of the nonhomogenous equation

$$\mathcal{L}\phi := a \cdot \mathcal{D}^2\phi + b \cdot \mathcal{D}\phi + c\phi - \partial\phi/\partial t = f \tag{NH}$$

that vanish at infinity. Again we assume conditions (C_1), (C_2), (C_3) on the coefficients of $\mathcal{L}$.

Theorem 8. *Suppose conditions (C_1), (C_2), (C_3) hold on the coefficients of $\mathcal{L}$ and that $f \in B_{-\tau,\alpha}$ for some $\tau > 2$. Then there exists a unique entire classical solution ϕ of (NH) such that ϕ vanishes at infinity, and moreover if $2 < \tau < n+2$ we have the bound*

$$|||\phi|||_{2-\tau} \leq M(n, \alpha, \delta, \tau, M_1, M_2)\|f\|_{-\tau,\alpha}.$$

Proof: The uniqueness of the solution follows from the maximum principle since the difference of two solutions would solve the homogeneous equation and vanish at infinity. In order to establish the existence of a solution, we consider the Banach space B defined by

$$B = \left\{\phi \in B_{2-\tau,1} : D_x\phi \in B_{1-\tau,\alpha},\ D_x^2\phi \in B_{-\tau,\alpha},\ D_t\phi \in B_{-\tau,\alpha}\right\}$$

with norm

$$\|\phi\|^*_{\tau,\alpha} = |||\phi|||_{2-\tau,\alpha} = \|\phi\|_{2-\tau,1} + \sum \|D_x\phi\|_{1-\tau,\alpha} + \sum \|D_x^2\phi\|_{-\tau,\alpha} + \|D_t\phi\|_{-\tau,\alpha}$$

where the summations are over all partial derivatives. That B is a Banach space follows from Lemma 2(h). We consider the family of linear operators $\mathcal{L}_r$, $0 \leq r \leq 1$, defined by

$$\mathcal{L}_r = (1-r)\Big[\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \frac{\partial}{\partial t}\Big] + r\mathcal{L},$$

setting $a_r = (1-r)I + ra$, $b_r = rb$ and $c_r = rc$, we have

$$\mathcal{L}_r\phi = a_r \cdot \mathcal{D}^2\phi + b_r \cdot \mathcal{D}\phi + c_r\phi - \partial\phi/\partial t.$$

We will show that $\mathcal{L}_r$ maps B into $B_{-\tau,\alpha}$. We first note that

$$\|a_r\|_{-\delta,\alpha} = \|r(a-I) + I\|_{-\delta,\alpha} \leq r\|a - I\|_{-\delta,\alpha} + \sqrt{n} \leq M_1 + \sqrt{n}.$$

We also have

$$\|b_r\|_{-1-\delta,\alpha} = r\|b\|_{-1-\delta,\alpha} \leq M_1 \ \text{ and } \ \|c_r\|_{-2-\delta,\alpha} = r\|c\|_{-2-\delta,\alpha} \leq M_1,$$

so

$$\begin{aligned}\|\mathcal{L}_r\phi\|_{-\tau,\alpha} &\leq (M_1 + \sqrt{n})\|\mathcal{D}^2\phi\|_{-\tau,\alpha} + M_1\|\mathcal{D}\phi\|_{1-\tau,\alpha} + M_1\|\phi\|_{2-\tau,\alpha} + \|\partial\phi/\partial t\|_{-\tau,\alpha} \\ &\leq (M_1 + \sqrt{n})\|\phi\|^*_{\tau,\alpha}.\end{aligned}$$

Hence, $\mathcal{L}_r$ maps B into $B_{-\tau,\alpha}$ and in fact $\mathcal{L}_r$ is a bounded linear operator from B to $B_{-\tau,\alpha}$. We also get from (C_3) that $c_r = rc \leq 0$ in $\mathbb{R}^{n+1}$, and from (C_2) we have for $\bar{x} \in \mathbb{R}^{n+1}$ and ξ a real $n \times n$ vector

$$a_r(\bar{x})\xi \cdot \xi = (1-r)|\xi|^2 + ra(\bar{x})\xi \cdot \xi \geq (1-r)|\xi|^2 + rM_2|\xi|^2 \geq \min\{1, M_2\}|\xi|^2.$$

Hence, Theorem 7 can be applied and for all $\phi \in B$ and $0 \leq r \leq 1$, we have

$$\|\phi\|^*_{\tau,\alpha} = |||\phi|||_{2-\tau,\alpha} \leq M(n, \alpha, \delta, \tau, M_1, M_2)\|\mathcal{L}_r\phi\|_{-\tau,\alpha}.$$

According to Theorem 5.2 in [7], the Schauder continuation method applies. Theorem 6 asserts that the operator

$$\mathcal{L}_0 = \sum_{i=1}^{n} \frac{\partial^2 \phi}{\partial x_i^2} - \frac{\partial \phi}{\partial t}$$

maps B onto $B_{-\tau,\alpha}$. Hence, any $\mathcal{L}_r, 0 \leq r \leq 1$ does the same and in particular $\mathcal{L}_1 = \mathcal{L}$ maps B onto $B_{-\tau,\alpha}$.

Before we study the behavior of entire solutions of (NH) at infinity, it will be useful to study the behavior of entire solutions of the heat equation. We shall show that the class of entire solutions of the heat equation in B_m is the same as the class of polynomial solutions of the heat equation of degree at most m in x. We shall for brevity use the notation

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

Lemma 9. *(a) If $P(x,t)$ is a polynomial solution of the heat equation of degree no greater than m in x and ℓ any nonnegative integer, then*

(i) *P has degree at most $[m/2]$ in t and therefore $P \in B_m$,*

(ii) *$P \in B_{m,1}$,*

(iii) *$D_x P \in B_{m-1,1}$,*

(iv) *$D_x^\ell P \in B_{m-\ell,1}$.*

b) If ϕ is an entire classical solution of the heat equation and $\phi \in B_m$ for some nonnegative integer m then ϕ is a polynomial of degree at most m in x and degree at most $[m/2]$ in t.

Proof: (a) i) The derivative

$$\frac{\partial \phi}{\partial t} = \sum_{i=1}^{n} \frac{\partial^2 \phi}{\partial x_i^2}$$

will for fixed t be a polynomial of degree at most $m - 2$ in x, and since $\partial\phi/\partial t$ also solves the heat equation,

$$\frac{\partial^2 \phi}{\partial t^2} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\Big(\sum_{i=1}^{n} \frac{\partial^2 \phi}{\partial x_j^2}\Big)$$

is a polynomial of degree at most $m-4$ in x and $(\partial^{[m-2]}\phi)/(\partial t^{[m-2]})(x,t)$ is a polynomial of degree at most $m - 2[m/2] = 0$ or 1 in x, so $(\partial^{[m/2]+1}\phi)/(\partial t^{[m/2]+1})(x,t)$ is identically zero. This implies that for fixed x, $(\partial^{[m/2]}\phi)/(\partial t^{[m/2]})$ is a constant and thus ϕ is a polynomial of degree at most $[m/2]$ in t.

ii) Any derivative $\partial P/\partial x_i$ solves the heat equation and either constant or of degree at most $m-1$ in x and degree at most $[\frac{m-1}{2}]$ in t, so $|D_x P(x,t)|/[\bar{x}]^{(m-1)}$ is bounded and, therefore, $D_x P \in B_{m-1}$. $D_t P$ solves the heat equation and has degree at most $([\frac{m}{2}] - 1) \leq [\frac{m-2}{2}]$ in t and degree at most $m-2$ in x so $D_t P$ is in B_{m-2}. By Lemma 2(e), $P(x,t) \in B_{(m-1)}$.

iii) Apply part ii) to each $\partial P/\partial x_1$.

iv) Apply part ii) to each $D_x^\ell P$.

b) Observe that $\phi \in B_m$ implies $|\phi(\bar{x})| \leq \|\phi\|_m [\bar{x}]^m$ for $\bar{x} \in \mathbb{R}^{n+1}$. Let $\phi \in B_m$ be an entire solution of the heat equation. Using Theorem 3 with $f = 0$, $a = I_n$, $b = 0$, $c = 0$, we get $|||\phi|||_{m,\alpha} \leq M\|\phi\|_m$ for any $\alpha \in (0,1)$ where M is a suitable constant. In particular all the norms $\|D_x\phi\|_{m-1}$ are finite. $D_x\phi$ also solves the heat equation so all the norms $\|D_x^2\phi\|_{m-2}$

are finite. We continue in this fashion and get that $\|D_x^{m+1}\phi\|_{-1}$ is finite. Thus any $D_x^{m+1}\phi$ is a solution of the heat equation and

$$|D_x^{m+1}\phi(\bar{x})| \le \|D_x^{M+1}\phi\|_{-1}(1+|t|+|x|^2)^{-1/2},$$

i.e., $D_x^{m+1}\phi$ vanishes at infinity. Hence, $D_x^{m+1}\phi$ is identically zero. This implies that any $D_x^m\phi$ is constant as a function of x, and for fixed t, ϕ is a polynomial in x of degree at most m; and hence, for fixed x, ϕ is a polynomial of degree at most $[m/2]$ in t. Thus

$$\phi(x,t) = \sum_{|\xi|\le m} x^\xi f_\xi(t) = \sum_{i=0}^{[m/2]} t^i g_i(x).$$

We show that the functions $g_i(x)$ are polynomials in x. Note that

$$\phi(x,0) = g_0(x) = \sum_{|\xi|\le m} x^\xi f_\xi(0)$$

so $g_0(x)$ is a polynomial in x of degree at most m.

$$\Delta\phi(x,0) = \sum_{|\xi|\le m} \Delta(x^\xi) f_\xi(0) = g_1(x) = \frac{\partial\phi}{\partial t}(x,0)$$

so $g_1(x)$ is a polynomial in x of degree at most $m-2$. Similarly for $1 \le i \le [m/2]$,

$$(\Delta)^i\phi(x,0) = \sum_{|\xi|\le m} (\Delta)^i(x^\xi) f_\xi(0) = i!\,g_i(x) = \frac{\partial^i\phi}{\partial t^i}(x,0),$$

so $g_i(x)$ is a polynomial in x of degree at most $m-2i$. Thus

$$\phi(x,t) = \sum_{i=0}^{[m/2]} t^i \sum_{|\xi|\le m-2i} r_\xi x^\xi$$

where r_ξ are real constants.

We introduce a new condition on the coefficients of the operator $\mathcal{L}$.

(D_1) There exist constants δ, α, Λ, m, with $\delta > 0$, $0 < \alpha < 1$, $\Lambda < 0$, m a positive integer, such that

$$\|a-I\|_{-m-\delta,\alpha},\ \ \|b\|_{-m-1-\delta,\alpha},\ \ \|c\|_{-m-2-\delta,\alpha} \le \Lambda\,.$$

When we replace (C_1) by (D_1) the coefficients $a_{i,j}$, b_i and c as well as the Hölder quotients $|a_{i,j}(\bar{x})-a_{i,j}(\bar{y})|/d(\bar{x},\bar{y})^\alpha$, $|b_i(\bar{x})-b_i(\bar{y}))|/d(\bar{x},\bar{y})^\alpha$ and $|c(\bar{x})-c(\bar{y})|/d(\bar{x},\bar{y})^\alpha$ are required to decay faster at infinity by a factor of $[\bar{x}]^{-m}$.

Theorem 10. *a) Suppose that conditions (D_1), (C_2), (C_3) hold on the coefficients of $\mathcal{L}$, and that $f \in B_{-\tau,\alpha}$ for some $\tau > 2$. Then, for any polynomial solution P of the heat equation, there exists a unique entire classical solution ϕ of (NH) such that $\phi(x,t) - P(x,t) \longrightarrow 0$ as $(x,t) \longrightarrow \infty$.*

b) If ϕ is an entire classical solution of (NH) with $\phi \in B_m$, then there exists a unique polynomial $P(x,t)$ of degree at most m in x with $\Delta P - (\partial P/\partial t) = 0$ such that $\phi(x,t) - P(x,t) \longrightarrow 0$ as $(x,t) \longrightarrow \infty$.

Proof: a) Uniqueness is a consequence of the maximum principle. We have

$$\mathcal{L}P = \sum_{i=1}^{n} \frac{\partial^2 P}{\partial x_i^2} - \frac{\partial P}{\partial t} + \Big(\mathcal{L} - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\partial}{\partial t}\Big)P$$
$$= a \cdot \mathcal{D}^2 P + b \cdot \mathcal{D}P + cP - \partial P/\partial t - I \cdot \mathcal{D}^2 P + \partial P/\partial t = (a - I) \cdot \mathcal{D}^2 P + b \cdot \mathcal{D}P + cP,$$

$$\|\mathcal{L}P\|_{-2-\delta,\alpha} \le \|(a-I)\cdot \mathcal{D}^2 P\|_{-2-\delta,\alpha} + \|b \cdot \mathcal{D}P\|_{-2-\delta,\alpha} + \|cP\|_{-2-\delta,\alpha}$$
$$\le \|(a-I)\|_{-m-\delta,\alpha}\|\mathcal{D}^2 P\|_{m-2,\alpha} + \|b\|_{-m-1-\delta,\alpha}\|\mathcal{D}P\|_{m-1,\alpha} + \|c\|_{-m-2-\delta,\alpha}\|P\|_{m,\alpha},$$

so $\mathcal{L}P \in B_{-2-\delta,\alpha}$. Noting that for small $\epsilon > 0$, $f - \mathcal{L}P \in B_{-(2+\epsilon),\alpha} \subset B_{-\tau,\alpha}$ and that (D_1) implies (C_1), we have by Theorem 8 a unique entire classical solution of $\mathcal{L}\psi = f - \mathcal{L}P$ such that $\psi \longrightarrow 0$ as $\bar{x} \longrightarrow \infty$. We set $\phi = \psi + P$ to obtain the desired solution of (NH).

b) Uniqueness of P is clear for if P_1 and P_2 were two polynomial solutions of the heat equation satisfying the required condition, then the difference $P_1 - P_2$ would vanish at infinity giving $P_1 \equiv P_2$.

To show existence of P, we observe that if ϕ is a classical solution of (NH) with $\phi \in B_m$, then by Theorem 3, $\phi \in B_{m,1}$, $D_x\phi \in B_{m-1,\alpha}$, $D_x^2\phi \in B_{m-2,\alpha}$. Further

$$\Delta\phi - \frac{\partial \phi}{\partial t} = \mathcal{L}\phi - (\mathcal{L} - (\Delta - \frac{\partial}{\partial t}))\phi = f - (a-I)\cdot \mathcal{D}^2\phi - b \cdot \mathcal{D}\phi - c\phi.$$

We claim that $\mathcal{K}(\Delta\phi - \frac{\partial\phi}{\partial t})$ has a continuous derivative with respect to t and continous second derivatives with respect to the x_i at all points $\bar{x}$ in $\mathbb{R}^{n+1}$, and that

$$(\Delta - \frac{\partial}{\partial t})(\mathcal{K}(\Delta\phi - \frac{\partial\phi}{\partial t})) = \Delta\phi - \frac{\partial\phi}{\partial t}$$

and further that $\mathcal{K}(\Delta\phi - \frac{\partial\phi}{\partial t})$ vanishes at infinity. To establish the claim we need to show that $(\Delta\phi - \frac{\partial\phi}{\partial t})$ is in $B_{-2-\gamma,\alpha}$ for some $\gamma > 0$ so that the hypotheses for Lemma 5 are satisfied. We have

$$\|\Delta\phi - \frac{\partial\phi}{\partial t}\|_{-2-\gamma,\alpha}$$
$$\le \|f\|_{-2-\gamma,\alpha} + \|(a-I)\cdot\mathcal{D}^2\phi\|_{-2-\gamma,\alpha} + \|b\cdot\mathcal{D}\phi\|_{-2-\gamma,\alpha} + \|c\phi\|_{-2-\gamma,\alpha}$$
$$\le \|f\|_{-2-\gamma,\alpha} + \|(a-I)\|_{-m-\gamma,\alpha}\|\mathcal{D}^2\phi\|_{m-2,\alpha} + \|b\|_{-m-1-\gamma,\alpha}\|\mathcal{D}\phi\|_{m-1,\alpha}$$
$$+ \|c\|_{-m-2-\gamma,\alpha}\|\phi\|_{m,\alpha}$$
$$\le \|f\|_{-2-\gamma,\alpha} + \Lambda\|\mathcal{D}^2\phi\|_{m-2,\alpha} + \Lambda\|\mathcal{D}\phi\|_{m-1,\alpha} + \Lambda\|\phi\|_{m,\alpha} < \infty$$

for small enouth γ, since $f \in B_{-\tau,\alpha}$ and $\tau > 2$. The claim is now established. The function $P = \phi - \mathcal{K}(\Delta\phi - \frac{\partial\phi}{\partial t})$ therefore, satisfies the heat equation $\Delta\phi - \frac{\partial\phi}{\partial t} = 0$ in $\mathbb{R}^{n+1}$. Since $\phi \in B_m$ and $\mathcal{K}(\Delta\phi - \frac{\partial\phi}{\partial t})$ in B_0 it follows that $P \in B_m$ and so by Lemma 9b, P is a polynomial in x and t with degree at most m in x. Further since $\phi - P = \mathcal{K}(\Delta\phi - \frac{\partial\phi}{\partial t})$, it follows that $\phi(x,t) \longrightarrow 0$ as $(x,t) \longrightarrow \infty$.

Corollary 11. *Suppose that conditions (D_1), (C_2), (C_3) hold on the coeffieints of $\mathcal{L}$. Then the entire classical solutions of (H) which are in B_m form a finite-dimensional vector space. There is a one-to-one correspondence between this space and the space of all polynomial solutions to the heat equation of degree no greater than m determined by Theorem 10.*

We finally remark that the space of all polynomial solutions to the heat equation of degree no greater than m is spanned by Widder's heat polynomials [16]. There is one heat polynomial for each monomial $x_1^{\alpha(1)} x_2^{\alpha(2)} \cdots x_n^{\alpha(n)}$ so the dimension of the space of entire classical solutions of (H) which are in B_m is the same as the dimension of the space of all polynomials in $x_1, x_2, \cdots, x_n$ of degree no greater than m.

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REFERENCES

[1] E. Bohn and L.K. Jackson, *The Liouville theorem for a quasi-linear elliptic partial differential equation,* Trans. Amer. Math. Soc., 104 (1962), 392-397.

[2] H. Begehr and G.N. Hile, *Schauder estimates and existence theory for entire solutions of linear elliptic equations,* Proc. Roy. Soc. Edinburgh (to appear).

[3] F.G. Dressel, *The fundamental solutions of the parabolic equation,* Duke Math. J., 7 (1940), 186-203.

[4] F.G. Dressel, *The fundamental solutions of the parabolic equation II,* Duke Math. J., 13 (1946), 61-70.

[5] A. Friedman, "Partial Differential Equations of Parabolic Type," Englewood Cliffs, N.J., Prentice-Hall, 1964.

[6] D. Gilbarg and J. Serrin, *On isolated singularities of solutions of second order elliptic differential equations,* J. D'Analyse Mathematique, 4 (1954-56), 309-340.

[7] D. Gilbarg and N.S. Trudinger, "Elliptic Partial Differential Equations of Second Order," Springer-Verlag, Berlin, 1977.

[8] S. Hidebrant and K. Widman, *Sätze vom Liouvilleschen Typ für quasilinear elliptische Gleichungen und Systeme,* Nachr. Akad. Wiss. Gottingen Math.-Phys. Kl., II, 4 (1979), 41-59.

[9] G.N. Hile, *Entire solutions of linear elliptic equations with Laplacian principal part,* Pacific J. Math., 62 (1976), 127-140.

[10] A.V. Ivanov, *Local estimates for the first derivatives of solutions of quasilinear second order elliptic equations and their applications to Liouville type theorem,* Sem. Steklov Math. Inst. Leningrad, 30 (1972), 40-50. Translated in J. Sov. Math., 4 (1975), 335-344.

[11] L. Peletier and J. Serrin, *Gradient bounds and Liouville theorems for quasilinear elliptic equations,* Annali Scuola Norm Sup. Pisa (IV) 5 (1978), 65-104.

[12] M. Protter and H. Weinberger, "Maximum Principles in Differential Equations," Prentice-Hall, Englewood Cliffs, N.J., 1967.

[13] R. Redheffer, *On the inequality $\Delta u \geq f(u, |\,\mathrm{grad}\, u|)$,* J. Math. Annal. Appl., 1 (1960), 277-299.

[14] J. Serrin, *Entire solutions of nonlinear Poisson equations,* Proc. London Math. Soc., 24 (1972), 348-366.

[15] J. Serrin, *Liouville theorems and gradient bounded for quasilinear elliptic systems,* Archive Rat. Mech. Anal., 66 (1977), 295-310.

[16] D.V. Widder, "The Heat Equation," Academic Press, New York, San Francisco, London 1975.