Differential and Integral Equations, Volume 1, Number 2, April 1988, pp. 133–141

OSCILLATION OF EMDEN-FOWLER SYSTEMS*

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(Submitted by: F.V. Atkinson)

Abstract. The oscillation theory of a certain form of systems of two first-order nonlinear differential equations is studied. This form includes in particular the classical Emden-Fowler equations. The well-known oscillation criteria of Atkinson, Belohorec, and Waltman are generalized.

1. Introduction. In the papers [6-8], D.D. Mirzov studies the Emden-Fowler system

$$\begin{aligned} u_1' &= a_1(t)|u_2|^{\lambda_1} \operatorname{sign} u_2 \\ u_2' &= -a_2(t)|u_1|^{\lambda_2} \operatorname{sign} u_1 , \end{aligned} \tag{1.1}$$

with $a_1(t) \geq 0$ or $a_2(t) \geq 0$. A solution is said to be continuable if it exists on the whole half-infinite interval $[0, \infty)$. A continuable solution is said to be oscillatory if it has an infinite number of zeros with ∞ as the only accumulation point. The system (1.1) is said to be oscillatory if every pair of continuable solutions, $u_1(t)$ and $u_2(t)$, are oscillatory.

When $a_1(t) > 0$ and $\lambda_1 = 1$, the system reduces to the classical Emden-Fowler equation:

$$\Big(\frac{u_1'}{a_1(t)}\Big)' + a_2(t)\,|u_1|^{\lambda_2} \operatorname{sign} u_1 = 0 \,. \tag{1.2}$$

Mirzov generalizes many of the well-known oscillation criteria for (1.2) to cover (1.1).

Received September 10, 1987.
*This work was supported by the Applied Mathematical Sciences subprogram of the Office of Energy Research, U.S. Department of Energy, under contract W-31-109-Eng-38.
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AMS(MOS) Subject Classifications: 34C10.

In this paper we consider the more general nonlinear system

$$\begin{aligned} x' &= a_1(t) f_1(y) \\ y' &= -a_2(t) f_2(x) \end{aligned} \tag{1.3}$$

and show how similar generalizations can be achieved. We assume that the functions f_i are continuous and that

$$u f_i(u) > 0 \quad \text{for} \quad u \neq 0. \tag{1.4}$$

Further conditions will be imposed in the appropriate sections.

In Sections 2 and 3 we assume that both a_1 and a_2 are non-negative, and we establish generalizations of Atkinson's theorem for superlinear equations and Belohorec's theorem for sublinear equations respectively. In §4 we assume only that one of the a_i's is non-negative and generalizes Waltman's theorem using techniques of integral inequalities. In §5 we give a counter-example to show that Waltman's result is no longer true if both coefficients a_1 and a_2 can assume negative values for arbitrarily large t.

Notice that if $a_1(t) \geq 0$, the oscillation of y follows from that of x. Indeed, if x oscillates, then x' assumes both positive and negative values for large t. By the first equation in (1.3), $f(y)$ must assume both positive and negative values for large t. It follows that y oscillates.

2. Superlinear case: Atkinson's theorem. Atkinson has shown in [1] that for the equation (1.2), if $a_1(t) \equiv 1$, $a_2(t) \geq 0$, and $\lambda_2 > 1$, then a necessary and sufficient condition for oscillation is that

$$\int_0^\infty t a_2(t)\, dt = \infty\,. \tag{2.1}$$

We will generalize this result to the system (1.3).

In this Section we assume that

$$a_1(t) \geq 0 \qquad a_2(t) \geq 0, \tag{2.2}$$

but neither one vanishes identically in any half-line $[a, \infty)$. when one of the a_i's does vanish in a neighborhood of ∞, then either x or y becomes a constant for large t. Obviously that is not a very interesting case.

Besides condition (1.4), we also assume that f_1 and f_2 are C^1 functions and that they satisfy the following "superlinear" conditions:

$$f_1'(u) > 0 \quad \text{for } u \geq 0 \tag{2.3}$$

$$f_2'(u) \geq \quad \text{for } u \geq 0 \tag{2.4}$$

$$\int^{\pm\infty} \frac{du}{f_2(u)} < \infty\,. \tag{2.5}$$

Define

$$A(t) = \int_0^t a_1(s)\, ds\,. \tag{2.6}$$

Theorem 1. *Suppose that conditions (1.4) and (2.2)-(2.5) are satisfied. Suppose further that*

$$\lim_{t\to\infty} A(t) = \infty \tag{2.7}$$

and

$$\int_0^\infty A(t)a_2(t)\,dt = \infty\,. \tag{2.8}$$

Then the system (1.3) is oscillatory.

Proof: As pointed out in §1, the assumption (2.2) allows us to infer the oscillation of x from that of y and vice versa. Suppose that (1.3) is not oscillatory, so that x and y are eventually of one sign. Without loss of generality we may assume that $x(t) > 0$ for $t \geq 0$, and that $y(t)$ is either ≤ 0 or ≥ 0 for $t \geq 0$.

Let us consider the former case first. The second equation of (1.3) implies that y is decreasing. Since $y(t) \leq 0$, it approaches either $-\infty$ or a finite negative value as $t \to \infty$. It follows that, likewise, $f_1(y(t))$ approaches either $-\infty$ or a negative value as $t \to \infty$. Together with (2.7), this implies that

$$\int_0^\infty a_1(t)f_1(y(t))\,dt = -\infty\,.$$

Now integrating the first equation in (1.3) gives

$$x(T) = x(0) + \int_0^T a_1(t)f_1(y(t))\,dt \to -\infty\,,$$

contradicting the assumption that $x(t) \geq 0$.

To handle the second case, we define $w(x) = f_1(y)f_2(x)$. The following equation holds as a consequence of (1.3):

$$w' + a_1(t)f_2'(x)w^2 + a_2(t)f_1'(y) = 0. \tag{2.9}$$

By (2.4) the second term is non-negative. The assumption that $x(t) \geq 0$ and $y(t) \geq 0$ implies that $y(t)$ decreases to a non-negative constant. By (2.3) and compactness, there is a positive constant c such that

$$f_1'(u) \geq c > 0 \quad \text{for} \quad u \in [0, y(0)]. \tag{2.10}$$

We thus have

$$w'(t) \leq -ca_2(t)\,. \tag{2.11}$$

Multiplying the two sides of (2.11) by $A(t)$ and integrating from $t = 0$ to $t = T$, and applying the integration by parts formula to the left-hand side, we obtain the inequality

$$A(T)w(T) - \int_0^T a_1(t)w(t)\,dt \leq -c\int_0^T A(t)a_2(t)\,dt\,. \tag{2.12}$$

Notice that the right-hand side tends to $-\infty$ as $T \to \infty$, while the first term on the left-hand side is positive. thus if we can show that the second term is bounded, we have a contradiction and our theorem is proved. Using the first equation in (1.3), we see that the second term can be evaluated as follows and the required assertion is hence a consequence of (2.5).

$$\int_0^T a_1(t)w(t)\,dt = \int_0^T \frac{x'(t)}{f_2(x)}\,dt = \int_0^{x(T)} \frac{du}{f_2(u)} < \infty\,. \tag{2.13}$$

If $A(t) \geq kt$ for some constant $k > 0$ (for instance, if $a_1(t)$ is bounded from below by a positive constant), then condition (2.8) reduces to the Atkinson condition (2.1).

3. Sublinear case: Belohorec's theorem. Belohorec has shown in [2] that for equation (1.2), if $a_1(t) \equiv 1$, $a_2(t) > 0$ and $0 < \lambda_2 < 1$, then a necessary and sufficient condition for oscillation is that

$$\int_0^\infty t^{\lambda_2}\, a_2(t)\,dt = \infty\,. \tag{3.1}$$

This result is generalized to the system (1.3) in this section.

Again we assume that $a_1(t)$ and $a_2(t)$ are non-negative and non-trivial in any half-line, and that (2.7) holds. The functions f_i, on the other hand, satisfy the following "sublinear" conditions:

$$f_i'(u) \geq 0 \qquad i = 1,2 \tag{3.2}$$

$$f_2(uv) \geq f_2(u)f_2(v) \quad \text{for all} \quad v \geq 1,\ u > 0 \tag{3.3}$$

$$\int_0^{\pm 1} \frac{du}{f_2 \circ f_1(u)} < \infty\,, \tag{3.4}$$

where $f_2 \circ f_1$ denotes the composite function $f_2 \circ f_1(u) = f_2(f_1(u))$.

Examples of functions satisfying (3.3) are $f_2(x) = |x|^{\gamma-1}\,x$, $0 < \gamma < 1$, and $f_2(x) = 1 - e^{-x}$.

Theorem 2. *Suppose that (1.40), (2.2), (2.7), and (3.2)-(3.4) are satisfied. suppose further that*

$$\int_0^\infty f_2(A(t))a_2(t)\,dt = \infty \tag{3.5}$$

where $A(t)$ is defined as in (2.6). Then the system (1.3) is oscillatory.

Proof: Let us first show that if $t_0 > 0$ is any fixed value, then

$$\int_{t_0}^\infty f_2\big(A(t) - A(t_0)\big)a_2(t)\,dt = \infty\,. \tag{3.6}$$

Since $A(t) \to \infty$ as $t \to \infty$, there is a $t_1 > t_0$ such that

$$A(t) \geq \max\{1,\, 2A(t_0)\} \quad \text{for} \quad t > t_1\,. \tag{3.7}$$

Then

$$A(t) - A(t_0) \geq \frac{1}{2}A(t) \quad \text{for} \quad t > t_1\,. \tag{3.8}$$

By (3.2) and (3.3), for $t > t_1$,

$$\begin{aligned} f_2\big(A(t) - A(t_0)\big) &\geq f_2(\tfrac{1}{2}A(t)) \\ &\geq f_2(\tfrac{1}{2})f_2(A(t))\,. \end{aligned} \tag{3.9}$$

Now (3.6) follows from (3.9) and (3.5).

Suppose now that (1.3) is not oscillatory, so that $x(t)$ and $y(t)$ are of one sign for $t > t_0$. Using (3.6), we may do a translation to shift t_0 to 0. This way we may assume without loss of generality that $x(t) \geq 0$ and $y(t)$ is of one sign for all $t \geq 0$. The case $y(t) \leq 0$ can be disposed of in exactly the same way as in the proof of Theorem 1.

We can thus assume that $y(t) \geq 0$ for $t > 0$. Analyzing (1.3) shows that x is an increasing and y is a decreasing function. Integrating the first equation in (1.3) yields

$$\begin{aligned} x(t) &= x(0) + \int_0^t a_1(s) f_1(y(s))\, ds \\ &\geq \int_0^t a_1(s) f_1(y(t))\, ds \\ &= A(t) f_1(y(t)) . \end{aligned} \tag{3.10}$$

It follows that, for all t large enough such that $A(t) \geq 1$,

$$\begin{aligned} f_2(x(t)) &\geq f_2\big(A(t) f_1(y(t))\big) \\ &\geq f_2(A(t)) f_2\big(f_1(y(t))\big). \end{aligned} \tag{3.11}$$

Using the second equation in (1.3), we obtain

$$y'(t) \leq -a_2(t) f_2(A(t)) f_2 \circ f_1(y(t)) \tag{3.12}$$

or

$$-\frac{y'(t)}{f_2 \circ f_1(y(t))} \geq a_2(t) f_2(A(t)). \tag{3.13}$$

Recall that $y(t)$ is decreasing. Integrating the above inequality, we obtain

$$\int_{y(T)}^{y(0)} \frac{du}{f_2 \circ f_1(u)} \geq \int_0^T a_2(t) f_2(A(t))\, dt . \tag{3.14}$$

The right-hand side tends to ∞ as $T \to \infty$ while the left-hand side remains bounded by (3.4) as $y(T) \to 0$. This contradiction completes the proof of the theorem.

4. Generalization of Waltman's theorem. Waltman has shown in [9] that for the equation (1.2), if $a_1(t) > 0$ and $\lambda_2 > 1$, then a sufficient condition for oscillation is that

$$\int_0^\infty a_2(t)\, dt = \infty . \tag{4.1}$$

Notice that the function $a_2(t)$ need not be non-negative for this result to hold. Later Wong [10] improves the result by replacing the nonlinear term $|u_1|^{\lambda_2} \operatorname{sign} u_1$ with any non-decreasing function $f(u_1)$ (as well as with more general forms involving the first derivative of the unknown function u_1). In [5] we obtain the same result using the techniques of integral inequality. In this section we would like to show that the same technique can be employed to extend the result to (1.3).

We assume that

$$a_1(t) \geq 0, \tag{4.2}$$

but that $a_2(t)$ can assume both positive and negative values.

For the functions f_i we assume besides (1.4)

$$\liminf_{u\to\pm\infty} |f_1(u)| \neq 0 \tag{4.3}$$

$$f_2'(u) \geq 0 \quad \text{for} \quad u \geq 0\,. \tag{4.4}$$

Theorem 3. *Suppose that (1.4) and (4.2)-(4.4) are satisfied. Suppose further that*

$$\lim_{T\to\infty} \int_0^T a_i(t)\,dt = \infty \qquad i = 1,2\,. \tag{4.5}$$

Then the system (1.3) is oscillatory.

Proof: Without loss of generality we may assume that $x(t) > 0$ for $t \geq 0$. Dividing the second equation in (1.3) by $f_2(x)$ and integrating over $[0,t]$, we obtain

$$-\int_0^t \frac{y'(s)}{f_2(x(s))}\,ds = \int_0^t a_2(s)\,ds = B(t)\,. \tag{4.6}$$

Using integration by parts on the first term, we derive

$$-\frac{y(t)}{f_2(x(t))} = B(t) + \Big(-\frac{y(0)}{f_2(x(0))}\Big) + \int_0^t \frac{y(s)f_2'(x)x'(s)}{f_2^2(x)}\,ds\,. \tag{4.7}$$

The last term is non-negative as $f_2' \geq 0$ and $y(s)x'(s) = a_1(s)y(s)f_1(y) \geq 0$ by (1.4) and (4.2). By (4.5), there exists a value t_0 such that the sum of the first two terms on the left-hand side of (4.7) is at least 1, for all $t \geq t_0$. We thus have the integral inequality

$$\frac{-y(t)}{f_2(x(t))} \geq 1 + \int_{t_0}^t \Big[-\frac{f_2'(x)x'(s)}{f_2^2(x)}\Big](-y(s))\,ds \quad \text{for all} \quad t \geq t_0\,. \tag{4.8}$$

In particular, we infer that $y(t) < 0$ for $t \geq t_0$. The theory of integral inequalities allows us to compare $-y(t)$ with the solution $U(t)$ of the following integral equation:

$$\frac{U(t)}{f_2(x(t))} = 1 + \int_{t_0}^t \Big[-\frac{f_2'(x)x'(s)}{f_2^2(x)}\Big](U(s))\,ds \quad \text{for all} \quad t \geq t_0 \tag{4.9}$$

to conclude that

$$-y(t) \geq U(t) \qquad \text{for all} \quad t \geq t_0\,. \tag{4.10}$$

By differentiating (4.9), we see easily that $U'(t) = 0$, from which follows that $U(t) = 1$. Thus (4.10) reduces to

$$y(t) \leq -1 \qquad \text{for all} \quad t \geq t_0\,. \tag{4.11}$$

From the hypothesis (4.3), we have

$$f_1(y(t)) \leq \sup_{y \leq -1} f_1(y) = k < 0. \tag{4.12}$$

Thus, by (4.5),

$$\int_{t_0}^{t} a_1(s) f_1(y(s))\, ds \leq k \int_{t_0}^{t} a_1(s)\, ds \to -\infty\,. \tag{4.13}$$

Integrating the first equation in (1.3) gives

$$x(t) - x(t_0) = \int_{t_0}^{t} a_1(s) f_1(y(s))\, ds. \tag{4.14}$$

The fact that the righ-thand side tends to $-\infty$ contradicts our assumption that x remains positive for all t. This completes the proof of the theorem.

5. A counterexample and remarks. In all the theorems proved in this paper, some positivity condition has to be imposed on one of the coefficients $a_i(t)$. We would like to show with an example that some such condition is inevitable.

Let us just look at the linear case, namely, when $f_i(u) = u$ for $i = 1, 2$. The new variable $r(t) = -y(t)$ satisfies the Riccati differential equation

$$r'(t) = a_2(t) + a_1(t) r^2(t), \tag{5.1}$$

or, upon integration, the Riccati integral equation

$$r(t) = Q(t) + \int_0^t a_1(s) r^2(s)\, ds, \tag{5.2}$$

where

$$Q(t) = r(0) + \int_0^t a_2(s)\, ds. \tag{5.3}$$

The question whether (1.3) has a solution x of one sign (let us say positive) is equivalent to the question whether (5.2) has a continuous solution on $[0, \infty)$.

In the following we are going to construct step functions $Q(t)$ and $a_2(t)$ such that (5.2) does have a solution on $[0, \infty)$, if we allow a_i to change sign, even though condition (4.5) is satisfied. Although our smoothness condition on a_i precludes such step functions, and the solution has jumps at each of the points $3k$, $3k + 1$, modification of our example by smoothing out the abrupt jumps can easily lead to an acceptable counterexample.

We use the following a_1 :

$$a_1(t) = \begin{cases} -1 & t \in [3k,\, 3k+1) \\ 1 & t \in [3k+1,\, 3k+3) \end{cases}$$

Obviously,

$$\lim_{T\to\infty}\int_0^T a_1(t)\,dt=\infty\,.$$

Let us now construct our $Q(t)$. In each $[3k,\,3k+1)$, we choose $Q(t)$ to be a constant $\alpha=\alpha(k)$ so large that, for instance,

$$\alpha+\int_0^{3k} a_1(s)r^2(s)\,ds=k. \tag{5.4}$$

Later, we are going to show that the number $\int_0^{3k} a_1(s)r^2(s)\,ds$ is negative for all k. Hence,

$$\alpha(k)\ge k\to\infty \qquad \text{as } \; k\to\infty\,. \tag{5.5}$$

In the interval $[3k,\,3k+1)$, equation (5.2) reduces to

$$r(t)=k-\int_{3k}^{t} r^2(s)\,ds,$$

which has the simple solution

$$r(t)=[t-3k+1/k]^{-1}.$$

Note that the right-hand side of the above formula is well defined, as the denominator does not vanish in $[3k,\,3k+1)$. From (5.2), we have

$$\int_0^{3k+1} a_1(s)r^2(s)\,ds=r(3k+1)-\alpha=\frac{k}{k+1}-\alpha<0. \tag{5.6}$$

In $[3k+1,\,3k+3)$, we take $Q(t)$ to be the negative of the number in (5.6). Taking into account equation (5.5), we see that

$$\lim_{t\to\infty} Q(t)=\infty.$$

In $[3k+1,\,3k+3)$, (5.2) reduces to the simple equation

$$r(t)=\int_{3k+1}^{t} r^2(s)\,ds,$$

which has the trivial solution

$$r(t)=0 \quad \text{ for } \; t\in[3k+1,\,3k+3).$$

Thus,

$$\int_0^{3(k+1)} a_1(s)r^2(s)\,ds=\int_0^{3k+1} a_1(s)r^2(s)\,ds<0. \tag{5.7}$$

We have started with the assumption that the above inequality is true when the upper limit of integration is $3k$ and conclude that the same is true for $3(k+1)$. This induction step allows us to infer that (5.7) holds for all $k>0$.

Continuing our process over each interval, we obtain a solution $r(t)$ defined on the whole half-line $[0, \infty)$, and so we have an example of a non-oscillatory system (1.3) for which (4.5) is satisfied.

It is natural to ask whether some relaxation on the positivity requirement is possible. As shown by the above example, it is not sufficient to assume only that at each t one of the a_i's is non-negative.

As seen above, the Riccati equation is an important tool in the study of the oscillation of linear equations. Another case in which this is true is the study of (1.1) when $\lambda_1\lambda_2 = 1$. This is the so-called half-linear case. With the introduction of the variable $r(t) = -u_2/|u_1|^{\lambda_2} \operatorname{sign} u_1$, we get the Riccati equation

$$r'(t) = a_2(t) + \lambda_2 a_1(t) r^{\gamma}(t), \tag{5.8}$$

where $\gamma = (\lambda_2 + 1)/\lambda_2$. All Sturmian comparison-type theorems continue to hold, and oscillation criteria can easily be obtained via the traditional methods.

The sufficiency part of Atkinson and Belohorec's theorems continues to hold even when the coefficient is allowed to change sign, although the proofs are much more complicated. Furthermore, extensions have been obtained by various authors. See, for instance, the references quoted in the recent papers [3-5]. It will be nice if the same facts can be extended to the system (1.3).

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