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# AUTOMORPHISMS AND BACKLUND TRANSFORMATIONS

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**Abstract.** Bäcklund transformations for completely integrable systems are obtained as gauge transformations of the connection associated with the system. These gauge transformations depend projectively on the wave functions, and the differential equation for the gauge transformation leads in a natural way to the Riccati equations for the inhomogeneous coordinates of the wave functions. In some cases, such as the AKNS system, the Bäcklund transformation is associated with a simple automorphism of the underlying Lie algebra. The Bäcklund/gauge transformation for the modified Lax equations is constructed, and the associated automorphism found by Fordy is discussed. An automorphism for Bäcklund transformations for the generalized wave equation, discussed by Beals and Tenenblat, is given.

## 1. Introduction.

There has been considerable interest in Bäcklund transformations for completely integrable systems in recent years, and the subject has been approached from a number of different directions. In one point of view, completely integrable systems are viewed as integrability conditions for a flat connection, and the Bäcklund transformation is obtained via a gauge transformation that intertwines with the components of the connection. The gauge-theoretic point of view has been pursued by numerous researchers (cf. [4], [5], [8], [10], [11], [13], [14]). It is the analog of the Darboux transformation in the theory of the Schrödinger equation and Bäcklund transformations for the KdV equation.

In another point of view, Riccati equations for the ratios of the components of the wave functions are derived from the linear equations for the wave functions, and the Bäcklund transformations are obtained from suitable automorphisms of these Riccati equations. This approach has been developed by Chen [6], Fordy [7], Winternitz [18], and Harnad, St. Aubin, and Shnider [9], among others.

In fact, the gauge transformations depend projectively on the wave functions, in a sense we shall explain here, and are found to satisfy a matrix Riccati equation. When rewritten in terms of the wave functions, the equations satisfied by the gauge transformation lead naturally to the Riccati equations for the wave functions as well as the automorphism associated with the Bäcklund transformation. These facts follow quite simply from the results in [14], and will be explained below.

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**2. Gauge transformations and Bäcklund transformations.** Consider the pair of operators

$$D_x(z,Q) = \frac{\partial}{\partial x} - zJ - Q \qquad D_t(z,Q) = \frac{\partial}{\partial t} - z^r K - B_r(z,Q)$$

where $J$ and $K$ are diagonal matrices and $Q$ is a matrix-valued function of $x$ and $t$ with vanishing diagonal elements: $Q_{ii} = 0$. We assume $Q \in L_1(R)$, and in addition sometimes require $Q$ to have smooth derivatives up to some order. We shall assume that $J$, $K$ and $Q$ take their values in some semi-simple Lie algebra $\mathbf{g}$. The term $B_r$ is a polynomial in $z$, $Q$, and its derivatives up to order $r-1$. For the correct choice of the polynomials $B_r$ the condition $[D_x\,,\,D_t] = 0$ leads to a completely integrable equation in the matrix $Q$. (cf., for example, [12], [17]).

**Definition 2.1.** A gauge transformation $G$ from the potential $Q_1$ to $Q_2$ is a matrix valued function $G(x,t,z)$, bounded in $x$ on $(-\infty,\infty)$ and analytic in $z$ such that

$$GD_x(z,Q_1) = D_x(z,Q_2)G \qquad GD_t(z,Q_1) = D_t(z,Q_2)G. \tag{2.1}$$

It follows immediately from (2.1) that

$$[D_x(z,Q_1)\,,\,D_t(z,Q_1)] = 0 \qquad \Rightarrow \qquad [D_x(z,Q_2)\,,\,D_t(z,Q_2)] = 0$$

so if $Q_1$ satisfies the nonlinear evolution equation, so does $Q_2$; in other words, such gauge transformations preserve solutions of the nonlinear differential equation.

The following theorem was proved in [14] (cf. also [2]).

**Theorem 2.1.** *Let $G$ be a gauge transformation which is a linear function of $z$. Then $G$ has the form*

$$G = \Phi(z-C)\Phi^{-1}, \qquad \Phi = \|\phi_1, \phi_2, \ldots, \phi_n\|$$

*where $C$ is a diagonal matrix, $C = diag(z_1, z_2, \ldots, z_n)$ and the wave functions $\phi_j$ satisfy the simultaneous equations*

$$D_x(z_j, Q_1)\phi_j = 0 \qquad D_t(z_j, Q_1)\phi_j = 0.$$

The potentials $Q_1$ and $Q_2$ are related by

$$Q_2 = Q_1 + [G, J].$$

As $x \to \pm\infty$, $G$ tends to the limits

$$\lim_{x\to-\infty} G(x,t,z) = z - C \qquad \lim_{x\to\infty} G(x,t,z) = z - \tau C \tau^{-1}$$

where $\tau$ is a permutation matrix.

From the first equation in (2.1) we get the following differential equation for the gauge transformation:

$$\frac{\partial G}{\partial x} = z[J,G] + Q_2 G - G Q_1\,.$$

Substituting the specific form $G = z - G_0$ into this equation, where $G_0 = \Phi C \Phi^{-1}$, we get the pair of equations

$$\begin{aligned} Q_2 - Q_1 &= [J\,,\,G_0] \\ \frac{\partial G_0}{\partial x} &= Q_2 G_0 - G_0 Q_1\,. \end{aligned}$$

Using the first equation we may eliminate either $Q_2$ or $Q_1$ from the second equation. In that case we obtain the pair of equations

$$\begin{aligned} \frac{\partial G_0}{\partial x} &= [Q_1, G_0] + [J, G_0]G_0 \\ \frac{\partial G_0}{\partial x} &= [Q_2, G_0] + G_0[J, G_0]\,. \end{aligned} \tag{2.2}$$

Note that the transformation $Q_1 \to Q_2$ is effected by flipping $G_0$ to the other side of the commutator $[J, G_0]$. This flip can sometimes be obtained from an automorphism of the underlying Lie algebra **g**, as we shall see in the examples below. Note also that equations (2.2) are matrix Riccati equations for $G_0$.

**3. Bäcklund transformations for AKNS systems.** Bäcklund transformations for $2 \times 2$ AKNS systems have been discussed by numerous authors ([4,5,6,8,10,11,13]). Let us show here how the gauge transformation may be computed directly from Theorem 2.1 above. We consider a $2 \times 2$ system with

$$Q = \begin{bmatrix} 0 & q \\ -q & 0 \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\,. \tag{3.1}$$

Let $\sigma$ be the matrix

$$\sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\,. \tag{3.2}$$

Observe that $\sigma D_x(z, Q)\sigma^{-1} = D_x(-z, Q)$ since $\sigma Q \sigma^{-1} = Q$, and $\sigma J \sigma^{-1} = -J$. Let $\phi_0$ be a wave function with wave number $z_0$, i.e., $D_x(z_0, Q)\phi_0 = 0$. Then $\sigma\phi_0$ is a wave function with wave number $-z_0$. We form the matrix $\Phi = \|\phi_0\,,\,\sigma\phi_0\|$ and note that the gauge transformation $G$ depends projectively on the wave functions $\phi_0$ and $\sigma\phi_0$. That is, $G$ is unchanged if we factor out a diagonal matrix on the right of $\Phi$. Let us write

$$\phi_0 = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}\,.$$

Then

$$\Phi = \begin{bmatrix} \phi_1 & -\phi_2 \\ \phi_2 & \phi_1 \end{bmatrix} = \begin{bmatrix} 1 & -u \\ u & 1 \end{bmatrix} \begin{bmatrix} \phi_1 & 0 \\ 0 & \phi_1 \end{bmatrix}$$

where $u = \phi_2/\phi_1$; hence $\Phi$ is equivalent projectively to

$$\begin{bmatrix} 1 & -u \\ u & 1 \end{bmatrix} \sim \frac{1}{(1+u^2)^{1/2}} \begin{bmatrix} 1 & -u \\ u & 1 \end{bmatrix} = \begin{bmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{bmatrix}$$

where $u = \tan\gamma$. Thus the gauge transformation for the $2 \times 2$ AKNS system with a skew symmetric potential as in (3.1) is

$$G = OAO^t$$

where $O$ is the rotation matrix with angle $\gamma$ and $A = z - z_0 J$. The gauge transformation for this case was first found by Newell [11] by another method. It was given in the present form in [13].

Now let us study the automorphism associated with this Bäcklund transformation. First note that

$$G_0 = z_0 OJO^t = z_0 \begin{bmatrix} \cos 2\gamma & -\sin 2\gamma \\ -\sin 2\gamma & -\cos 2\gamma \end{bmatrix}.$$

A straightforward calculation shows that the pair of equations (2.2) reduce to

$$\gamma_x = q_1 - z_0 \sin 2\gamma \qquad \gamma_x = q_2 + z_0 \sin 2\gamma.$$

The Bäcklund transformation is given by $q_1 \to q_2$. In terms of the variable $u = \tan\gamma$ these relations can be written as

$$q_1 = F(z_0, u) \qquad q_2 = F(z_0, -1/u)$$

where

$$F(z_0, u) = \frac{u_x + 2z_0 u}{1 + u^2}.$$

We thus see that for the $2 \times 2$ AKNS system with skew symmetric potential, the Bäcklund transformation is affected by the autoromorphism $u \to -1/u$. Since $u = \phi_2/\phi_1$, this transformation is obtained from $\phi \to \sigma\phi$. Note that the inner automorphism $X \to \sigma X \sigma^{-1}$ is an automorphism of the Lie algebra $\mathbf{g} = \mathrm{sl}\,(2, R)$, whose matrices appear in the connection $\{D_x\,,\, D_t\}$.

Moreover, the transformation $u \to -1/u$ has a simple geometric interpretation on the projective space $P_1(R)$ : namely, the line through the wave function $\phi$ is transformed into its orthogonal complement.

Similar results hold for the case where the potential $Q$ is skew-Hermitian. This case arises, for example, in the nonlinear Schrödinger equation. In that case $G = UAU^*$, where $U$ is a unitary matrix; and the automorphism associated with the Bäcklund transformation is $u \to 1/u^*$. The projective space $P_1(C)$ is equivalent to the Riemann sphere, and the automorphism is equivalent to mapping a point $u$ onto its antipodal point. (cf. [13]).

**4. Modified Lax equations.** We now construct a Bäcklund - gauge transformation for the modified Lax equations (cf. Fordy, [7]). We shall show that the Riccati equations obtained by Fordy follow directly from the matrix Riccati equations (2.2) for $G_0$. In the $2\times 2$ case the Bäcklund - gauge transformation was constructed using the inner automorphism generated by $\sigma$ in (3.2). A similar construction will be given in this case.

Consider the linear operator $D_x(z, Q) = \partial_x - zJ - Q$ where

$$J = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & \\ & & \cdot & & \\ & & \cdot & & \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} q_1 & 0 & 0 & \dots \\ 0 & q_2 & 0 & \dots \\ & & \cdot & \\ & & & \dots \\ 0 & 0 & \dots & q_n \end{bmatrix}.$$

The linear system of equations $D_x(z,Q)\Psi = 0$ comes from an $n^{\text{th}}$ order scalar equation $L\Psi = z^n\Psi$ by an appropriate factorization of the scalar operator $L$ as $L = (\partial - q_n)\dots(\partial - q_1)$. The matrix $Q$ is of trace zero whenever $L$ has the form $L = \partial^n + u_{n-2}\partial^{n-2k} + \cdots + u_0$; and we shall take this to be the case. Thus the matrix coefficients of $D_x$ lie in the Lie algebra $\mathrm{sl}(n,R)$.

We construct a gauge transformation using an inner automorphism of $\mathrm{sl}(n,R)$ which leaves $Q$ invariant. Note that any diagonal matrix $\sigma \in SL(n,R)$ (capital letters denote the Lie group) leaves $Q$ invariant: $\sigma Q\sigma^{-1} = Q$. We choose $\sigma = \mathrm{diag}\,(1,\omega,\omega^2,\dots,\omega^{n-1})$; the reason for this particular choice will be explained shortly.

In order to apply Theorem 2.1 we must work in a basis (gauge) in which $J$ is diagonal. The algebraic computations are easier, however, in the original gauge; so it will be necessary to make the initial computations in the "diagonal" gauge and then transform back. We denote all quantities in the gauge in which $J$ is diagonal by a $\sim$ .

The eigenvalues of $J$ are $1,\omega,\omega^2,\dots,\omega^{n-1}$, where $\omega = e^{2\pi i/n}$. The diagonalizing matrix $P$ is

$$P = \frac{1}{\sqrt{n}}\begin{bmatrix} 1 & 1 & 1 & \dots & \\ 1 & \omega & \omega^2 & \omega^3 & \dots \\ 1 & \omega^2 & \omega^4 & \omega^6 & \dots \\ & & \cdot & & \\ 1 & \omega^{n-1} & \dots & & \end{bmatrix}.$$

The entries of $P$ are

$$P_{ij} = \frac{1}{\sqrt{n}}\omega^{(i-1)(j-1)} \qquad i,j = 1,\dots,n.$$

It is easily seen that $P^t = P$ and that $PP^* = 1$. In fact,

$$(PP^*)_{ij} = \frac{1}{n}\sum_{k=1}^{n}\omega^{(i-1)(k-1)}\omega^{(k-1)(j-1)} = \frac{1}{n}\sum_{k=1}^{n}(\omega^{i-j})^{k-1} = \delta_{ij}\,.$$

(Note that $\omega^k = \omega^{-k}$ and $\sum_{k=0}^{n-1}\omega^{lk} = 0 \quad$ for any $\ l \neq 0$). A quick check shows that $P$ diagonalizes $J : P^*JP = \tilde{J} = \mathrm{diag}\,(1,\omega,\omega^2,\dots,\omega^{n-1})$.

We now compute $\Phi$ in the $\sim$ gauge and then transform back. Let $\tilde{\sigma} = \|\mathbf{e}_2,\mathbf{e}_3,\dots,\mathbf{e}_1\|$, *viz.*

$$\tilde{\sigma} = \begin{bmatrix} 0 & 0 & 0 & \dots & & 1 \\ 1 & 0 & 0 & \dots & & 0 \\ 0 & 1 & 0 & \dots & & 0 \\ \vdots & & & & & \\ 0 & \dots & & \dots & 1 & 0 \end{bmatrix}.$$

Then $\tilde{\sigma}\tilde{J}\tilde{\sigma}^{-1} = (\omega)^{n-1}\tilde{J}$; and $\tilde{\sigma}\tilde{Q}\tilde{\sigma}^{-1} = \tilde{Q}$. To see this, note first that $\sigma = P\tilde{\sigma}P^* = \mathrm{diag}\,(1,\omega,\omega^2,\dots,\omega^{n-1})$; hence $\sigma Q\sigma^{-1} = Q$. The invariance of $Q$ under the action of $\sigma$ is preserved by the transformation to the $\sim$ gauge.

It follows that $\tilde{\sigma}D_x(z,\tilde{Q})\tilde{\sigma}^{-1} = D_x(\omega^{n-1}z,\tilde{Q})$; so if $\tilde{\phi}$ is a wave function with wave number $z_0(D_x(z_0,\tilde{Q})\tilde{\phi} = 0)$, then $\tilde{\sigma}^j\tilde{\phi}$ will be a wave function with wave number $\omega^{j(n-1)}z_0$.

The diagonal matrix $\sigma$ we mentioned above that leaves $Q$ invariant is $\sigma = P\tilde{\sigma}P^* = \operatorname{diag}(1, \omega^{n-1}, \ldots, \omega)$. Of course, there are several other diagonal matrices that could have been chosen. But this choice of $\sigma$ ensures the correct asymptotic behavior of $\tilde{\Phi}$ as $x \to -\infty$. Note that $\tilde{\sigma}e_j = e_{j+1} \pmod n$. The gauge transformation $\tilde{G}$ is given by $\tilde{G} = \tilde{\Phi}(z - z_0\tilde{C})\tilde{\Phi}^{-1}$ where

$$\tilde{C} = \operatorname{diag}(1, \omega^{n-1}, \ldots, \omega^{(n-1)(n-1)}) = (\tilde{J})^{n-1} \qquad \tilde{\Phi} = \|\tilde{\phi}, \tilde{\sigma}\tilde{\phi}, \ldots, \tilde{\sigma}^{n-1}\tilde{\phi}\|.$$

Recall that $\tilde{G} \to z - \tilde{C}$ as $x \to -\infty$; we must therefore choose $\tilde{\Phi}$ so that it tends projectively to a diagonal matrix as $x \to -\infty$. By choosing $z_0$ so that $\operatorname{Re} z_0 < \operatorname{Re} \omega^j z_0$ for $j = 1, \ldots, n$, we have

$$\tilde{\phi} \sim \exp(xz_0)(e_1 + o(1)) \qquad \text{as } x \to -\infty;$$

hence

$$\tilde{\sigma}^j\tilde{\phi} \sim \exp(xz_0)(e_{j+1} + o(1)) \qquad \text{as } x \to -\infty.$$

Therefore the matrix $\tilde{\Phi}$ constructed above tends projectively to the identity matrix as $x \to -\infty$; that is, $\tilde{\Phi} = \exp\{xz_0\}(1 + o(1))$ as $x$ tends to $-\infty$.

The gauge transformation $G$ in the original gauge is given by

$$G = \Phi(z - C)\Phi^{-1}$$

where

$$\Phi = P\tilde{\Phi}P^* \qquad \text{and} \qquad C = P\tilde{C}P^* = P\tilde{J}^{n-1}P^* = J^{n-1}.$$

Now

$$\begin{aligned} \Phi = P\tilde{\Phi}P^* &= P\|\tilde{\phi}, \tilde{\sigma}\tilde{\phi}, \ldots\|P^* \\ &= \|P\tilde{\phi}, P\tilde{\sigma}P^*P\tilde{\phi}, \ldots\|P^* \\ &= \|\phi, \sigma\phi, \ldots\|P^*. \end{aligned}$$

Since $\sigma = P\tilde{\sigma}P^* = \operatorname{diag}(1, \omega, \ldots, \omega^{n-1})$, we find after a straightforward calculation that

$$\Phi = \sqrt{n}\,\operatorname{diag}(\phi_1, \phi_2, \ldots, \phi_n)$$

where

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{bmatrix}.$$

Hence

$$\Phi^{-1} = \frac{1}{\sqrt{n}}\operatorname{diag}(\phi_1^{-1}, \phi_2^{-1}, \ldots, \phi_n^{-1}).$$

Finally, $G = z - z_0 G_0$, where $G_0 = \Phi C \Phi^{-1} = \Phi J^{n-1}\Phi^{-1}$. A simple computation shows that

$$G_0 = \begin{bmatrix} 0 & 0 & \ldots & & \xi_n \\ \xi_1 & 0 & \ldots & & 0 \\ 0 & \xi_2 & \ldots & & 0 \\ & & \ldots & & \\ 0 & 0 & \ldots & \xi_{n-1} & 0 \end{bmatrix} \qquad \text{where} \qquad \xi_j = \frac{\phi_{j+1}}{\phi_j}.$$

We now return to the Riccati equations (2.2) for $G_0$. Writing the initial and transformed potentials as

$$Q_j = \operatorname{diag}(q_{j,1}, \ldots, q_{j,n}) \qquad j = 1, 2$$

equations (2.2) lead immediately to the pair of Riccati equations

$$\frac{d\xi_j}{dx} = \xi_j(q_{1,j+1} - q_{1,j}) + z_0\xi_j(\xi_{j+1} - \xi_j) \tag{4.1a}$$

and

$$\frac{d\xi_j}{dx} = \xi_j(q_{2,j+1} - q_{2,j}) + z_0\xi_j(\xi_j - \xi_{j-1}). \tag{4.1b}$$

These two equations are equivalent (with $\phi_j = \log \xi_j$), respectively, to the Riccati equations (4.11) and (4.13) given by Fordy. They may be obtained directly from the equations for the wave functions as follows. In the equation $D_x(z_0, Q)\phi = 0$ set $\xi_j = \phi_{j+1}/\phi_j$; then the Riccati equations (4.1a) for $\xi_j$ follow immediately by a direct substitution into the linear equations. To get the second equation, Fordy first applies the automorphism $X \to -X^t$; this is an automorphism of any Lie algebra. The new equation for the wave function is then

$$\frac{\partial \phi}{\partial x} + zJ^t\phi + Q\phi = 0$$

or

$$\frac{\partial \phi_j}{\partial x} + z\phi_{j-1} + q_j\phi_j = 0.$$

The second equation (4.1b) is then obtained by setting $\xi_j = \phi_j/\phi_{j+1}$.

The discrete automorphism found by Fordy which takes (4.1a) into (4.1b) is

$$X \to -X^t \qquad \xi_j \to 1/\xi_j \qquad Q_1 \to Q_2.$$

There is an important difference between this case and that of the AKNS system discussed in §3. There the automorphism $X \to -X^t$ is an inner automorphism given by conjugation with the matrix $\sigma$ defined in (3.2): on $\mathrm{sl}(2, R)$, $\sigma X \sigma^{-1} = -X^t$. Under the transformation $X \to \sigma X \sigma^{-1} = -X^t$, the wave functions transform automatically as $\Psi \to \sigma\Psi$; and the Bäcklund transformation is accomplished solely by $u \to -1/u$. As we saw in §3, this has a simple geometric interpretation on the projective sphere $P_1(R)$.

In the present case, the transformation from (4.1a) to (4.1b) is *not* obtained by the transformation $\xi_j \to 1/\xi_j$ alone; one must also make the transformation $X \to -X^t$ in the wave equations. Thus the Bäcklund transformation for the modified Lax equations is not affected by a simple geometric transformation of $P^{n-1}(R)$ by itself, as in the $2 \times 2$ case.

One final comment: note that $J^n = 1$, and therefore, since $G_0 = \Phi J^{n-1}\Phi^{-1}$, $G_0^n = 1$ also. Consequently, $z^n - 1 = (z - G_0)(z^{n-1} + z^{n-2}G_0 + \cdots + G_0^{n-1})$, and

$$(z - G_0)^{-1} = \frac{z^{n-1} + z^{n-2}G_0 + \cdots + G_0^{n-1}}{z^n - 1}.$$

Since the scalar $(z^n - 1)^{-1}$ factors through the differential operator $D_x(z, Q)$, the gauge transformation for the inverse Bäcklund transformation (which takes $Q_2$ to $Q_1$) can be taken to be the $(n-1)^{\mathrm{st}}$ degree function $(z^{n-1} + z^{n-2}G_0 + \cdots + G_0^{n-1})$.

**5. Generalized wave and sine-Gordon equations.** The classical geometric transformation due to Bäcklund for surfaces of constant negative Gaussian curvature has been generalized by Terng and Tenenblat to n dimensional surfaces (cf. [15], [16]). The generalized Bäcklund transformation is expressed analytically in the form of a matrix Riccati equation, which in turn can be "linearized" (by passing to homogeneous coordinates) to obtain a $2n \times 2n$ system of linear equations [1]. The generalized sine-Gordon equations (GSGE) arise in the embedding of a hyperbolic n-dimensional manifold in $R^{2n-1}$; while the generalized wave equation (GWE) arises in the embedding of a flat n ($\geq 3$) dimensional manifold in the unit sphere $S^{2n-1}$ in $R^{2n}$.

I would like to thank Dick Beals for explaining his work with Tenenblat to me, and making available a copy of [3].

The GWE and GSGE arise as compatibility conditions for overdetermined systems of linear equations. For example, in the case of GWE, which is the simpler of the two cases, the linear system is [1,3]

$$\frac{\partial \Psi}{\partial x_j} = zJ_j\Psi + C_j\Psi \tag{5.1}$$

where

$$J_j = \begin{bmatrix} 0 & e_j \\ e_j & 0 \end{bmatrix} \qquad \text{and} \qquad C_j = \begin{bmatrix} \alpha_j & 0 \\ 0 & \gamma_j \end{bmatrix}.$$

Here 0 is the $n \times n$ zero matrix; $e_j$ is the $n \times n$ matrix with zeroes everywhere except the $jj^{\text{th}}$ entry, which is 1; and $\alpha_j$ and $\gamma_j$ are $n \times n$ skew symmetric matrices. The matrices $\alpha_j$ and $\gamma_j$ are related to the coefficients of the first and second fundamental forms of the embedded flat surface. $J_j$ and $C_j$ are $2n \times 2n$ matrices.

The consistency equations for the overdetermined system (5.1) lead to a set of nonlinear partial differential equations for $\alpha_j$ and $\gamma_j$. The original geometric Bäcklund transformation derived by Terng and Tenenblat coincides with the "Bäcklund transformation" for integrable systems constructed in Theorem 2.1. The associated gauge transformation, along with a number of interesting properties, has been obtained by Beals and Tenenblat [3]. Let us look at that Bäcklund/gauge transformation from the point of view of the associated automorphism. It bears an interesting relationship with the $2 \times 2$ case.

In order to apply the results of Theorem 2.1, the equations are transformed into a form in which the matrix coefficient of $z$ is diagonal. As in the case of the modified Lax equations, we label all quantities in this gauge with a $\sim$ . The transformation to the $\sim$ gauge is accomplished by the (orthogonal) matrix

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

where, as usual, 1 here denotes the $n \times n$ identity matrix. Thus $\Psi = U\tilde{\Psi}$ and

$$\tilde{J}_j = U^t J_j U = \begin{bmatrix} e_j & 0 \\ 0 & -e_j \end{bmatrix} \qquad \tilde{Q}_j = U^t Q_j U = \begin{bmatrix} \alpha_j + \gamma_j & \gamma_j - \alpha_j \\ \gamma_j - \alpha_j & \alpha_j + \gamma_j \end{bmatrix}.$$

We shall show that the inner automorphism given by conjugation with respect to

$$\sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

where 1 denotes the $n \times n$ identity matrix, generates the Bäcklund transformation for GWE. It is easily seen that $\sigma J_j \sigma^{-1} = -J_j$ and $\sigma Q_j \sigma^{-1} = Q$; so $\sigma D_x(z, Q_j)\sigma^{-1} = D_x(-z, Q_j)$.

In the $\sim$ gauge,

$$\tilde{\sigma} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Beals and Tenenblat proved

**Lemma 5.1.** ([3]) *Let $\tilde{\Psi}$ be the fundamental solution of the equations $\tilde{D}_x(z, \tilde{Q})$ normalized so that* $\exp(-zx \cdot J)\tilde{\Psi} \to 1$ *as $x \to -\infty$ (here $x \cdot J = \sum_j x_j J_j$). Then*

(i) $\tilde{\sigma}\tilde{\Psi}\tilde{\sigma}^{-1} = (\tilde{\Psi}^t)^{-1} = \tilde{\Psi}(-z, x)$.

(ii) *Let $v + v^t = 0$ and define*

$$\tilde{\Phi} = \tilde{\Psi}(x, z_0)\begin{bmatrix} 1 & 0 \\ v & 0 \end{bmatrix} + \tilde{\Psi}^\sigma(x, z_0)\begin{bmatrix} 0 & v \\ 0 & 1 \end{bmatrix}$$

*(here $\tilde{\Psi}^\sigma = \tilde{\sigma}\tilde{\Psi}\tilde{\sigma}^{-1}$). Then $\tilde{\sigma}\tilde{\Phi}\tilde{\sigma}^{-1} = \tilde{\Phi}$; and consequently $\tilde{\Phi}$ has the form*

$$\tilde{\Phi} = \begin{bmatrix} \tilde{p} & \tilde{q} \\ \tilde{q} & \tilde{p} \end{bmatrix}.$$

(iii) $\tilde{u} = \tilde{q}\tilde{p}^{-1}$ *satisfies* $\tilde{u}^t = -\tilde{u}$.

As a consequence of (i), $\Psi$ satisfies the condition

$$\tilde{\Psi}\tilde{\sigma}\tilde{\Psi}^{-1} = \tilde{\sigma};$$

such matrices form a Lie group (isomorphic to, $\mathrm{O}(n, n)$). Thus, the symmetry of the potentials $\tilde{Q}_j$ leads to the fact that the fundamental solution is a section of a subgroup of $\mathrm{SL}(2n, R)$. The matrix $\tilde{u}$ is an $n \times n$ matrix that plays the role of the "inhomogeneous coordinate" for the Bäcklund transformation.

Now let us transform back to the original gauge. The wave functions are related by $\Psi = U\tilde{\Psi}$; so

$$\begin{bmatrix} p \\ q \end{bmatrix} = U\begin{bmatrix} \tilde{p} \\ \tilde{q} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \tilde{p} - \tilde{q} \\ \tilde{p} + \tilde{q} \end{bmatrix};$$

hence

$$u = qp^{-1} = \left(\frac{\tilde{p} + \tilde{q}}{\sqrt{2}}\right)\left(\frac{\tilde{p} - \tilde{q}}{\sqrt{2}}\right)^{-1} = (1 + \tilde{u})(1 - \tilde{u})^{-1}.$$

As a consequence of (iii), it follows that $u^t u = 1$.

In the $2 \times 2$ case the components of the wave function $\Psi$ were regarded as homogeneous coordinates of point in $P_1(R)$ (actually, a *section* of $P_1(R)$, since $\Psi$ is a function of $x$ and $t$). The parameter $u = \Psi_1/\Psi_2$ was an inhomogeneous coordinate of $P_1(R)$, and satisfied a Riccati equation. The automorphism $\sigma$ of §3, which was the Cartan automorphism of $\mathrm{sl}(2, R)$, induced an action on $P_1(R)$ which generated the Bäcklund transformation.

In the present case, the $2n \times n$ vector

$$\begin{bmatrix} p \\ q \end{bmatrix}$$

may be regarded as a point in the Grassmannian $\mathrm{Gr}\,(2n, n)$; and the $n \times n$ matrix $u = qp^{-1}$ is an inhomogeneous coordinate on this Grassmannian.

Let us derive the matrix Riccati equation for $u$. The associated gauge transformation is

$$\tilde{G} = \tilde{\Phi}(z - z_0 \tilde{J})\tilde{\Phi}^{-1} \qquad \text{or} \qquad G = \Phi(z - z_0 J)\Phi^{-1}$$

where

$$J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \Phi = U\tilde{\Phi}U^{-1} = \sqrt{2}\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}.$$

It is easily found that

$$G = z - z_0 \begin{bmatrix} 0 & u^{-1} \\ u & 0 \end{bmatrix}.$$

Substituting $G$ into equation (2.2), we get the matrix Riccati equations for $u$ :

$$\frac{\partial u}{\partial x_j} = z_0(e_j - ue_j u) + \alpha_j u - u\gamma_j.$$

These equations are identical to those obtained directly from the linear equations for $p$ and $q$ :

$$\frac{\partial p}{\partial x_j} = z_0 e_j q + \alpha_j p$$
$$\frac{\partial q}{\partial x_j} = z_0 e_j p + \gamma_j q$$

by setting $u = qp^{-1}$.

Under the automorphism $\sigma$ we have $(p, q) \to (p, -q)$, hence $u \to -u$ and $G_0 \to -G_0$. Therefore, under $\sigma$ the first equation goes into

$$[\hat{Q}_j, G_0] = \frac{\partial G_0}{\partial x_j} + z_0[J_j, G_0]G_0$$

where we have also made the transformation $Q_j \to \hat{Q}_j$. Now it is easy to check that

$$[J_j, G_0]G_0 = -G_0[J_j, G_0]$$

so the new potential $\hat{Q}_j$ satisfies

$$[\hat{Q}_j, G_0] = \frac{\partial G_0}{\partial x_j} - z_0 G_0[J_j, G_0]$$

which is the same equation satisfied by $Q_2^j$.

In order to complete the proof that $\sigma$ is in fact the automorphism of the Bäcklund transformation we still need to prove that the equation $[G_0, w] = f$ uniquely determines $w$ given $f$. However, $\operatorname{ad} G_0$ has a nontrivial kernel. In fact

$$\ker \operatorname{ad} G_0 = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \gamma \end{bmatrix} \mid u\alpha - \gamma u = 0 \right\}$$

*viz.* $\alpha$ and $\gamma$ are similar: $u\alpha u^{-1} = \gamma$; so such potentials must be eliminated from consideration. I have so far not been able to resolve this point.

The gauge transformation and the Riccati equations can also be calculated in the $\sim$ gauge, though the Riccati equations are more complicated. For $\tilde{G}$ one finds:

$$\begin{aligned} \tilde{G} &= \tilde{\Phi}(z - z_0\tilde{J})\tilde{\Phi}^{-1} \\ &= \begin{bmatrix} \tilde{p} & \tilde{q} \\ \tilde{q} & \tilde{p} \end{bmatrix} (z - z_0 J) \begin{bmatrix} \tilde{p} & \tilde{q} \\ \tilde{q} & \tilde{p} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & \tilde{u} \\ \tilde{u} & 1 \end{bmatrix} (z - z_0\tilde{J})(1 - \tilde{u}^2)^{-1} \begin{bmatrix} 1 & \tilde{u} \\ -\tilde{u} & 1 \end{bmatrix} \end{aligned}$$

where $u = qp^{-1}$. Carrying out these computations we get

$$\tilde{G} = z - \frac{z_0}{1 - \tilde{u}^2} \begin{bmatrix} 1 + \tilde{u}^2 & -2\tilde{u} \\ 2\tilde{u} & -(1 + \tilde{u}^2) \end{bmatrix}.$$

Putting $u = \tanh\gamma$ (since $u$ is skew symmetric, we can define $\gamma = \tanh^{-1} u$ by the spectral decomposition theorem) we see that

$$\tilde{G} = z - z_0 \begin{bmatrix} \cosh 2\gamma & -\sinh 2\gamma \\ \sinh 2\gamma & -\cosh 2\gamma \end{bmatrix}$$

where $\gamma$ is an $n \times n$ matrix.

An extensive analysis of matrix Riccati equations and their role in Bäcklund transformations for nonlinear sigma models has been given by Harnad, St. Aubin, and Shnider [9]. Winternitz [18] has also discussed the theory of Bäcklund transformations from the point of view of matrix Riccati equations.

**6. Transformation of the scattering data.** The gauge transformation is an essential tool in calculating the transformation of the scattering data under the Bäcklund transformation. In fact, it is precisely the analog of the "Darboux transformation" which plays the corresponding role in the theory of the KdV equation. (cf. [2], [8], [10]). The Bäcklund transformation introduces new poles (or increases the order of the poles already there) into the scattering data at the points $z_1 \dots z_n$. The scattering data is represented by matrices, essentially matrix coefficients in a principal parts expansion of the fundamental wave function at the poles. The calculation of the new matrix coefficients corresponding to the poles introduced into the scattering data by the Bäcklund transformation was worked out by Zurkowski in his thesis for the general case [13, 19]. We shall illustrate the process in a simple case here.

The scattering data is formulated in terms of the singularities of the wave function $m(x,z)$ constructed by Beals and Coifman. A solution of $D_x(z,Q)\Psi = 0$ is constructed of the form $\Psi = me^{xzJ}$, where $m$ is a solution of the differential equation

$$\frac{\partial m}{\partial x} = z[J,m] + Qm \tag{6.1}$$

bounded on $-\infty < x < \infty$, and normalized to tend to the identity matrix as $x \to -\infty$. Let $J = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ and define the sectors

$$\Omega_\nu = \left\{ z \,|\, \operatorname{Re} z(\lambda_{\nu^{-1}(j)} - \lambda_{\nu^{-1}(k)}) < 0 \;\text{ for }\; j < k \right\}$$

where $\nu$ is a permutation in $S_n$. Then $m$ is meromorphic in the sectors $\Omega_\nu$ and has jumps across the boundaries of these sectors. The poles of $m$ are a consequence of the "boundary condition" that $m$ be bounded on the entire real line. Zurkowski proved

**Theorem 6.1.** *Let $m$ have an isolated pole at $z_0 \in \Omega_\nu$. Then $m$ can be factored as*

$$m(x,z) = \eta(x,z)V^x(z)$$

*where $\eta$ is a solution of (6.1), regular at $z_0$ which tends to 1 as $x$ tends to $-\infty$, but which is not necessarily bounded on the entire real line, and*

(i) $V^x = e^{xzJ}V_0(z)e^{-xzJ}$.

(ii) $V_0(z) = I + L_0(z)$, *where $\nu L_0 \nu^{-1}$ is strictly lower triangular.*

(iii) *$L_0$ is a polynomial in $(z - z_0)^{-1}$ that tends to zero as $z \to \infty$.*

The matrix $I + L_0$ is called the *principal factor* of $m$ at $z_0$. The poles of $m$ and the principal factors of $m$ at those poles constitute the discrete part of the scattering data. In addition, there is a continuous component of the scattering data related to the jumps of $m$ across the boundaries of the regions $\Omega_v$ :

$$V_\nu^x(\xi) = m(x,\xi+)m^{-1}(x,\xi-)$$

where $\xi \in \Sigma_\nu$, the boundary of one of the sectors $\Omega_\nu$, and $m(x,\xi\pm)$ denote the limits of $m$ as $\xi$ is approached from the counter clockwise and clockwise directions, respectively.

The transformation of the scattering data under the gauge transformation $G = \Phi A \Phi^{-1}$, where $A = z - C$ is easily computed for the jump conditions and the principal factors at the simple poles of $m$. It is simply conjugation by the matrix $A$ : under the Bäcklund transformation $Q_1 \to Q_2$,

$$V_{1,\nu} \to V_{2,\nu} = A(\xi)V_{1,\nu}A^{-1}(\xi) \qquad \xi \in \Sigma_\nu,$$

and

$$V_{1,0} \to V_{2,0}(z) = A(z)V_{1,0}A(z)^{-1}.$$

The transformation of the principal factors at the higher order poles is slightly more involved (cf. [13, 19]).

The gauge transformation $G$ introduces new poles at the eigenvalues of $C$, or increases by one the order of any poles of the scattering data of $Q_1$ already there. One would like to calculate the principal factors of $m(x,z,Q_2)$ at the new poles in terms of the parameters of the gauge transformation $G$. The complete solution to this problem was worked out by Zurkowski:

**Theorem 6.2.** *Suppose $Q_1$ has poles at $z_1, \ldots, z_n$ with principal factors $V_{1,j}$. (with $V_{1,j}$ the identity matrix if $m(x, z, Q_1)$ is regular at $z_j$) Then $m(x, z, Q_2)$ has poles at $z_1, \ldots, z_n$ with principal factors given by*

$$V_{2,j}A(z) = (A(z) + T_j)V_{1,j}, \qquad z_j \in \Omega_\nu,$$

*where $T_j$ is a constant matrix and $\nu T_j \nu^{-1}$ is strictly lower triangular.*

Let us decribe the solution in the simple case of the $3 \times 3$ modified Lax equations. The sectors $\Omega_\nu$ are depicted in the diagram below. (In diagonal form, $J = \mathrm{diag}\,(1, \omega, \omega^2)$).

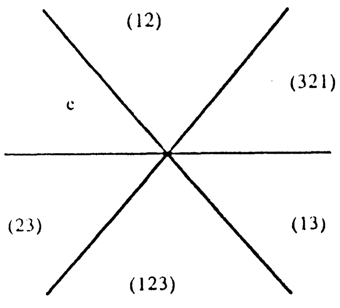


**Fig 6.1** The sectors $\Omega_\nu$ for the matrix $J = \mathrm{diag}\,(1, \omega, \omega^2)$, labelled by the permutation $\nu$.

The gauge transformation, in the $\sim$ gauge, is

$$\tilde{G} = \tilde{\Phi}\tilde{A}\tilde{\Phi}^{-1} \qquad \text{where} \qquad \tilde{\Phi} = \|\tilde{\phi}, \tilde{\sigma}\tilde{\phi}, \tilde{\sigma}^2\tilde{\phi}\|,$$

$$\tilde{A}(z) = z - \tilde{C}, \qquad \tilde{C} = \mathrm{diag}\,(1, \omega^2, \omega) = J^2,$$

$$\tilde{\sigma} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then $\tilde{\sigma}\mathbf{e}_1 = \mathbf{e}_2$, $\tilde{\sigma}\mathbf{e}_2 = \mathbf{e}_3$, $\tilde{\sigma}\mathbf{e}_3 = \mathbf{e}_1$. If $z_0 \in \Omega_e$, then $\mathrm{Re}\, z_0 < \mathrm{Re}\,\omega z_0 < \mathrm{Re}\,\omega^2 z_0$ and $\tilde{\phi} = \exp\{xz_0\}(\mathbf{e}_1 + \mathrm{o}(1))$ as $x \to -\infty$. Consequently,

$$\tilde{\Phi} = \exp\{xz_0\}(1 + \mathrm{o}(1)) \qquad \text{as} \qquad x \to -\infty.$$

We assume that $m(x, z, Q_1)$ is regular at $z_1 \ldots z_n$; then $V_{1,j} = 1$. The principal factor of $m(x, z, Q_2)$ at $z_0$ is given by

$$V_{2,0}A(z) = A(z) + T_0.$$

Since $z_0$ has been chosen to lie in the sector $\Omega_\nu$ for $\nu =$ identity, $T_0$ is strictly lower triangular. Writing $V_{2,0}(z) = 1 + L_{2,0}(z)$, where

$$L_{2,0}(z) = \frac{1}{z - z_0} \begin{bmatrix} 0 & 0 & 0 \\ a' & 0 & 0 \\ b' & c' & 0 \end{bmatrix},$$

we have $L_{2,0}(z)A(z) = T_0$ where $T_0$ is the constant matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \beta & \gamma & 0 \end{bmatrix}.$$

The equation $L_{2,0}(z)A(z) = T_0$ is then

$$\frac{1}{z-z_0}\begin{bmatrix} 0 & 0 & 0 \\ a'(z-z_0) & 0 & 0 \\ b'(z-z_0) & c'(z-\omega^2 z_0) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \beta & \gamma & 0 \end{bmatrix}.$$

These equations hold identically in $z$ if and only if $\gamma = c' = 0$, $\alpha = a'$, and $\beta = b'$. Thus the principal factor $V_{2,0}$ is uniquely determined by the matrix $T_0$. The matrix $T_0$ is in turn free and, along with the $T$ matrices corresponding to the poles at $\omega^2 z_0$ and $\omega z_0$, parameterizes the Bäcklund transformation.

The matrix $T_0$ is related to the wave function $\phi_0$ with wave number $z_0$ as follows:

$$\phi_0 = m(x, z_0, Q_1)\exp\{xz_0 J\}\mathbf{a}_0$$

where $\mathbf{a}_0$ is the solution to the linear system of equations

$$(A(z_0) + T_0)\mathbf{a}_0 = 0.$$

Writing

$$\mathbf{a}_0 = \begin{bmatrix} 1 \\ a \\ b \end{bmatrix}$$

we find

$$\phi_0 = m(x, z_0)\exp\{xz_0 J\}\mathbf{a}_0 = m(x, z_0)\begin{bmatrix} \exp\{xz_0\} \\ a\exp\{x\omega z_0\} \\ b\exp\{x\omega^2 z_0\} \end{bmatrix} \tag{6.2}$$

and

$$\begin{bmatrix} 0 & 0 & 0 \\ \alpha & z_0(1-\omega^2) & 0 \\ \beta & 0 & z_0(1-\omega) \end{bmatrix}\begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = 0,$$

hence

$$\alpha = -z_0(1-\omega^2)a \qquad \beta = -z_0(1-\omega)b.$$

The parameters $a$ and $b$ determine relative components of the three linearly independent wave functions with wave number $z = z_0$ used in constructing the Bäcklund/gauge transformation. A similar situation prevails in the $2 \times 2$ case, or in the case of KdV equation. For example, in the $2 \times 2$ case one constructs the Bäcklund transformation from a Riccati equation for the function

$$u = \frac{\psi_2}{\psi_1} \qquad \text{where} \qquad \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = m(x, z)e^{xzJ}\begin{bmatrix} 1 \\ a \end{bmatrix}.$$

The parameter $a$ can be chosen to vary with time so that the wave function $\Psi$ also satisfies the linear time dependent wave equations. Then the new solution obtained by the Bäcklund transformation will also satisfy the nonlinear evolution equation given by $[D_x, D_t] = 0$ (for details, cf. [14].).

Putting all the above computations together, the principal factor of $m(x, z, Q_2)$ at $z_0$ is

$$V_{2,0}(z) = I + \frac{1}{z - z_0} \begin{bmatrix} 0 & 0 & 0 \\ -a(z_1 - z_2) & 0 & 0 \\ -b(z_1 - z_3) & 0 & 0 \end{bmatrix}$$

where $\phi_0$ is given by (6.2).

As $x \to +\infty$, we know by Theorem 2.1, that $\tilde{\Phi}$ tends projectively to a permutation matrix. For example, when $Q_1 = 0$ we may calculate $\tilde{\Phi}$ explicitly; for then

$$\tilde{\phi} = \begin{bmatrix} \exp\{xz_0\} \\ \alpha \exp\{x\omega z_0\} \\ \beta \exp\{x\omega^2 z_0\} \end{bmatrix}$$

and

$$\tilde{\Phi} = \begin{bmatrix} \exp\{xz_0\} & \beta \exp\{x\omega^2 z_0\} & \alpha \exp\{x\omega z_0\} \\ \alpha \exp\{x\omega z_0\} & \exp\{xz_0\} & \beta \exp\{x\omega^2 z_0\} \\ \beta \exp\{x\omega^2 z_0\} & \alpha \exp\{x\omega z_0\} & \exp\{xz_0\} \end{bmatrix}.$$

As $x \to \infty$,

$$\tilde{\Phi} = \beta \exp\{x\omega^2 z_0\} \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + o(1) \right\}.$$

Thus the permutation matrix $\tau$ of Theorem 2.1 is $\tau = \|\mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_2\|$.

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