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ON A THIRD-ORDER THREE-POINT BOUNDARY VALUE PROBLEM AT RESONANCE

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Abstract. D. Krajcinovic modelled the static deflection of a three-layered elastic beam by a linear third order, three-point, boundary value problem involving only the first order and third order derivatives of the displacement. This paper is concerned with the existence and uniqueness of solutions of some third order, three-point, nonlinear boundary value problems which are at resonance, in the sense that the associated linear boundary value problem has non-trivial solutions. The methods used involve second-order integro-differential boundary value problems and a use of the Leray-Schauder continuation theorem.

1. Introduction. This paper is motivated by the model equation describing the deflection of a three-layer beam formed by parallel layers of different materials given by D. Krajcinovic in [8]. According to Krajcinovic, the state of static deflection of an equally-loaded three-layered beam is described by

$$\begin{cases} u''' - k(K_2A_e - K_1^2)u' + a = 0, \\ u'(0) = u'(1) = u(\frac{1}{2}) = 0, \end{cases} \tag{1.1}$$

where K_1, K_2 are shear parameters, A_e is the cross-sectional area of the beam and k, a are other physical parameters related to the elasticity of the layers. Krajcinovic assumed that $K_2A_e - K_1^2 > 0$. It is clear from (1.1) that the assumption of $K_2A_e - K_1^2 > 0$ is rather restrictive, and interesting physical situations arise when $K_2A_e - K_1^2$ is in fact negative. To this end, the author with A.R. Aftabizadeh and J.M. Xu [2] recently studied the following nonlinear third-order three-point boundary value problem:

$$\begin{cases} u''' + f(u')u'' = g(x, u, u', u'') + e(x), \\ u'(0) = u'(1) = u(\eta) = 0, \quad 0 \le \eta \le 1, \end{cases} \tag{1.2}$$

where $f : \mathbf{R} \to \mathbf{R}$ is continuous, $g : [0,1] \times \mathbf{R}^3 \to \mathbf{R}$ satisfies Caratheodory's conditions, $e(x) \in L^1[0,1]$ and g satisfies some additional supplementary conditions. The results of [2] for (1.2), when particularized to the case of (1.1), allow one to consider the possibility that

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$K_2 A_e - K_1^2$ may be negative. However, the results of [2] cannot be applied to the following boundary value problem

$$\begin{cases} u''' + \pi^2 u' = e(x), \\ u'(0) = u'(1) = u(\eta) = 0, \quad 0 \le \eta \le 1. \end{cases} \tag{1.3}$$

It should be noted that, for a given $e(x) \in L^1[0,1]$, a necessary condition for the solvability of (1.3) is that

$$\int_0^1 e(x) \sin \pi x \, dx = 0. \tag{1.4}$$

The purpose of this paper is to obtain existence and uniqueness theorems for the following third order three-point boundary value problems:

$$\begin{cases} -u''' - \pi^2 u' + g(x, u, u', u'') = e(x), \\ u'(0) = u'(1) = u(\eta) = 0, \quad 0 \le \eta \le 1, \end{cases} \tag{1.5}$$

and

$$\begin{cases} u''' + \pi^2 u' + g(x, u, u', u'') = e(x), \\ u'(0) = u'(1) = u(\eta) = 0, \quad 0 \le \eta \le 1, \end{cases} \tag{1.6}$$

where $e(x) \in L^1[0,1]$ satisfies the condition (1.4) and $g : [0,1] \times \mathbf{R}^3 \to \mathbf{R}$ satisfies Caratheodory's conditions and a boundedness condition. Existence of a solution for (1.5) is obtained when g additionally satisfies the condition

$$g(x, u, v, w)v \ge 0, \quad x \in [0,1], \; u, \, v, \, w \in \mathbf{R}. \tag{1.7}$$

For the existence of a solution for (1.6), g needs to satisfy (1.7) and also the condition there exists a constant $\beta \ge 0$ such that

$$\limsup_{|v| \to \infty} \frac{g(x, u, v, w)}{v} = \beta < 3\pi^2, \tag{1.8}$$

uniformly for $(x, u, w) \in [0,1] \times \mathbf{R}^2$.

We note that the boundary value problems (1.5), (1.6) are at resonance since the linear boundary value problem

$$\begin{cases} u''' + \pi^2 u' = 0, \\ u'(0) = u'(1) = u(\eta) = 0, \quad 0 \le \eta \le 1, \end{cases}$$

has a non-trivial solution, namely,

$$u(x) = \int_\eta^x \sin \pi t \, dt = \frac{1}{\pi}(\cos \pi\eta - \cos \pi x).$$

There has been a considerable amount of interest in third-order three-point boundary value problems in recent years. We refer the reader to [1]-[5], [7] and [12]-[14], concerning some of the recent work on these problems.

Existence theorems. Let X, Y denote the Banach spaces $X = C^1[0,1]$, $Y = L^1[0,1]$ with their usual norms and let H denote the Hilbert space $L^2[0,1]$. Let Y_2 be the subspace of Y spanned by the function $\sin \pi x$; i.e.,

$$Y_2 = \{u \in Y \mid u(x) = \alpha \sin \pi x \ \text{ a.e. for some } \alpha \in \mathbf{R}\}\,, \tag{2.1}$$

and let Y_1 be the subspace of Y such that $Y = Y_1 \bigoplus Y_2$. (Here and in the following, $\bigoplus$ denotes the direct sum). We note that for $u \in Y$ we can write

$$u(x) = u(x) - \left(2\int_0^1 u(t)\sin \pi t\, dt\right)\sin \pi x + \left(2\int_0^1 u(t)\sin \pi t\, dt\right)\sin \pi x\,, \tag{2.2}$$

$x \in [0,1]$. We define the canonical projection operators $P : Y \to Y_1$, $Q : Y \to Y_2$ by

$$\begin{aligned} P(u) &= u(x) - \left(2\int_0^1 u(t)\sin \pi t\, dt\right)\sin \pi x,\\ Q(u) &= \left(2\int_0^1 u(t)\sin \pi t\, dt\right)\sin \pi x, \end{aligned} \tag{2.3}$$

for $u \in Y$. Clearly, $Q = I - P$, where I denotes the identity mapping on Y, and the projection P and Q are continuous. Now let $X_2 = X \cap Y_2$. Clearly, X_2 is a closed subspace of X. Let X_1 be the closed subspace of X such that $X = X_1 \bigoplus X_2$. We note that $P(X) \subset X_1$, $Q(X) \subset X_2$ and the projections $P \mid X : X \to X_1$, $Q \mid X : X \to X_2$ are continuous. Similarly, we obtain $H = H_1 \bigoplus H_2$ and the continuous projections $P \mid H : H \to H_1$, $Q \mid H : H \to H_2$. In the following, X, Y, H, P, Q, etc will refer to Banach spaces, Hilbert space and the projections as defined above, and we shall not distinguish between P, $P \mid X$, $P \mid H$ (resp. Q, $Q \mid X$, $Q \mid H$) and will depend on the context for proper meaning.

Also for $u \in X$, $v \in Y$, let $(u, v) = \int_0^1 u(x)v(x)\, dx$ denote the duality pairing between X and Y. We note that for $u \in X$, $v \in Y$ so that $u = Pu + Qu$, $v = Pv + Qv$, we have

$$(u, v) = (Pu, Pv) + (Qu, Qv). \tag{2.4}$$

Define a linear operator $L : D(L) \subset X \to Y$ by setting

$$D(L) = \{u \in X \mid u'(t) \in AC[0,1],\ \ u(0) = u(1) = 0\}\,, \tag{2.5}$$

and for $u \in D(L)$

$$Lu = -u'' - \pi^2 u\,. \tag{2.6}$$

(Here $AC[0,1]$ denotes the space of real-valued absolutely-continuous functions on $[0,1]$). Now, for $u \in D(L)$ we have

$$(Lu, u) = -\int_0^1 u'' u\, dt - \pi^2 \int_0^1 u^2\, dt = \int_0^1 u'^2\, dt - \pi^2 \int_0^1 u^2\, dt \geq 0, \tag{2.7}$$

in view of Wirtinger's inequality.

Let now, for $e \in Y_1$; i.e., $e(x) \in L^1[0,1]$ such that $\int_0^1 e(x)\sin \pi x\, dx = 0$, $u = Ke$, denote the unique solution of the linear boundary value problem

$$\begin{gathered} -u'' - \pi^2 u = e(x),\\ u(0) = u(1) = 0, \end{gathered}$$

such that $\int_0^1 u(x)\sin\pi x\,dx = 0$. It is immediate that $K : Y_1 \to X_1$ is a bounded linear mapping such that for

$$u \in Y,\ \ KP(u) \in D(L),\ \ LKP(u) = P(u),\ \ \text{and}\ \ (KP(u), P(u)) \geq 0. \tag{2.8}$$

Also we see, using Fourier-series and Parseval inequality, for $u \in H_1$, (i.e., $u \in L^2[0,1]$ with $\int_0^1 u(x)\sin\pi x\,dx = 0$), that

$$(Ku, u) \leq \frac{1}{3\pi^2}\|u\|_H^2\,, \tag{2.9}$$

with equality holding if and only if u has the form

$$u(x) = \alpha \sin 2\pi x\,.$$

for some $\alpha \in \mathbf{R}$.

Definition 2.1. *$g : [0,1] \times \mathbf{R}^3 \to \mathbf{R}$ satisfies Caratheodory's conditions for $L^1[0,1]$ (resp. $L^2[0,1]$) if $g(x,\cdot,\cdot,\cdot)$ is continuous on $\mathbf{R}^3$ for a.e. $x \in [0,1]$, $g(\cdot,u,v,w)$ is measurable on $[0,1]$ for every $(u,v,w) \in \mathbf{R}^3$ and for each $r \in \mathbf{R}$, there is a function $\alpha_r(x) \in L^1[0,1]$ (resp. $L^2[0,1]$) such that $|g(x,u,v,w)| \leq \alpha_r(x)$ for a.e. x in $[0,1]$, u, v, $w \in \mathbf{R}$ with $|u| \leq r$, $|v| \leq r$, $|w| \leq r$.*

Theorem 2.2. *Let $g : [0,1] \times \mathbf{R}^3 \to \mathbf{R}$ satisfy Caratheodory's conditions for $L^1[0,1]$ and*

$$g(x,u,v,w)v \geq 0 \quad \textit{for} \quad (x,u,v,w) \in [0,1] \times \mathbf{R}^3\,. \tag{2.10}$$

Suppose further that for every bounded set B of $\mathbf{R}$, g is bounded on $[0,1] \times B^2 \times \mathbf{R}$. Then, for each $e \in Y = L^1[0,1]$ with $\int_0^1 e(x)\sin\pi x\,dx = 0$ and a given $\eta \in [0,1]$, the boundary value problem

$$\begin{cases} -u''' - \pi^2 u' + g(x,u,u',u'') = e(x), \\ u'(0) = u'(1) = u(\eta) = 0, \end{cases} \tag{2.11}$$

has at least one solution u in $X = C^2[0,1]$.

Proof: We first observe that it suffices to show that the following integro-differential boundary value problem

$$\begin{cases} -y'' - \pi^2 y + g\left(x, \displaystyle\int_\eta^x y(t)\,dt, y(x), y'(x)\right) = e(x), \\ y(0) = y(1) = 0, \end{cases} \tag{2.12}$$

has at least one solution y in $X = C^1[0,1]$. Indeed, if $y(x)$ is a solution for (2.12) then, $u(x) = \int_\eta^x y(t)\,dt$ is a solution for (2.11).

Defining, now, $N : X \to Y$ for $y \in X$ by setting

$$(Ny)(x) = g\left(x, \int_\eta^x y(t)\,dt,\ y(x),\ y'(x)\right), \quad x \in [0,1]$$

we see that (2.12) reduces to the functional equation

$$Ly + Ny = e \tag{2.13}$$

in X with $e \in Y_1$, where $L : D(L) \subset X \to Y$ is the linear mapping defined by (2.6). Now to solve the equation (2.13) it suffices to solve the system of equations

$$\begin{cases} Py + KPNy = e_1, \\ \qquad QNy = 0, \end{cases} \tag{2.14}$$

$y \in X$, $e_1 = Ke$ (here P, Q are as defined by (2.3) and $K : Y_1 \to X_1$ is as defined by (2.8)). We observe that it is easy to see using Arzela-Ascoli theorem that $KPN : X \to X_1$ is a well-defined compact mapping and $QN : X \to X_2$ is a bounded mapping.

Now, (2.14) is clearly equivalent to the single equation

$$Py + QNy + KPNy = e_1, \tag{2.15}$$

which has the form of a compact perturbation of the Fredholm operator P of index zero. We can therefore apply the version given in [11], (Theorem 1, Corollary 1) or [9] (Theorem IV.4) or [10], of the Leray-Schauder continuation theorem which ensures the existence of a solution for (2.15) if the set of solutions of the family of equations

$$Py + (1-\lambda)Qy + \lambda QNy + \lambda KPNy = \lambda e_1, \tag{2.16}$$

$\lambda \in (0,1)$ is, a priori, bounded in X independently of λ. We notice that equation (2.16) is equivalent to the system of equations

$$\begin{cases} \qquad Py + \lambda KPNy = \lambda e_1, \\ (1-\lambda)Qy + \lambda QNy = 0. \end{cases} \tag{2.17}$$

Also, for every $k \geq 0$, there is a constant $C(k) \geq 0$ such that

$$(Ny, y) \geq k\|Ny\|_Y - C(k), \quad y \in X. \tag{2.18}$$

Indeed, for $y \in X$, $k \geq 0$,

$$\begin{aligned} (Ny, y) &= \int_0^1 g\left(x, \int_\eta^x y(t)\,dt,\, y(x),\, y'(x)\right) y(x)\,dx \\ &= \int_{|y(x)|\geq k} \left| g\left(x, \int_\eta^x y(t)\,dt,\, y(x),\, y'(x)\right)\right| |y(x)|\,dx \\ &\quad + \int_{|y(x)|<k} \left| g\left(x, \int_\eta^x y(t)\,dt,\, y(x),\, y'(x)\right)\right| |y(x)|\,dx \\ &\geq k\|Ny\|_Y - C(k)\,, \end{aligned}$$

for some constant $C(k)$ (here we used the assumptions that g satisfies Caratheodory's conditions and (2.10)).

Suppose now, for $\lambda \in (0,1)$, $y_\lambda \in X$ is a solution for (2.17). We observe that it follows easily using (2.17) and (2.18) that there exists a constant C, independent of $\lambda \in (0,1)$, such that

$$\|Ny_\lambda\|_Y \leq C, \quad \|Py_\lambda\|_X \leq C, \quad \lambda \in (0,1)\,. \tag{2.19}$$

It only remains to prove that there is a constant C_1, independent of $\lambda \in (0,1)$ such that

$$\|Qy_\lambda\|_X \leq C_1, \quad \lambda \in (0,1)\,.$$

To prove this, we suppose, on the other hand, that the set

$$\{\|Qy_\lambda\|_X : \lambda \in (0,1)\} \quad \text{is unbounded.} \tag{2.20}$$

Now, we have from (2.19) that

$$\|(Py_\lambda)'\|_\infty \le C, \quad \lambda \in (0,1). \tag{2.21}$$

We next use the well-known estimate

$$\left|\frac{v(x)}{\sin \pi x}\right| \le \frac{1}{2} \max_{s\in[0,1]} |v'(s)|,$$

for $v \in C^1[0,1]$, $v(0) = v(1) = 0$; to get

$$|(Py_\lambda)(x)| \le \frac{1}{2} C \sin \pi x, \tag{2.22}$$

for $x \in [0,1]$, $\lambda \in (0,1)$ and some constant C independent of λ.

Now, we see from (2.20) that there is a sequence $\lambda_n \in (0,1)$ such that

$$\|Qy_{\lambda_n}\|_X = (\pi + 1)\left|2\int_0^1 y_{\lambda_n}(t)\sin \pi t\, dt\right| \to \infty,$$

as $n \to \infty$. We may now assume that

$$\int_0^1 y_{\lambda_n}(t) \sin \pi t\, dt \to \infty,$$

as $n \to \infty$, and thus find an n_0 such that

$$\int_0^1 y_{\lambda_n}(t)\sin \pi t\, dt \ge \frac{1}{4} C \quad \text{for} \quad n \ge n_0, \tag{2.23}$$

here C is the same as the C in (2.22). It now follows using (2.22) and (2.23) that for $n \ge n_0$, for $x \in [0,1]$,

$$\begin{aligned} y_{\lambda_n}(x) = Qy_{\lambda_n}(x) + Py_{\lambda_n}(x) &= \left(2\int_0^1 y_{\lambda_n}(t)\sin \pi t\, dt\right)\sin \pi x + Py_{\lambda_n}(x) \\ &\ge \frac{C}{2}\sin \pi x - \frac{C}{2}\sin \pi x \ge 0. \end{aligned}$$

Hence, we have from our assumption (2.10) that

$$g\left(x, \int_\eta^x y_{\lambda_n}(t)\, dt, y_{\lambda_n}(x), y'_{\lambda_n}(x)\right) \ge 0,$$

for $x \in [0,1]$ and $n \ge n_0$ which in turn implies that

$$(QNy_{\lambda_n}, Qy_{\lambda_n}) \ge 0 \quad \text{for} \quad n \ge n_0\,. \tag{2.24}$$

But the second equation in (2.17) implies that $(QNy_{\lambda_n}, Qy_{\lambda_n}) < 0$ for every n and so (2.24) gives a contradiction. Similarly, $\int_0^1 y_{\lambda_n}(t)\sin \pi t\, dt \to -\infty$ also leads to a contradiction. Hence, the set in (2.20) is bounded, and accordingly, the set of solutions of (2.16) are, a priori, bounded by a constant independent of $\lambda \in (0,1)$. This completes the proof of the theorem.

Theorem 2.3. *Let $g : [0,1] \times \mathbf{R}^3 \to \mathbf{R}$ satisfy Caratheodory's conditions for $L^2[0,1]$ and*

$$g(x,u,v,w)v \geq 0 \quad \text{for} \quad (x,u,v,w) \in [0,1] \times \mathbf{R}^3 . \tag{2.25}$$

Suppose further that there exists a constant $\beta \geq 0$ such that

$$\limsup_{|v|\to\infty} \frac{g(x,u,v,w)}{v} \leq \beta < 3\pi^2 \tag{2.26}$$

uniformly for $(x,u,w) \in [0,1] \times \mathbf{R}^2$. Then, for each $e \in Y = L^1[0,1]$ with $\int_0^1 e(x) \sin \pi x \, dx = 0$ and a given $\eta \in [0,1]$, the boundary value problem

$$\begin{cases} u''' + \pi^2 u' + g(x,u,u',u'') = e(x), \\ u'(0) = u'(1) = u(\eta) = 0, \end{cases} \tag{2.27}$$

has at least one solution $u \in X = C^2[0,1]$.

Proof: As in the proof of Theorem 2.2, it suffices to show that the following integro-differential boundary value problem

$$\begin{cases} y'' + \pi^2 y + g\left(x, \displaystyle\int_\eta^x y(t)\,dt,\, y,\, y'\right) = e(x), \\ y(0) = y(1) = 0 \end{cases} \tag{2.28}$$

has at least one solution y in $X = C^1[0,1]$. Indeed, if $y(x)$ is a solution for (2.28), then $u(x) = \int_\eta^x y(t)\,dt$, is a solution for (2.27).

Let us set $\tilde{L} = -L$ and $\tilde{K} = -K$ where $L : D(L) \subset X \to Y$ is the linear operator defined by (2.5), (2.6) and $K : Y_1 \to X_1$ is the bounded linear mapping as in (2.8), (2.9). Accordingly, we have for

$$u \in Y,\ \ \tilde{K}P(u) \in D(\tilde{L}),\ \ \tilde{L}\tilde{K}P(u) = P(u), \tag{2.29}$$

and for $u \in H_1$,

$$(\tilde{K}u,\, u) \geq -\frac{1}{3\pi^2}\|u\|_H^2 . \tag{2.30}$$

Defining now, $N : X \to Y$ for $y \in X$ by setting

$$(Ny)(x) = g\left(x, \int_\eta^x y(t)\,dt, y(x),\, y'(x)\right), \quad x \in [0,1],$$

we see that (2.28) reduces to the functional equation

$$\tilde{L}y + Ny = e,$$

in X with $e \in Y_1$. Now, as in the proof of Theorem 2.2, it suffices to show that the set of solutions of the system of equations

$$\begin{cases} Py + \lambda\tilde{K}PNy = \lambda e_1 \\ (1-\lambda)Qy + \lambda QNy = 0, \end{cases} \tag{2.31}$$

where $e_1 = \tilde{K}e$ is a priori bounded in X independently of $\lambda \in (0,1)$. Choosing $\epsilon > 0$ such that $\beta + \epsilon < 3\pi^2$, we first note that our assumptions on g imply that there is a constant $C(\epsilon) > 0$ such that

$$(Ny, y) \geq \frac{1}{\beta+\epsilon}\|Ny\|_H^2 - C(\epsilon), \tag{2.32}$$

for $y \in X$. Suppose now, for $\lambda \in (0,1)$, $y_\lambda \in X$ is a solution for (2.31). Then we have, using (2.31), that

$$(Py_\lambda,\, PNy_\lambda) + \lambda(\tilde{K}PNy_\lambda,\, PNy_\lambda) = \lambda(e_1,\, PNy_\lambda),$$
$$(1-\lambda)(Qy_\lambda,\, QNy_\lambda) + \lambda(QNy_\lambda,\, QNy_\lambda) = 0,$$

and hence, we get, using (2.30), that

$$\begin{cases} (Py_\lambda,\, PNy_\lambda) - \dfrac{1}{3\pi^2}\|PNy_\lambda\|_H^2 \leq \lambda(e_1,\, PNy_\lambda), \\ (Qy_\lambda,\, QNy_\lambda) \leq 0. \end{cases} \tag{2.33}$$

It now follows from (2.32) and (2.33) that

$$\begin{aligned} \frac{1}{\beta+\epsilon}\|Ny_\lambda\|_H^2 - C(\epsilon) &\leq (Ny_\lambda,\, y_\lambda) = (Py_\lambda,\, PNy_\lambda) + (Qy_\lambda,\, QNy_\lambda) \\ &\leq \frac{1}{3\pi^2}\|PNy_\lambda\|_H^2 + \lambda(e_1, PNy_\lambda) \\ &\leq \frac{1}{3\pi^2}\|Ny_\lambda\|_H^2 + C_0\|e_1\|_X\|Ny_\lambda\|_Y \\ &\leq \frac{1}{3\pi^2}\|Ny_\lambda\|_H^2 + C_0\|e_1\|_X\|Ny_\lambda\|_H\,, \end{aligned}$$

$C_0 \geq 0$ is such that $\|Pu\|_Y \leq C_0\|u\|_Y$ for $u \in Y$. This gives then,

$$\left(\frac{1}{\beta+\epsilon} - \frac{1}{3\pi^2}\right)\|Ny_\lambda\|_H^2 \leq C_0\|e_1\|_X\|Ny_\lambda\|_H + C(\epsilon)\,.$$

This shows that there is a constant $C > 0$, independent of $\lambda \in (0,1)$ such that

$$\|Ny_\lambda\|_H \leq C\,. \tag{2.34}$$

Also, we obtain from the first equation in (2.31) that

$$\begin{aligned} \|Py_\lambda\|_X &\leq \|\tilde{K}PNy_\lambda\|_X + \|e_1\|_X \leq \|\tilde{K}\|\,\|PNy_\lambda\|_Y + \|e_1\|_X \\ &\leq C_0\|\tilde{K}\|\,\|Nu_\lambda\|_Y + \|e_1\|_X \leq C_0\|\tilde{K}\|\,\|Nu_\lambda\|_H + \|e_1\|_X \\ &\leq C_0\|\tilde{K}\|C + \|e_1\|_X \equiv C_1\,. \end{aligned}$$

The boundedness of $\{\|Qu_\lambda\|_X : \lambda \in (0,1)\}$ follows as in the proof of Theorem 2.2 above. We have thus proved that the set of solutions of (2.31) is, a priori, bounded in X independently of $\lambda \in (0,1)$ and the proof of the theorem is complete.

3. Uniqueness theorems. Let us observe that for $\eta \in [0,1]$ the boundary value problem

$$\begin{cases} u''' + \pi^2 u' = 0 \\ u'(0) = u'(1) = u(\eta) = 0 \end{cases} \tag{3.1}$$

does not have a unique solution, since $u(x) = A(\cos\pi x - \cos\pi\eta)$ with A arbitrary is a solution for (3.1). So if the boundary value problems (1.5), (1.6) have unique solutions, this uniqueness has to be generated by some special nature of the function $g(x,u,v,w)$ used in *(1.5) and* (1.6).

Theorem 3.1. *Let $g : [0,1] \times \mathbf{R}^3 \to \mathbf{R}$ satisfy Caratheodory's conditions for $L^1[0,1]$ and*

(i) for $x \in [0,1]$, u_i, v_i, $w_i \in \mathbf{R}$,

$$(g(x,u_1,v_1,w_1) - g(x,u_2,v_2,w_2))\,(v_1 - v_2) \geq 0\,; \tag{3.2}$$

(ii) $g(x,0,0,0) \equiv 0$.

Suppose further, that for every bounded subset B of $\mathbf{R}$, g is bounded on $[0,1] \times B^2 \times \mathbf{R}$. Suppose for $e \in Y = L^1[0,1]$ with $\int_0^1 e(x) \sin \pi x\, dx = 0$ and a given $\eta \in [0,1]$, u_1, u_2 are any two solutions of the boundary value problem

$$\begin{cases} -u''' - \pi^2 u' + g(x,u,u',u'') = e(x), \\ u'(0) = u'(1) = u(\eta) = 0. \end{cases} \tag{3.3}$$

Then $y = u_1 - u_2$ is a solution of (3.1); i.e., any two solutions of (3.3) differ by a solution of (3.1).

Proof: We first observe that the boundary value problem (3.3) has at least one solution u in $C^2[0,1]$ in view of Theorem 2.2. If now, u_1 and u_2 are any two solutions of (3.3), we see by setting $y = u_1 - u_2$, that

$$\begin{cases} -y''' - \pi^2 y' + g(x,u_1,u_1',u_1'') - g(x,u_2,u_2',u_2'') = 0, \\ y'(0) = y'(1) = y(\eta) = 0. \end{cases} \tag{3.4}$$

Multiplying now the equation in (3.4) by $y' = u_1' - u_2'$ and integrating on the interval $[0,1]$, we get

$$\begin{aligned} 0 &= -\int_0^1 y'''y'\,dx - \pi^2 \int_0^1 y'^2\,dx + \int_0^1 [g(x,u_1,u_1',u_2'') - g(x,u_2,u_2',u_2'')]\,(u_1' - u_2')\,dx \\ &\geq \int_0^1 \left[(y'')^2 - \pi^2 y'^2\right]\,dx \geq 0, \end{aligned}$$

in view of our assumption (3.2) and the Wirtinger's inequality ([6]). Hence,

$$\int_0^1 \left[y''^2 - \pi^2 y'^2\right]\,dx = 0,$$

which implies that $y' = A \sin \pi x$ for some constant A and $y = \frac{A}{\pi}(\cos \pi\eta - \cos \pi x)$. Thus, $y = u_1 - u_2$ is a solution of (3.1) and the proof of the theorem is complete.

Theorem 3.2. *Let $g : [0,1] \times \mathbf{R}^3 \to \mathbf{R}$ be as in Theorem 2.2. Suppose that there exist functions $a(x) \in C^1[0,1]$, $b(x)$, $c(x) \in C[0,1]$ and constants a_0, b_0, $c_0 \in \mathbf{R}$, such that for a.e. $x \in [0,1]$,*

$$a'(x) \leq a_0, \quad b(x) \geq b_0, \quad c(x) \geq -c_0,$$

and for every u_i, v_i, $w_i \in \mathbf{R}$, $i = 1,2$ and a.e. $x \in [0,1]$

$$\begin{aligned} &(g(x,u_1,v_1,w_1) - g(x,u_2,v_2,w_2))\,(v_1 - v_2) \\ &\quad \geq a(x)(w_1 - w_2)(v_1 - v_2) + b(x)(v_1 - v_2)^2 + C(x)|u_1 - u_2|\,|v_1 - v_2|\,. \end{aligned} \tag{3.5}$$

with

$$\pi a_0 + 4M_\eta c_0 < 2\pi b_0 \,, \tag{3.6}$$

where $M_\eta = \max\{\eta,\, 1-\eta\}$. *Then, for each* $e \in Y = L^1[0,1]$ *with* $\int_0^1 e(x)\sin \pi x\, dx = 0$ *and a given* $\eta \in [0,1]$, *the boundary value problem (2.11) has exactly one solution* u *in* $X = C^2[0,1]$.

Proof: Let $u_1(x)$ and $u_2(x)$ be any two solutions of (2.11). Setting $y = u_1 - u_2$, we then get

$$\begin{cases} -y''' - \pi^2 y' + g(x,u_1,u_1',u_1'') - g(x,u_2,u_2',u_2'') = 0 \\ y'(0) = y'(1) = y(\eta) = 0\,. \end{cases} \tag{3.7}$$

Now we get from (3.7) that

$$\begin{aligned} 0 &= -\int_0^1 y'''y'\, dx - \pi^2 \int_0^1 y'^2\, dx + \int_0^1 [g(x,u_1,u_1',u_1'') - g(x,u_2,u_2',u_2'')]\,(u_1'-u_2')\, dx \\ &\ge \int_0^1 (y''^2 - \pi^2 y'^2)\, dx + \int_0^1 a(x)(u_1''-u_2'')(u_1'-u_2')\, dx + \int_0^1 b(x)(u_1'-u_2')^2\, dx \\ &\quad + \int_0^1 c(x)|u_1-u_2|\,|u_1'-u_2'|\, dx \\ &\ge -\frac{1}{2}\int_0^1 a'(x)(u_1'-u_2')^2\, dx + b_0 \int_0^1 (u_1'-u_2')^2\, dx - c_0 \int_0^1 |u_1-u_2|\,|u_1'-u_2'|\, dx \\ &\ge -\frac{1}{2} a_0 \|u_1'-u_2'\|_2^2 + b_0\|u_1'-u_2'\|_2^2 - c_0 \|u_1-u_2\|_2\, \|u_1'-u_2'\|_2 \\ &\ge \left(-\frac{1}{2}a_0 + b_0 - \frac{2}{\pi}M_\eta c_0\right) \|u_1'-u_2'\|_2^2\,, \end{aligned}$$

where we have used the inequality $\|y\|_2 \le \frac{2}{\pi} M_\eta \|y'\|_2$ if $y(\eta) = 0$ (see e.g. Lemma 1.3 of [2]). It now follows, using (3.6), that

$$\|y'\|_2 = \|u_1'-u_2'\|_2 = 0,$$

and hence, $y = u_1 - u_2 = 0$ a.e. in $[0,1]$ (since $\|y\|_2 \le \frac{2}{\pi} M_\eta \|y'\|_2$) and so $u_1 - u_2 = 0$ in $[0,1]$ because of continuity of u_1 and u_2. Hence the theorem.

Theorem 3.3. *Let* $g : [0,1] \times \mathbf{R}^3 \to \mathbf{R}$ *satisfy Caratheodory's conditions for* $L^2[0,1]$. *Suppose* $0 \le \beta < 3\pi^2$ *is such that*

$$(g(x,u_1,v_1,w_1) - g(x,u_2,v_2,w_2))\,(v_1 - v_2) \ge \frac{1}{\beta}\,(g(x,u_1,v_1,w_1) - g(x,u_2,v_2,w_2))^2\,. \tag{3.8}$$

for a.e. x *in* $[0,1]$ *and all* u_1, v_i, $w_i \in \mathbf{R}$, $i = 1,2$. *Also, let* $g(x,0,0,0) \equiv 0$ *for* x *in* $[0,1]$. *Suppose for* $e \in Y = L^1[0,1]$ *with* $\int_0^1 e(x)\sin \pi x\, dx = 0$ *and given* $\eta \in [0,1]$ u_1 *and* u_2 *are any two solutions of the boundary value problem*

$$\begin{cases} u''' + \pi^2 u' + g(x,u,u',u'') = e(x), \\ u'(0) = u'(1) = u(\eta) = 0\,. \end{cases} \tag{3.9}$$

Then $y = u_1 - u_2$ is a solution of (3.1); i.e., any two solutions of (3.9) differ by a solution of (3.1).

Proof: It is immediate from Theorem 2.3 that (3.9) has at least one solution u in $C^2[0,1]$. Let now, u_1 and u_2 be any two solutions of (3.9). We then see by setting $y = u_1 - u_2$, that

$$\begin{cases} y''' + \pi^2 y' + g(x, u_1, u_1', u_1'') - g(x, u_2, u_2', u_2'') = 0, \\ y'(0) = y'(1) = y(\eta) = 0. \end{cases} \tag{3.10}$$

Multiplying now the equation in (3.10) by $y' = u_1' - u_2'$ and integrating on the interval $[0,1]$, we get

$$\begin{aligned} 0 &= \int_0^1 y''' y'\, dx + \pi^2 \int_0^1 y'^2\, dx + \int_0^1 [g(x, u_1, u_1', u_1'') - g(x, u_2, u_2', u_2'')]\,(u_1' - u_2')\, dx \\ &\geq - \int_0^1 y''^2\, dx + \pi^2 \int_0^1 y'^2\, dx + \frac{1}{\beta} \int_0^1 [g(x, u_1, u_1', u_1'') - g(x, u_2, u_2', u_2'')]^2\, dx \\ &\geq -\frac{1}{3\pi^2} \int_0^1 (y''' + \pi^2 y')^2\, dx + \frac{1}{\beta} \int_0^1 [g(x, u_1, u_1', u_1'') - g(x, u_2, u_2', u_2'')]^2\, dx \\ &= (\frac{1}{\beta} - \frac{1}{3\pi^2}) \int_0^1 (y''' + \pi^2 y')^2\, dx\,, \end{aligned}$$

and so $y''' + \pi^2 y' = 0$ with $y'(0) = y'(1) = y(\eta) = 0$. Hence, y is a solution of (3.1) and the proof of the theorem is complete.

Remark 3.4 We note that in the above proof, if u_1 and u_2 are any two solutions of (3.9), then

$$g(x, u_1, u_1', u_1'') = g(x, u_2, u_2', u_2'') \tag{3.11}$$

for a.e. x in $[0,1]$. If now, g in Theorem 3.3 is such that (3.11) implies $u_1(x) = u_2(x)$ a.e. and hence for every $x \in [0,1]$, then (3.9) has exactly one solution. So if we desire uniqueness of a solution for (3.9), it has to come from additional properties on g such that (3.11) implies $u_1(x) = u_2(x)$ for a.e. x in $[0,1]$. For example, we can ask that g in Theorem 3.3 additionally satisfies conditions (3.5), (3.6) of Theorem 3.2.

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