A NONLINEAR EQUATION WITH PIECEWISE CONTINUOUS ARGUMENT

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Abstract. Asymptotic and qualitative behavior of solutions is established for the equations (1) $x'(t) = \mu x(t) (1 - x([t]))$, (2) $x'(t) = \mu x(t) (1 - x(2[(t + 1)/2]))$, where $\mu$ is a positive parameter. Comparison is made with the continuous logistic equation (3) $x'(t) = \mu x(t)(1 - x(t))$ and the discrete logistic equation (4) $x_n = \mu x_{n-1}(1 - x_{n-1})$. One result is that (1) and (4) can exhibit complicated dynamics and (2) and (3) cannot.

I. Introduction. This paper is devoted to a study of two scalar non-linear differential equations of the logistic form, in which one of the arguments is $t$ and the other argument is a piece-wise continuous function of $t$. Specifically, the equations are as follows:

$$x'(t) = \mu x(t) (1 - x([t])), \quad x(0) = c_0, \quad t \geq 0,$$

(1.1)

$$x'(t) = \mu x(t) (1 - x(2[(t + 1)/2])), \quad x(0) = c_0, \quad t \geq 0,$$

(1.2)

Here, $x'$ is the derivative of $x$, $[t]$ denotes the greatest integer function, $[t] = n$ when $n \leq t < n + 1$ where $n$ is an integer, and $\mu$ and $c_0$ are real parameters.

Equations with arguments less than $t$, such as $[t]$ in (1.1), may be regarded as special types of functional differential equations with retarded argument. These have been studied in some linear and nonlinear cases by Cooke and Wiener [2,3]. Equations of the neutral type with this kind of argument have also been discussed by these authors in [5], and equations of the advanced type were investigated by Shah and Wiener [8]. Cooke and Wiener also introduced an example of an equation with argument $2[(t + 1)/2]$, which is alternately advanced and retarded [4].

It is possible to think of (1.1) and (1.2) as semi-discretizations of

$$x'(t) = \mu x(t) (1 - x(t)), \quad x(0) = c_0, \quad t \geq 0,$$

(1.3)

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the standard logistic differential equation, which is a well-known model for the dynamics of growth of a one-species population. This interpretation of (1.1) and (1.2) is motivated by the fact that the continuous variable \( t \) is replaced by the discrete variable \([t]\) or \(2[(t+1)/2]\) at one place in (1.3), while it is not disturbed at the other place. As we show below, this makes it possible to obtain recurrence relations for the values of the solution at integer points and hence to determine the expression of the solution of (1.1) and (1.2), as well as to obtain a complete characterization of the qualitative behavior of all solutions.

Suppose that we think of \( t \) as representing time. Then in (1.1), the rate of change of \( x(t) \) always depends on \( x(t) \) and \( x \) evaluated at the previous (or equal) time \([t]\). This "bias" toward previous times makes it plausible to think that the behavior of the solution of (1.1) may be similar to that of the solution of the (full) discrete form of the logistic equation

\[
x_n = \mu x_{n-1}(1 - x_{n-1}), \quad x_0 = c_0, \quad n = 1, 2, \ldots
\]  

Equation (1.4) has been extensively studied [1, 6, 7] as a paradigm of complicated dynamics (chaos). In section 3, we show that (1.1) also presents the same pattern of complicated dynamics as (1.4).

Now, in (1.2), the argument is half the time less than \( t \) and half the time greater than \( t \) and thus it is not "biased toward the past". It turns out that the behavior of its solutions is much closer to that of the differential equation (1.3), as is shown in section 4. It may prove useful in other situations to introduce such "unbiased" discretization. Indeed, it is known to numerical analysts that "implicit" formulas often are more stable than "explicit" formulas in solving differential equations.

In section 2, we present some preliminary results and definitions that will be used in the sequel. In section 3, we discuss Eq. (1.1) and show that it presents, for some values of \( \mu \), periodic solutions of integer periods which satisfy the full Sarkovskii order. Finally, in section 4, we deal with Eq. (1.2) and show that it has an asymptotically stable equilibrium which attracts all its positive solutions for all positive values of \( \mu \).

We wish to encourage the reader to compare by himself the behavior pattern of the solutions of (1.1) and (1.2) with those of (1.4) and (1.3), respectively, as they are presented in section 2.

2. Preliminary results and definitions. Here, we give the precise notion of a solution of either (1.1) or (1.2), and we state other definitions and well known results that will be needed in the next sections.

**Definition 1.** A solution of Eq. (1.1) (Eq. 1.2)) is a function \( x(t) \) which

(i) is defined and continuous for all \( t > 0 \),

(ii) its derivative \( x'(t) \) exists everywhere in this interval except, possibly, at the integers where, nevertheless, one-sided derivatives exist, and

(iii) wherever \( x'(t) \) exists, it satisfies (1.1) ((1.2)).

As occurs with functional differential equations, the problem of continuing \( x(t) \) for \( t < 0 \) as a solution is also pertinent in the present situation. Therefore, we state:
Definition 2. A solution $x(t)$ of either Eq. (1.1) or Eq. (1.2) is said to be backward continuable if one can extend to the left the interval of definition of $x(t)$ in such a way that $x(t)$ comes to satisfy the requirements of Def. 1, also within this new enlarged domain. A solution for which it is impossible to obtain a backward continuation is said to be noncontinuable.

We now recall that a periodic function $x(t)$ is one for which there exists a $T > 0$ such that $x(t) = x(t + T)$ for all $t$. We call $T$ the period of $x$ and refer to such a function as “T-periodic”. Whenever convenient one may think of $T$ as being the least possible period of $x$. Note that discrete functions defined on the integers that are periodic must have integer periods. An equilibrium of a differential equation (or of a discrete equation, such as (1.4)) is a constant solution of it and hence it is periodic of any period $T > 0$. $x(t) \equiv 0$ and $x(t) \equiv 1$ are the only (at the same time) equilibria of Eqs. (1.1), (1.2) and (1.3), while $x_n \equiv 0$ and $x_n \equiv (\mu - 1)/\mu$, $\mu \neq 1$, are the only equilibria of Eq. (1.4). A solution, $y(t)$, of either one of the above equations ($y_n$, in the case of (1.4)) is said to be asymptotically stable if for any other solution $x(t)$ ($x_n$, in case of (1.4)) such that $x(0)$ is sufficiently close to $y(0)$ ($x_0$ is sufficiently close to $y_0$), one has that $|x(t) - y(t)| \to 0$ as $t \to \infty$ ($|x_n - y_n| \to 0$ as $n \to \infty$), and it is usual, in this case, to say that the solution $x$ is attracted by $y$. A good description of the dynamics of a differential equation such as the above ones (or of a discrete equation such as (1.4)) is obtained when one completely describes the asymptotic behavior of its equilibria and periodic solutions. The following well known facts about equations (1.3) and (1.4) are listed here in order to facilitate to the reader the task of comparing the different kinds of behavior of the above given logistic equations.

Theorem 2.1. Eq. (1.3) has a unique solution for each given initial value $c_0$, and the equilibrium $x(t) \equiv 1$ attracts every positive solution for all $\mu > 0$.

Theorem 2.2. Eq. (1.4) can exhibit complicated dynamics.

We shall explain a little bit of this statement. Consider a discrete equation

$$x_n = f(\mu, x_{n-1}), \ x_0 = c_0,$$

(2.1)

where $f : I \times J \to J$ is continuous, $I$ and $J$ are intervals in the real line. When we say that (2.1) can exhibit complicated dynamics, we mean that there is a value of the parameter $\mu$ at which (2.1) has a (perhaps asymptotically stable) periodic solution with least period equal to 3. This means [1, 6, 7] that there is a sequence $\{\mu_n\}_{n=1}^{\infty}$ satisfying Sarkovskii's order

$$\mu_1 < \mu_2 < \ldots < \mu_{2^n} < \ldots < \mu_{2^2} < \mu_3 < \mu_{2^1} - 1 < \ldots < \mu_5 < \mu_3,$$

(2.2)

at which it has an (asymptotically stable) solution with least period equal to $n$. As a consequence of this, it follows that (2.1) has at infinitely many values of $\mu$, either sensitivity with respect to initial conditions or (asymptotically stable) homoclinic solutions (strange attractors). The result referring to the asymptotically stable solutions is particularly valid in the case of the $C^1$-unimodal maps. We say that $f(\mu, \cdot)$ is $C^1$-unimodal when there exists a point $r(\mu)$ in the interior of $J$ such that $f(\mu, x)$ is strictly increasing for $x < r(\mu)$ and strictly decreasing for $x > r(\mu)$ and $f(\mu, \cdot)$ is continuously differentiable in $J$. The important property of the $C^1$-unimodal maps is that if one proves that $r(\mu)$ becomes 3-periodic for a certain value of $\mu$ then (2.1) will have complicated dynamics for all nearby values of $\mu$. Thus the “can exhibit” locution of the statement of the theorem is in the sense that if the
parameter is allowed to change in an appropriate interval, then for infinitely many values of the parameter \( \mu \) in this interval, Eq. (1.4) will depict a kind of wild behavior. It is known that Eq. (1.4) has an asymptotically stable equilibrium at \( \mu_1 = 2 \), \( x_n = 1/2 \), and an asymptotically stable 2-periodic solution such that \( x_0 = 1/2 \) at \( \mu_2 = 3.23606 \ldots \) and an asymptotically stable 3-periodic solution at \( x_0 = 1/2 \) at \( \mu_3 = 3.83187 \ldots \).

3. The retarded equation. In this section we investigate Eq. (1.1) which, for definiteness, we repeat below:

\[
x'(t) = \mu x(t) (1 - x([t])), \quad x(0) = c_0.
\]  

Motivated by the applications of the logistic equation, we shall restrict our analysis to the case when the parameter \( \mu \) is positive and the parameter \( c_0 \) is nonnegative. The value \( c_0 = 0 \), of course, leads to the equilibrium \( x_n = 0 \).

Theorem 3.1. Eq. (1.1) has for each given \( \mu > 0 \) and each given \( c_0 \geq 0 \) a unique solution \( x(t) \).

Proof: In fact, it is immediately seen that the function \( x_n(t) = c_n \exp (\mu(1 - c_n)(t - n)) \), where \( n \) is an integer and \( c_n \) is any given real number, satisfies the condition

\[
x'_n(t) = \mu x_n(t) (1 - x_n([t])), \quad x_n(n) = c_n, \quad n \leq t \leq n + 1.
\]  

Hence, given \( c_0 \) and \( \mu \), if we can properly choose \( c_n, n = 1, 2, \ldots \) such that the function \( x : [0, \infty) \to \mathbb{R} \) given by \( x(t) = x_n(t) \) if \( t \in [n, n + 1) \), \( n = 0, 1, 2, \ldots \) becomes continuous, then \( x(t) \) will be a solution of (1.1). But \( x(t) \) as defined is continuous if, and only if

\[
c_n = c_{n-1} \exp (\mu(1 - c_{n-1})), \quad n = 1, 2, \ldots
\]

Thus, the problem of finding a solution of (1.1) is equivalent to the problem of finding a solution of the discrete equation (3.1) through a given initial value \( c_0 \). Notice, then, that this solution \( c_n \) exists and is unique due simply to the fact that the field \( f_\mu \) of (3.1),

\[
f_\mu(x) = x \exp (\mu(1 - x)), \quad x \in \mathbb{R},
\]

is a (well-defined) map. This establishes the existence and uniqueness of the solution of Eq. (1.1), as we wished.

In Fig. 1 we draw the graph of \( f_\mu (\mu > 1, \text{in this case}) \), which is based on the following elementary properties of \( f_\mu \). It is a continuously differentiable map from \([0, \infty)\) into itself which has a maximum at \( x = 1/\mu \), is strictly increasing in \([0, 1/\mu]\) and strictly decreasing in \([1/\mu, \infty)\), \( f_\mu(0) = 0 \) and \( f_\mu(1) = 1 \) and thus the only equilibria of (3.1) are \( c_n \equiv 0 \) (the trivial equilibrium, as we shall refer to it) and \( c_n \equiv 1 \) (the nontrivial equilibrium).

Now, since \( |f'_\mu(0)| = \exp \mu > 1 \) for \( \mu > 0 \) and \( |f'_\mu(1)| = |1-\mu| < 1 \), if and only if \( 0 < \mu < 2 \), we have:

Theorem 3.2. The trivial equilibrium of (1.1) is unstable for any \( \mu > 0 \) and the nontrivial equilibrium of the equation is asymptotically stable if and only if \( 0 < \mu < 2 \).

Note that \( c_{n-1} < 1 \) implies \( c_n > c_{n-1} \), and \( c_{n-1} > 1 \) implies that \( c_n < c_{n-1} \) which shows that the solutions of (3.1) (and hence of Eq. (1.1)) tend to oscillate around the nontrivial equilibrium. This leads us to inquire whether Eq. (3.1) has a nontrivial \( n \)-periodic solution. Indeed, we have
Theorem 3.3. Eq. (3.1) can exhibit complicated dynamics.

Proof: We note that \( f_\mu \) as defined in (3.2), \( \mu > 0 \) is a \( C^1 \)-unimodal map of the interval \([0, \infty)\), with \( r(\mu) = 1/\mu \). It is then but a simple matter of computation to check that at \( \mu^* = 3.11670 \ldots \) the solution through \( c_0 = r(\mu^*) \) is an asymptotically stable 3-periodic solution of (3.1), so that the result of the theorem follows.

Now, the simple observation that \( n \)-periodic (resp. homoclinic solutions and sensitivity with respect to initial conditions) solutions of Eq. (3.1) correspond one-to-one to \( n \)-periodic (resp. homoclinic solutions and sensitivity to initial conditions) solutions of Eq. (1.1), implies that this last equation will also have complicated dynamics for values of \( \mu \) that are smaller but sufficiently close to \( \mu^* \) and for all values of \( \mu \) larger than or equal to \( \mu^* \). And, the observation that (1.1) cannot have periodic solutions with a noninteger period due to the monotonicity in \([n, n+1)\) of the function \( x_n(t) \), defined in the proof of Theorem 3.1, allows us to colloquially (but precisely) say that the dynamics of Eq. (1.1) may get as complicated as the dynamics of Eq. (1.4), although this complication takes place at distinct values of the parameter \( \mu \). For instance, \( c_0 = r(\mu) \) becomes an (asymptotically stable) equilibrium of (1.1) at \( \mu_1 = 1 \), 2-periodic solution at \( \mu_2 = 2.25631 \ldots \) and a 3-periodic solution at \( \mu_3 = 3.11670 \ldots \).

The problem of the backward continuation of the solutions of (1.1) is as follows: (i) if \( c_0 > (1/\mu)\exp(\mu - 1) \), there is no backward continuation of the solution and (ii) if \( c_0 < (1/\mu)\exp(\mu - 1) \), there are infinitely many backward continuations.

4. The equation with advanced-retarded argument. In this section, we analyze the problem

\[ x'(t) = \mu x(t) \left( 1 - x(2[(t+1)/2]) \right), \quad x(0) = c_0, \quad (1.2) \]

in which \( c_0 \geq 0 \) and \( \mu > 0 \) are given parameters; this restriction on the range of the parameters being motivated, as in the previous case, by the interpretation of (1.2) as a population growth model.
The argument $2[(t+1)/2]$ is equal to $t-\tau(t)$, where $\tau(t) = t-2[(t+1)/2]$ is the displacement or deviation of the argument from $t$. Since $\tau(t)$ is of negative sign in $(2n-1,2n)$ and of positive sign in $(2n,2n+1)$, Eq. (1.2) is a functional differential equation of alternately retarded and advanced type. In this section, we first establish the existence and uniqueness of the solution of (1.2) and then show that its unique nontrivial equilibrium attracts all its positive solutions for all $\mu > 0$. Then, we describe the backward continuation properties of its solutions.

**Theorem 4.1.** Eq. (1.2) has for each given $\mu > 0$ and $c_0 \geq 0$ a unique solution.

**Proof:** In the interval $[-1,1)$, we have $[(t + 1)/2] = 0$ and therefore, the unique solution (in the usual sense of a solution of an ordinary differential equation) of (1.2) in this interval is $x_0(t) = c_0 \exp(\mu(1-c_0)t)$. Similarly, given any real number $c_{2n}$, the map

$$x_{2n}(t) = c_{2n} \exp(\mu(1-c_{2n})(t-2n)), \quad t \in [2n-1,2n+1],$$

is the unique map which satisfies the condition

$$x'(t) = \mu x(t) \left(1 - x(2[(t+1)/2])\right), \quad x(2n) = c_{2n},$$

in the interval $[2n-1,2n+1)$. Thus, the map $x: [0, \infty) \to \mathbb{R}$ defined by $x(t) = x_{2n}(t)$ if $t \in [2n-1,2n+1)$, $n = 0,1,2,\ldots$ will meet the requirements of Def. 1 provided that the numbers $c_{2n}$ are such that $x(t)$ becomes continuous at $t = 2n + 1$. This will be true if and only if $x_{2n}(2n+1) = x_{2n+2}(2n+1)$, or

$$c_{2n} \exp(\mu(1-c_{2n})) = c_{2n+2} \exp(-\mu(1-c_{2n+2})).$$

If this condition is satisfied, we can write

$$c_{2n+1} = x(2n + 1) = c_{2n} \exp(\mu(1-c_{2n}))$$

and the sequence $\{c_{2n}\}_{n=0}^{\infty}$ will unambiguously characterize the solution $x(t)$ of (1.2).

Our problem now reduces to the question of whether (4.3) determines, given $c_0$, a unique sequence $\{c_{2n}\}$. It will be useful to express this in terms of a new parameter $\xi_{2n} = c_{2n} + c_{2n+2}$. Then,

$$c_{2n+2} = c_{2n} \exp(\mu(2-\xi_{2n}))$$

and

$$\xi_{2n} = c_{2n} \left(1 + \exp(\mu(2-\xi_{2n}))\right).$$

Hence,

$$c_{2n} = \xi_{2n} / \left(1 + \exp(\mu(2-\xi_{2n}))\right).$$

These two equations yield two expressions for $c_{2n}$ which must agree in order for the sequence to be well defined. We must, therefore, have

$$f_\mu(\xi_{2n}) = g_\mu(\xi_{2n-2}) \; (= c_{2n}),$$

where

$$f_\mu(x) = x / (1 + \exp(\mu(2-x))), \quad g_\mu(x) = f_\mu(x) \exp(\mu(2-x)).$$
Thus, the sequence \( \{\xi_{2n}\}_{n=0}^{\infty} \) is well defined for a given \( \xi_0 \) if and only if the equation \( f_\mu(y) = g_\mu(x) \) can be solved for \( y \) for each given value of \( x \). Indeed, if such is the case, we can then give \( \xi_0 \) and recursively find \( \xi_{2n} \) from (4.7).

The graphs of \( f_\mu \) and \( g_\mu \) are sketched in Fig. 2. Note that \( f_\mu(0) = g_\mu(0) = 0 \), \( f_\mu(2) = g_\mu(2) = 1 \), \( f_\mu(x) \) is strictly increasing and asymptotic to \( x \) as \( x \to \infty \) and \( f_\mu(x) < x \) for \( x > 0 \); \( g_\mu \) has a unique maximum at a point \( \alpha > 0 \), \( f_\mu(x) < g_\mu(x) \) for \( 0 < x < 2 \) and \( g_\mu(x) \to 0 \) as \( x \to \infty \), \( g_\mu(x) \) is strictly increasing for \( 0 < x < \alpha \) and strictly decreasing for \( x > \alpha \). The situation of Fig. 2 is one for which \( \mu > 1 \).

Hence, it immediately follows that, given any \( x > 0 \), one can find a unique \( y \), such that \( f_\mu(y) = g_\mu(x) \), and thus the unique solution of (1.2) exists. This finishes the proof of the theorem.

\[ \text{Fig. 2 Graph of the functions } f_\mu(x) \text{ and } g_\mu(x) \text{ defined in Eq. (4.8).} \]

From Fig. 2 or from the defining equations, we see that if \( \xi_0 = 2 \), then \( \xi_{2n} \equiv 2 \) for \( n = 1, 2, \ldots \). Then \( c_{2n} \equiv 1 \) and from (4.4), \( c_{2n+1} \equiv 1 \). The corresponding solution is thus the nontrivial equilibrium \( x(t) \equiv 1 \). Below, we analyze the stability of this equilibrium. We note that \( x(t) \equiv 0 \) is the trivial equilibrium, and this corresponds to \( c_n \equiv 0 \). As in the case of Eq. (1.1), periodic solutions of (1.2) correspond one-to-one to periodic sequences \( c_n \) of the same period. In particular, Eq. (1.2) cannot have a noninteger-periodic solution. We shall show that \( \{\xi_{2n}\} \) defined by (4.7) cannot have a nontrivial 2-periodic solution. It follows from (4.6) that \( \{c_{2n}\} \) cannot be a nontrivial 2-periodic sequence and thus that Eq. (1.2) cannot have a nontrivial solution of period 4. Sarkovskii’s Theorem [1, 7] applied to (4.7) then implies that it (and hence, Eq. (1.2)) cannot have any \( p \)-periodic solution with \( p > 1 \) \((p > 2 \text{ in the case of Eq. (1.2)})\). Note that we cannot apply Sarkovskii’s Theorem to any of the equations (1.2), (4.3)-(4.6). We can just apply it to (4.7), which can be written in the form \( \xi_{2n} = f_\mu^{-1} \circ g_\mu(\xi_{2n-2}) \), of the type of Eq. (2.1).

**Theorem 4.2.** Eq. (4.7) does not have, for any \( \mu > 0 \), a nontrivial 2-periodic solution.

**Proof:** Suppose \( \xi_0 \neq 0 \). Let \( \xi_0 = x \) and \( \xi_2 = y \). If \( \{\xi_{2n}\}_{n=-\infty}^{\infty} \) were 2-periodic we would
have \( f_\mu(y) = g_\mu(x) \) and \( f_\mu(x) = g_\mu(y) \); i.e.,

\[
\begin{align*}
\frac{y}{1 + \exp(\mu(2 - y))} &= \frac{x \exp(\mu(2 - x))}{1 + \exp(\mu(2 - x))}, \\
\frac{y \exp(\mu(2 - y))}{1 + \exp(\mu(2 - y))} &= \frac{x}{1 + \exp(\mu(2 - x))}.
\end{align*}
\]

Multiplying the first equation above by \( \exp(\mu(2 - y)) \) and comparing the equation thus obtained with the second equation, one gets \( \exp(4 - x - y) = 1 \); that is, \( x + y = 4 \). Thus, we let \( x = 2 - \gamma \) and \( y = 2 + \gamma \) and substitute these values in the first equation in (4.9) in order to obtain:

\[
\frac{2 + \gamma}{1 + \exp(-\gamma \mu)} = \frac{(2 - \gamma) \exp(\gamma \mu)}{1 + \exp(\gamma \mu)}.
\]

Finally, multiplying the left-hand side of this equation by \( \exp(\gamma \mu)/\exp(\gamma \mu) \), one gets that \( 2 - \gamma = 2 + \gamma \), which implies that \( \gamma = 0 \). Hence, the only 2-periodic solutions of (4.7) are the trivial ones: \( \xi_{2n} \equiv 0 \) or \( \xi_{2n} \equiv 1 \), and this finishes the proof of the theorem.

Next, we have

**Theorem 4.3.** The equilibrium \( x(t) \equiv 1 \) of Eq. (1.2) is asymptotically stable for all \( \mu > 0 \).

**Proof:** In view of the expression of the solution of Eq. (1.2) given in the proof of Theorem 4.1 and the fact that the equilibrium \( x(t) \equiv 1 \) corresponds to \( \xi_{2n} \equiv 2 \), it suffices to prove that \( \xi_{2n} \rightarrow 2 \) as \( n \rightarrow \infty \) for any solution of (4.7), such that \( \xi_0 \) is sufficiently close to 2. And this will follow (from the general theory of discrete dynamical systems on the real line [1, 7]) whenever we have \( |g_\mu'(2)/f_\mu'(2)| < 1 \). But then, \( g_\mu'(2) = (\mu - 1)/2 \) and \( f_\mu'(2) = (\mu + 1)/2 \) readily imply the desired result.

We intend now to prove that \( x(t) \equiv 1 \) attracts not only a small neighborhood of it but that in fact, it attracts every positive solution of Eq. (1.2). Again, it suffices to prove that \( \xi_{2n} \equiv 2 \) attracts all positive solutions of (4.7).

**Lemma.** The sequence \( \xi_{2n} \equiv 2 \) attracts all other sequences that satisfy (4.7) and have \( \xi_0 > 0 \).

**Proof:** Indeed, let \( A \) be the maximal connected set containing \( \xi_0 = 2 \) and consisting of those \( \xi_0 \), such that \( \xi_{2n} \rightarrow 2 \) as \( n \rightarrow \infty \). Due to the above theorem’s proof, \( A \) is nonempty and therefore, due to the continuity of \( f_\mu \) and \( g_\mu \), it is an interval \((a, b)\), such that \( 0 < a < b < \infty \). Thus, either one of the following alternatives must hold: (i) \( f_\mu(a) = g_\mu(a), f_\mu(b) = g_\mu(b) \); (ii) \( f_\mu(a) = g_\mu(b), f_\mu(b) = g_\mu(a) \); and (iii) \( g_\mu(a) = g_\mu(b) = (f_\mu(a) (+) \) or \( f_\mu(b)) \). But, alternative (i) cannot hold, for the only two points that satisfy these relations are \( a = 0 \) and \( b = 2 \) and 2 is an interior point of \( A \). Alternative (ii) also cannot occur, since in this case \( a \) and \( b \) form a 2-periodic solution of (4.7) in contradiction to Theorem 4.2. So, it must be (iii) that holds. If \( a = 0 \) and \( b = \infty \), the theorem is proved. Otherwise, suppose \( 0 < a < b < \infty \). Then, since \( f_\mu(x) < g_\mu(x) \) for \( 0 < x < 2 \) and \( g_\mu(x) \rightarrow 0 \) asymptotically as \( x \rightarrow \infty \), we have that for any \( \xi_0 \in (0, a) \cup (b, \infty) \), \( \xi_{2n} \in (a, b) \) for some \( n \) sufficiently large. But then, the

\[ g_\mu^*(b) = \lim_{x \rightarrow b} g_\mu(x). \]
continuity of $f_\mu$ and $g_\mu$ this time implies that $a$ and $b$ are in $A$, a contradiction, and the result follows.

In order to give a brief account of the problem of the backward continuation of the solution of Eq. (1.2), we consider the transformation of coordinates given by

$$x = \xi \quad \text{and} \quad y = \xi + \eta - f_\mu(\xi),$$

where $f_\mu$ is given by (4.8). Under this change of coordinates, the equation $\eta = f_\mu(\xi)$ goes into the equation $y = x$, the diagonal in the $x,y$-plane, and $\eta = g_\mu(\xi)$ goes into $y = 2g_\mu(x)$, where $g_\mu$ is also given in (4.8). Therefore, Eq. (4.7) becomes $x_{2n} = 2g_\mu(x_{2n-2})$ and the situation here is similar to that of Eq. (1.1), section 3: if $x_0 > 2g_\mu(\tilde{x})$, where $\tilde{x}$ is the point of maximum of $g_\mu$, the corresponding solution of Eq. (1.2) is noncontinuable to the left of $t = -1$. If, on the other hand, $x_0 < 2g_\mu(\tilde{x})$ then the corresponding solution of Eq. (1.2) has infinitely many backward continuations.

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