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POSITIVITY AND A PRINCIPLE OF LINEARIZED STABILITY FOR DELAY-DIFFERENTIAL EQUATIONS

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Abstract. A principle of linearized stability is developed for the abstract delay differential equation $\dot{u}(t) = Bu(t) + \phi u_t$, $u_0 = f$, where B generates a strongly continuous linear semigroup and ϕ is Lipschitz continuous, and Fréchet- differentiable at an equilibrium point. Recent stability results for positive linear semigroups can then be used to obtain stability information for this equation.

I. Introduction. Let X be a Banach space with norm $\|\cdot\|$. For a fixed positive constant r_0, let $E = C([-r_0, 0], X)$ be the space of continuous functions mapping the interval $[-r_0, 0]$ to X. For $f \in E$, let $\|f\|_E = \sup_{s\in[-r_0,0]} \|f(s)\|$. We consider the abstract delay-differential equation

$$\dot{u}(t) = Bu(t) + \phi u_t, \qquad t \geq 0 \tag{FDE}$$
$$u_0 = f.$$

Here we assume that B generates a strongly continuous semigroup of bounded linear operators on X, and ϕ is a nonlinear Lipschitz continuous operator from E to X. (These hypotheses are made more precise later.) The functional u_t is defined by translation as $u_t(s) = u(t + s)$ for $s \in [-r_0, 0]$.

Equations which can be written in the abstract form (FDE) often serve as models for various phenomena where a certain state at time t depends on information about the state at an earlier time. Such equations appear in many diverse areas of physical and biological sciences.

Our goal is to use recent results from the theory of positive semigroups in order to study the stability properties of (FDE). Specifically, we will use a result of W. Desch and W. Schappacher [2] to develop a "principle of linearized stability" for (FDE) (under the additional assumption that ϕ is Fréchet- differentiable at an equilibrium point), and then use known stability results for positive linear semigroups in order to obtain stability information for (FDE).

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Motivation for the present research comes from a recent paper by A. Grabosch [5]. In [5] positive semigroup theory is used to study the stability of the solution semigroup corresponding to the abstract Cauchy problem

$$\begin{aligned} u(t) &= \phi u_t, \qquad t \geq 0 \\ u_0 &= f, \end{aligned} \tag{FE}$$

where ϕ is a nonlinear Lipschitz continuous operator from $E = L^1((-\infty, 0], X; e^{\eta s} ds)$ to X. The Desch-Schappacher result mentioned above is used to develop a "principle of linearized stability" for (FE).

An excellent reference for many of the recent results in positive semigroup theory is [9]. Applications of this theory to linear delay-differential equations can be found in [9, B-IV], [7] and [8]. Since our method of studying the stability of (FDE) involves the linearization of this equation we will use several known results for positive linear semigroups from these references.

We first recall some basic definitions and results from nonlinear semigroup theory (see, for example, the forthcoming book of J. Goldstein [3]). Basic results from linear semigroup theory can be found, for example, in the books by J. Goldstein [4] and A. Pazy [10].

Definition 1.1. *A strongly continuous semigroup on X is a family $(T(t))_{t \geq 0}$ of operators on X to X such that for t, $\tau \geq 0$ and $f \in X$,*

(i) $T(0)f = f$,

(ii) $T(t + \tau)f = T(t)T(\tau)f$,

(iii) $s \to T(s)f$ *is continuous on* $[0, \infty)$.

Theorem 1.2. (cf. Crandall-Liggett [1]). *Let $A : D(A) \subset X \to X$ be densely defined and satisfy*

(1) *for each $\alpha > 0$, the range of $(I - \alpha A)$ is all of X and there is a real number $w > 0$ such that $\|(I - \alpha A)^{-1} x_1 - (I - \alpha A)^{-1} x_2\| \leq \frac{1}{1 - \alpha w} \|x_1 - x_2\|$ for all x_1, $x_2 \in X$ and $0 < \alpha w < 1$.*

Then for each $f \in X$ and $t \geq 0$

$$T(t)f = \lim_{n \to \infty} (I - \frac{t}{n} A)^{-n} f$$

exists and defines a strongly continuous semigroup such that $\|T(t)x_1 - T(t)x_2\| \leq e^{wt} \|x_1 - x_2\|$ for all x_1, $x_2 \in X$ and $t \geq 0$.

If A satisfies the hypotheses of Theorem 1.3, then A is said to *generate* the semigroup $(T(t))_{t \geq 0}$. We note that if, in Theorem 1.2, A is linear, then this theorem becomes the Hille-Yosida generation theorem for strongly continuous linear semigroups.

We now give several definitions related to the study of the stability of an equilibrium of a semigroup.

Definition 1.3. *Let $(T(t))_{t \geq 0}$ be a semigroup on a Banach space X. Then $x_0 \in X$ is called an equilibrium of $(T(t))_{t \geq 0}$ if $T(t)x_0 = x_0$ for all $t \geq 0$.*

Definition 1.4. *An equilibrium $x_0 \in X$ of a semigroup $(T(t))_{t\geq 0}$ is called stable (in the sense of Liapunov) if for any neighborhood U of x_0 there exists a neighborhood V of x_0 such that $T(t)V \subset U$ for all $t \geq 0$. If, in addition, for any $x \in V$ there exists $\delta > 0$ and $M > 0$ such that $\|T(t)x - x_0\| \leq Me^{-\delta t}$, then the equilibrium x_0 is called an exponentially stable equilibrium of $(T(t))_{t\geq 0}$. If the M and δ above can be chosen independently of $x \in V$, then the equilibrium x_0 is called uniformly exponentially stable.*

It is well-known that the stability of an equilibrium of a linear semigroup $(T(t))_{t\geq 0}$ on a Banach space X can be studied by means of spectral properties of its generator. For example, if X is finite-dimensional the classical stability theorem of Liapunov tells us that the zero equilibrium of the solution semigroup $(e^{At})_{t\geq 0}$ corresponding to the linear equation $\dot{x} = Ax$ is uniformly exponentially stable if and only if $\operatorname{Re}\lambda < 0$ for every eigenvalue $\lambda \in \sigma(A)$ (the spectrum of A). On infinite-dimensional spaces the situation is more complicated. However, the spectral properties of the generator of a linear semigroup can still be useful in analyzing the stability of equilibria of the semigroup.

Definition 1.5. *Let A be the generator of a strongly continuous linear semigroup $(T(t))_{t\geq 0}$. The spectral bound of A, $s(A)$, is defined by*

$$s(A) = \sup\{Re\,\lambda : \lambda \in \sigma(A)\}.$$

The growth bound of $(T(t))_{t\geq 0}$ is defined by

$$\omega(A) = \omega(T(t)) = \inf\{w \in R : \text{ There exists } M \text{ such that } \|T(t)\| \leq Me^{wt} \text{ for all } t \geq 0\}.$$

(Here, and elsewhere in the paper, we use $\|\cdot\|$ to also denote the norm of a bounded linear operator on a Banach space.) Clearly, if $\omega(A) < 0$ the zero equilibrium of the semigroup is uniformly exponentially stable. If X is finite dimensional, $s(A) = \omega(A)$ (hence the classical Liapunov theorem). It is well-known that in general

$$s(A) \leq \omega(A) < +\infty,$$

but strict inequality may occur ([9, A-III, Sect. 1). However, as we will see below, one of the useful consequences of positivity of a linear semigroup is the ability to often conclude the uniform exponential stability of the zero equilibrium of the semigroup whenever $s(A) < 0$.

Since in many applications which are modelled by (FDE) only positive states have a reasonable physical interpretation, the theory of positive semigroups is a natural tool for the study of these equations.

Definition 1.6. *Let X be an ordered Banach space (for example, $C(K)$, K compact, $C_0(Y)$, Y locally compact, $L^p(\mu)$, $1 \leq p \leq \infty$, or, more abstractly, a Banach lattice (see [12], [9])). Let X_+ denote the positive cone of X, i.e., $X_+ = \{f \in X : f \geq 0\}$. Then an operator $\hat{B}$ (linear or nonlinear) with domain and range contained in X is a **positive operator** on X if $\hat{B}f \in X_+$ whenever $f \in X_+$. A semigroup $(T(t))_{t\geq 0}$ of (linear or nonlinear) operators on X is a positive semigroup if $T(t)f \in X_+$ whenever $f \in X_+$ and $t \in R_+$.*

We state below some of the important consequences of positivity for the stability analysis of linear semigroups. Other consequences relevant to the stability study of (FDE) will be considered in Section III.

In order to use known results we define the exponential (and uniform exponential) stability of a linear semigroup. This is a slightly different notion of exponential stability then that introduced in Definition 1.4.

Definition 1.7. *A strongly continuous linear semigroup $(T(t))_{t\geq 0}$ on a Banach space X, with generator A, is called*

(i) *exponentially stable if there exists $\gamma > 0$ such that $\lim_{t\to\infty} e^{\gamma t}\|T(t)f\| = 0$ for every $f \in D(A)$.*

(ii) *uniformly exponentially stable if there exists $\gamma > 0$ such that $\lim_{t\to\infty} e^{\gamma t}\|T(t)\| = 0$.*

This definition of exponential stability means the exponential convergence to zero of the solution $t \to u(t) = T(t)f$ of the Cauchy problem $u'(t) = Au(t)$, $u(0) = f$ for $f \in D(A)$. It is somewhat weaker than the definition of (Liapunov) exponential stability of the zero equilibrium of $(T(t))_{t\geq 0}$ given in Definition 1.4. On the other hand, the definition of uniform exponential stability of the semigroup given above is stronger than the definition of (Liapunov) uniform exponential stability of the zero equilibrium of $(T(t))_{t\geq 0}$ given in Definition 1.4. Thus if a semigroup is uniformly exponentially stable (in the sense of Definition 1.7) then the zero equilibrium of the semigroup is (Liapunov) uniformly exponentially stable (in the sense of Definition 1.4), and hence exponentially stable in this sense.

Theorem 1.8. ([9, C-IV, Theorem 1.3 and Theorem 1.1]). *Let A be the generator of a positive linear semigroup $(T(t))_{t\geq 0}$ on some Banach lattice X. Then the following properties are equivalent:*

a) *The semigroup $(T(t))_{t\geq 0}$ is exponentially stable.*

b) *The spectral bound $s(A)$ is less than zero.*

If, in addition, X is a space $C(K)$, K compact, $C_0(Y)$, Y locally compact, or $L^1(Y,\mu)$ or $L^2(Y,\mu)$ for some measure space (Y,μ), then the above properties are equivalent to

c) *The semigroup $(T(t))_{t\geq 0}$ is uniformly exponentially stable.*

The following proposition tells us that in order to determine $s(A)$ for a positive linear semigroup on a Banach lattice, it suffices to look at the real spectral values of A only.

Proposition 1.9. ([9, C-III, Corollary 1.4]). *Let $(T(t))_{t\geq 0}$ be a positive linear semigroup defined on a Banach lattice X, and A be its generator. Then $s(A) \in \sigma(A)$ unless $s(A) = -\infty$.*

In Section II we consider the nonlinear semigroup associated with (FDE) and some of its properties. In Section III we mention additional results which positivity gives for the study of the stability of linear delay- differential equations. Our main results on the development of a "principle of linearized stability" for (FDE) are given in Section IV, and an example to which our theory can be applied is given in Section V.

II. The nonlinear semigroup and its properties. We consider the problem (FDE), where B and ϕ satisfy the following hypotheses:

(H1) B is a closed and densely defined linear operator on X, and generates a strongly continuous linear semigroup $(S(t))_{t\geq 0}$ on X (in the sense of Theorem 1.2). Furthermore $\|S(t)\| \leq 1$ for $t \geq 0$ (that is, $(S(t))_{t\geq 0}$ is a contraction semigroup on X).

(H2) ϕ is a nonlinear operator from E to X and there is a positive constant L such that $\|\phi f - \phi g\| \leq L\|f - g\|_E$ for all $f, g \in E$.

Remark. The additional condition that $(S(t))_{t\geq 0}$ is a contraction semigroup can always be satisfied by renorming the Banach space X (see [4, Theorem 2.13]).

We define the operator $A = A_{B,\phi} : E \to E$ by

$$Af = f', \tag{2}$$
$$D(A) = \{f \in C^1([-r_0, 0], X) : f(0) \in D(B),\ f'(0) = Bf(0) + \phi f\}.$$

It is well-known that A generates a strongly continuous semigroup $T(t) = (T_{B,\phi}(t))_{t\geq 0}$ on E. In the following theorem we collect some of the known results about $T(t)$ which will be useful for the present analysis. The proof of statements (i) and (ii) follows from results in [13], and statement (iii) follows from [11, Theorem 2.1].

Theorem 2.1. *Assume that B and ϕ satisfy (H1) and (H2), and the operator A is defined by (2). Then the following assertions hold:*

(i) *The operator A generates a strongly continuous nonlinear semigroup $T(t) = (T_{B,\phi}(t))_{t\geq 0}$ on E.*

(ii) *The semigroup $T(t)$ satisfies*

$$T(t)f(0) = S(t)f(0) + \int_0^t S(t-s)\phi(T(s)f)\,ds \tag{3}$$

for $t \geq 0$ and every $f \in E$.

(iii) *$T(t)$ is a "translation semigroup", that is, $T(t)$ satisfies*

$$T(t)f(s) = \begin{cases} f(t+s) & \text{if } t+s \leq 0 \\ T(t+s)f(0) & \text{if } t+s \geq 0. \end{cases} \tag{4}$$

For $f \in E$, define $u : [-r_0, \infty) \to X$ by

$$u(t) = \begin{cases} f(t) & \text{if } -r_0 \leq t \leq 0 \\ T(t)f(0) & \text{if } t \geq 0. \end{cases} \tag{5}$$

Then $u(t)$ satisfies (3) for $t \geq 0$, and is called a "mild" or "weak" solution of (FDE).

In general $u(t)$, $t > 0$ may not belong to $D(B)$ and may not be differentiable (even if $f(0) \in D(B)$). The following proposition is from [13, Proposition 2.3]:

Proposition 2.2. *Assume in addition to (H1) and (H2) that ϕ is continuously differentiable from E to X and $f \in D(A)$. Assume also that there exists a positive constant L_1 such that $\|\phi'(f_1) - \phi'(f_2)\| \leq L_1\|f_1 - f_2\|_E$ for all $f_1, f_2 \in E$. (Here $\phi'(f) \in \mathcal{L}(E, X)$ denotes the Fréchet derivative of ϕ at f.) Then $u(t)$, as defined by (5), is a "strong" solution of (FDE), that is, u is right-sided differentiable at zero and continuously differentiable for $t > 0$, $u(t) \in D(B)$ for $t \geq 0$ and satisfies (FDE).*

We note that if ϕ is a bounded linear operator from E into X ($\phi \in \mathcal{L}(E, X)$) then the operator A defined by (2) generates a strongly continuous linear semigroup $T(t)$ satisfying the statements (ii) and (iii) of Theorem 2.1, and the function $u(t)$ defined by (5) yields a strong solution of (FDE) if $f \in D(A)$ (cf. [7, Theorem 1.1.]).

It is reasonable to expect the nonlinear semigroup $T(t)$ of Theorem 2.1 to inherit properties from both the semigroup $S(t)$ generated by the operator B, and from the operator ϕ in (FDE). One example of the effect of ϕ on $T(t)$ has been demonstrated in Proposition 2.2.

If instead of (H2) we require $\phi \in \mathcal{L}(E, X)$, then, as noted, A (defined by (2)) generates a linear semigroup $T(t)$ and (cf. [7, Proposition 2.1 and 2.2]) if $S(t)$ is norm continuous for $t > 0$ (that is, the function $t \to T(t)$ from $(0, \infty)$ into $\mathcal{L}(E)$ is norm continuous), then $T(t)$ is norm continuous for $t > r_0$ (that is, $T(t)$ is eventually norm continuous). If $S(t)$ is analytic (resp., compact for $t > 0$), then $T(t)$ is eventually differentiable (resp., eventually compact).

From [13, Proposition 2.4] we have the following.

Proposition 2.3. *Assume that (H1) and (H2) hold and in addition assume that $S(t)$ is compact for each $t > 0$. Then $(t, f) \to T(t)f$ is compact in f for each fixed $t > r_0$.*

As mentioned in the Introduction, the order properties of $T(t)$ and its generator A are important for the stability analysis of (FDE). The corresponding result of the following proposition for the case of $\phi \in \mathcal{L}(E, X)$ can be found in [9, B-IV, Proposition 3.5].

Proposition 2.4. *Assume that X is a Banach lattice, and in addition to (H1) and (H2), assume that ϕ is a positive operator and B generates a positive semigroup on X. Then the semigroup $T(t)$ (of Theorem 2.1) is positive.*

Proof: If X is a Banach lattice, then $E = C([-r_0, 0], X)$ is also a Banach lattice with the natural pointwise order and the supremum norm. The method of proof of the proposition is similar to that of the proof of [5, Proposition 3.5].

By Theorem 1.2 we know that

$$T(t)f = \lim_{n\to\infty} (I - \frac{t}{n}A)^{-n} f = \lim_{n\to\infty} (J_{t/n})^n f, \quad \text{for } f \in E,$$

where $J_\alpha f = (I - \alpha A)^{-1}$, $\alpha > 0$. Thus, in order to show the positivity of $T(t)$ it suffices to show the positivity of the operators J_α for α small. By (1) we know that for a given $g \in E$ there exists $f \in D(A)$ such that

$$f - \alpha A f = g \qquad \text{for all } \alpha > 0. \tag{6}$$

Since $Af = f'$ for $f \in D(A)$ this implies that $f' = \frac{1}{\alpha} f - \frac{1}{\alpha} g$, which leads to

$$f'(0) = \frac{1}{\alpha} f(0) - \frac{1}{\alpha} g(0) = Bf(0) + \phi f. \tag{7}$$

From (6) and (7) we have for $\theta \in [-r_0, 0]$

$$\begin{aligned} f(\theta) &= e^{\theta/\alpha} f(0) + \frac{1}{\alpha} \int_\theta^0 e^{(\theta - s)/\alpha} g(s)\, ds \\ f(0) &= (I - \alpha B)^{-1} (g(0) + \alpha \phi f). \end{aligned} \tag{8}$$

Let $F_\alpha : X \to X$ be defined by

$$F_\alpha x = (I - \alpha B)^{-1} \left(g(0) + \alpha \phi \left(e^{\theta/\alpha} x + \frac{1}{\alpha} \int_\theta^0 e^{(\theta - s)/\alpha} g(s)\, ds \right) \right). \tag{9}$$

An easy calculation (using (1)) shows that for x, $\hat{x}$ in X and α sufficiently small

$$\|F_\alpha x - F_\alpha \hat{x}\| \le \frac{L}{\frac{1}{\alpha} - w} \|x - \hat{x}\|.$$

Thus for α small enough that $\frac{1}{\alpha} > L + w$ we see that F_α is a strict contraction on X. Let $\bar{x}$ denote the unique fixed point of F_α. Then $\bar{x}$ yields the unique solution of (8) and hence (6) (for small α;) we call this solution $\bar{f}$. Since we are assuming that B generates a positive semigroup on X, this implies that $(I - \alpha B)^{-1}$ is a positive operator (for α sufficiently small) (cf. [9, B-II, Proposition 1.1]). Since ϕ is assumed to be positive we see by (9) that if $g \ge 0$ then F_α is a positive operator. Thus $\bar{x}$ is the unique fixed point of a positive operator, hence positive. This implies that $\bar{f}$ is also positive and, since $\bar{f} = J_\alpha g$ we can conclude that J_α, for α sufficiently small, is positive.

III. Positivity and stability for the linear case. In this section we will assume that (H1) holds and instead of (H2) we assume

$$\phi \in \mathcal{L}(E, X). \tag{$H2$}'$$

Then, as previously noted, the operator A defined by (2) generates a strongly continuous linear semigroup $T(t)$.

Definition 3.1. *For $\lambda \in \mathcal{C}$, $x \in X$, $g \in E$, we define the following operators:*

(i) $\epsilon_\lambda \otimes x \in E$ by $(\epsilon \otimes x)(s) = e^{\lambda s} \cdot x$, $s \in [-r_0, 0]$.

(ii) $H_\lambda \in \mathcal{L}(E)$ by $H_\lambda g(t) = \int_t^0 e^{\lambda(t-s)} g(s)\, ds$, $\quad t \in [-r_0, 0]$.

(iii) $\phi_\lambda \in \mathcal{L}(X)$ by $\phi_\lambda(x) = \phi(\epsilon_\lambda \otimes x)$.

The operators B and ϕ_λ generate semigroups on X since $\phi_\lambda \in \mathcal{L}(X)$, and the spectrum of $B + \phi_\lambda$ can be used to characterize the spectrum of A.

Theorem 3.2. ([9, B-IV, Proposition 3.4]). *Under the assumptions (H1) and (H2)$'$ the following statements hold:*

(i) *$\lambda \in \sigma(A)$ if and only if $\lambda \in \sigma(B + \phi_\lambda)$.*

(ii) *If $\lambda \in \rho(A)$ then the resolvent is given by*

$$R(\lambda, A)g = \epsilon_\lambda \otimes [R(\lambda,\, B + \phi_\lambda)(g(0) + \phi H_\lambda g)] + H_\lambda g \quad \text{for } g \in E.$$

A useful stability criterion for (FDE) (for the case of ϕ a bounded linear operator) is given in the following theorem:

Theorem 3.3. [9, B-IV, Theorem 3.7 and Corollary 3.8]. *Let X be a Banach lattice. Assume, in addition to (H1) and (H2)$'$, that B generates a positive semigroup and ϕ is a positive operator. For the generator A of the positive linear semigroup $T(t)$ on E and for $\lambda \in R$ the following statements hold:*

(a) *If $s(B + \phi_\lambda) < \lambda$, then $s(A) < \lambda$.*

(b) *If $s(B + \phi_\lambda) = \lambda$, then $s(A) = \lambda$.*

(c) *Suppose that B has compact resolvent and there exists $\mu \in R$ with $\sigma(B + \phi_\mu) \neq \emptyset$. Then*

$$s(B + \phi_\lambda) \leq \ (\text{or } \geq)\lambda \quad \text{if and only if} \quad s(A) \leq \ (\text{or } \geq)\lambda.$$

(In particular, $s(B + \phi_0) < 0$ if and only if $s(A) < 0$.)

(d) *The following are equivalent:*

(i) the semigroup generated by A is exponentially stable in E.

(ii) The semigroup generated by $B + \phi_0$ is exponentially stable in X.

We now state a result which allows one to often conclude the exponential stability of a positive linear semigroup $T(t)$ associated with (FDE) from the estimate $s(B + \phi_0) < 0$, independent of the form of the delay in (FDE). This is untypical of the stability behavior of general linear semigroups associated with delay-differential equations.

Definition 3.4. *A continuous map* $r : [-1,0] \to R_-$ *satisfying* $\min_{-1\le s\le 0} r(s) = -r_0$ *is called a delay function on* $[-r_0, 0]$. *If* ϕ *is a bounded linear operator from* $C([-1,0],X)$ *into* X, *the delayed operator* $\phi_r \in \mathcal{L}(E,X)$ *is defined by* $\phi_r f = \phi(f \circ r)$, $f \in E$.

Theorem 3.6. ([7, Theorem 4.3]). *Let* X *be a Banach lattice. Assume that* B *satisfies (H1) and generates a positive semigroup on* X, *and* ϕ *is a positive bounded linear operator from* $C([-1,0],X)$ *to* X. *If* $s(B+\phi_0) < 0$ *then* $s(A_{B,\phi_r}) < 0$ *for every delay function* r.

Proof: From Definitions 3.1 and 3.4 we have for $x \in X$, $(\phi_r)_0 x = \phi_r(1_{[-r_0,0]} \otimes x) = \phi(1_{[-1,0]} \otimes x) = \phi_0(x)$, that is, $(\phi_r)_0 = \phi_0$ is independent of the delay function r. Therefore the conclusion is obtained from Theorem 3.3(a).

If B does not generate a positive semigroup, or if ϕ is not positive, then we cannot conclude that the semigroup $T(t)$ is positive. However, if the semigroup generated by B is "dominated" (in the sense of Definition 3.7 below) by another semigroup and if ϕ is "dominated" by another operator, then the results of this section can still be used to study the stability of (FDE) in the linear case.

Definition 3.7. *Let* X *be a Banach lattice. If* ϕ_1, $\phi_2 \in \mathcal{L}(E,X)$ *then we say that* ϕ_1 *dominates* ϕ_2 *if* $|\phi_2 f| \le \phi_1 |f|$ *for all* $f \in E$. *If* $(T_1(t))_{t\ge 0}$, $(T_2(t))_{t\ge 0}$ *are semigroups on* E *we say* $(T_1(t))_{t\ge 0}$ *dominates* $(T_2(t))_{t\ge 0}$ *if* $|T_2(t)f| \le T_1(t)|f|$ *for every* $f \in E$, $t \ge 0$. *(Here* $|x| = \sup(x,-x)$ *for* $x \in X$.*)*

Clearly, the dominating operator or semigroup is always positive. We state one stability result for (FDE) involving dominating operators. For others we refer the reader to [7].

Theorem 3.8. ([7, Corollary 4.4]). *Let* X *be a Banach lattice. Let* B *be the generator of a linear semigroup* $S(t)$ *on* X *(that is,* B *satisfies (H1)) and let* $\phi \in \mathcal{L}(C([-1,0],X),X)$. *Assume that there exists a semigroup* $\tilde{S}(t)$ *on* X *with generator* $\tilde{B}$ *which dominates* $S(t)$, *and an operator* $\tilde{\phi}$ *dominating* ϕ. *Then the linear semigroup* $T_{B,\phi_r}(t)$ *is uniformly exponentially stable for all delay functions* r *if the spectral bound* $s(\tilde{B}+\tilde{\phi}_0) < 0$ *and one of the following conditions in satisfied:*

(i) $X = C(K)$, *K compact*

(ii) $\tilde{S}(t)$ *is norm continuous for* $t > 0$.

IV. Main results. We state the result of W. Desch and W. Schappacher which will be used to obtain a "principle of linearized stability" for (FDE).

Theorem 4.1. ([2, Theorem 2.1]). *Let* $T(t)(= (T(t))_{t\ge 0})$ *be a nonlinear semigroup on a Banach space* E *and let* $\bar{f}$ *be an equilibrium of* $T(t)$. *Suppose that* $T(t)$ *is Fréchet- differentiable at* $\bar{f}$ *with* $U(t) = T'(t,\bar{f})$ *(the Fréchet derivative of* $T(t)$ *at* $\bar{f}$*). Then* $U(t)$ *is a linear semigroup. If the zero equilibrium of the semigroup* $U(t)$ *is exponentially stable then* $\bar{f}$ *is exponentially stable with respect to* $T(t)$.

We will first characterize the linear semigroup $U(t)$ of Theorem 4.1 for the case where zero is an equilibrium of the nonlinear semigroup $T(t)$ associated with (FDE), and then reduce the case of a non-zero equilibrium to this case (cf. [5]).

Proposition 4.2. *Assume that B and ϕ satisfy (H1) and (H2). Assume, in addition, that $\phi(0) = 0$ and ϕ is Fréchet- differentiable at zero. Then zero is an equilibrium of $T(t) = (T_{B,\phi}(t))_{t\geq 0}$ (the nonlinear semigroup of Theorem 2.1), and the linearized semigroup (i.e., Fréchet derivative) of $T(t)$ is the semigroup $U(t) = (T_{B,\phi'(0)}(t))_{t\geq 0}$, where $\phi'(0) \in \mathcal{L}(E, X)$ denotes the Fréchet derivative of ϕ at zero. Here $U(t)$ is the semigroup with generator $A_{B,\phi'(0)}f = f'$ and domain $D(A_{B,\phi'(0)} = \{f \in C^1(-r_0, 0], X) : f(0) \in D(B),\ f'(0) = Bf(0) + \phi'(0)f\}$.*

Proof.: We first note that mild solutions of (FDE), as given by Theorem 2.1, are unique. This follows from the exponential estimate for $T(t)x$ given by [13, Proposition 3.1]. If $u(t)$ is a strong solution of (FDE), then from semigroup theory we know that $u(t)$ is given by (5), and $T(t)f \in D(A)$ whenever $f \in D(A)$. Also,

$$\begin{aligned} T(t+s)f(0) - f(0) &= \left[\int_0^{t+s} A(T(\tau)f)\,d\tau\right](0) = \int_0^{t+s}\left[\frac{d}{d\tau}(T(\tau)f)\right](0)\,d\tau \\ &= \int_0^{t+s}[B(T(\tau)f)(0) + \phi(T(\tau)f)]\,d\tau. \end{aligned} \tag{10}$$

If $u(t) = h$, $t \geq 0$, $h \in X$, where h satisfies $Bh + \phi(1 \otimes h) = 0$ (using the notation of Definition 3.1), then $T(t)f = 1 \otimes h$, $t \geq 0$, satisfies (10). Thus, by (4) and the uniqueness of solutions we can conclude that $T(t)(1 \otimes h) = 1 \otimes h$, $t \geq 0$, that is, $1 \otimes h$ is an equilibrium of the semigroup $T(t)$. In particular, since $B0 + \phi(1 \otimes 0) = 0$ if $\phi(0) = 0$, this implies that the constant function zero in E is an equilibrium of $T(t)$.

In order to prove the proposition we will show that for any $\epsilon > 0$ there exists $\delta > 0$ such that for all $f \in E$ with $\|f\|_E < \delta$, and all $t \geq 0$,

$$\|T_{B,\phi}(t)f - T_{B,\phi'(0)}(t)f\|_E < \epsilon\|f\|_E.$$

From (2) and (3) we have

$$\begin{aligned} T(t)f(s) &= \begin{cases} f(t+s) & \text{if } t+s \leq 0 \\ T(t+s)f(0) & \text{if } t+s \geq 0 \end{cases} \\ &= \begin{cases} f(t+s) & \text{if } t+s \leq 0 \\ S(t+s)f(0) + \int_0^{t+s} S(t+s-\tau)\phi(T(\tau)f)\,d\tau & \text{if } t+s \geq 0. \end{cases} \end{aligned} \tag{11}$$

By applying (11) to the semigroups $(T_{B,\phi}(t))_{t\geq 0}$ and $(T_{B,\phi'(0)}(t))_{t\geq 0}$ respectively we obtain, for $0 \leq t \leq r_0$,

$$\begin{aligned} &\|T_{B,\phi}(t)f - T_{B,\phi'(0)}(t)f\|_E \\ = &\sup_{s\in[-r_0,0]} \|T_{B,\phi}(t)f(s) - T_{B,\phi'(0)}(t)f(s)\| \\ = &\sup_{s\in[-t,0]} \left\| \int_0^{t+s} S(t+s-\tau)\left[\phi(T_{B,\phi}(\tau)f) - \phi'(0)(T_{B,\phi'(0)}(\tau)f)\right]\,d\tau\right\|. \end{aligned}$$

If $t > r_0$,

$$\begin{aligned} &\|T_{B,\phi}(t)f - T_{B,\phi'(0)}(t)f\|_E = \\ = &\sup_{s\in[-r_0,0]} \left\| \int_0^{t+s} S(t+s-\tau)\left[\phi(T_{B,\phi}(\tau)f) - \phi'(0)(T_{B,\phi'(0)}(\tau)f)\right]\,d\tau\right\|. \end{aligned}$$

Thus for all $t \geq 0$,

$$
\begin{aligned}
&\|T_{B,\phi}(t)f - T_{B,\phi'(0)}(t)f\|_E \qquad (12)\\
&\leq \sup_{s\in[-r_0,0]} \| \int_0^{t+s} S(t+s-\tau)\left[\phi(T_{B,\phi}(\tau)f) - \phi'(0)(T_{B,\phi'(0)}(\tau)f)\right] d\tau\| \\
&\leq \sup_{s\in[-r_0,0]} \sup_{\tau\in[0,t+s]} \|S(t+s-\tau)\| \int_0^{t+s} \|\phi(T_{B,\phi}(\tau)f) - \phi'(0)(T_{B,\phi}(\tau)f) \\
&\quad + \phi'(0)(T_{B,\phi}(\tau)f) - \phi'(0)(T_{B,\phi'(0)}(\tau)f)\| \, d\tau \\
&\leq \sup_{\tau\in[0,t]} \|S(\tau)\| \left[\int_0^t \|\phi(T_{B,\phi}(\tau)f) - \phi'(0)(T_{B,\phi}(\tau)f)\| \, d\tau \right.\\
&\quad \left. + \|\phi'(0)\| \int_0^t \|T_{B,\phi}(\tau)f - T_{B,\phi'(0)}(\tau)f\|_E \, d\tau\right].
\end{aligned}
$$

Let $\epsilon > 0$. Since ϕ is Fréchet- differentiable at zero with Fréchet derivative $\phi'(0)$ there exists a $\delta_1 > 0$ such that $\|T_{B,\phi}(\tau)f\|_E < \delta_1$ implies that

$$\|\phi(T_{B,\phi}(\tau)f) - \phi'(0)(T_{B,\phi}(\tau)f)\| \leq \epsilon\|T_{B,\phi}(\tau)f\|_E.$$

By [13, Proposition 3.1] $\|T_{B,\phi}(\tau)f - T_{B,\phi}(\tau)0\|_E \leq e^{(w+L)\tau}\|f\|_E$. Therefore there exists $\delta_0 > 0$ such that $\|f\|_E < \delta_0$ implies that $\|T_{B,\phi}(\tau)f\|_E = \|T_{B,\phi}(\tau)f - T_{B,\phi}(\tau)0\|_E < \delta_1$ for all $0 \leq \tau \leq t$. Thus $\|f\|_E < \delta_0$ implies that $\|T_{B,\phi}(\tau)f\| < \delta_1$, which further implies that $\|\phi(T_{B,\phi}(\tau)f) - \phi'(0)(T_{B,\phi}(\tau)f)\| \leq \epsilon\|T_{B,\phi}(\tau)f\|_E$ for all $0 \leq \tau \leq t$. Let $\|\phi'(0)\| = M$ and $\|f\|_E < \delta_0$. Since $\sup_{\tau\in[0,t]} \|S(\tau)\| \leq 1$ by (H2), we obtain from (12)

$$
\begin{aligned}
&\|T_{B,\phi}(t)f - T_{B,\phi'(0)}(t)f\|_E \\
&\leq M\int_0^t \|T_{B,\phi}(\tau)f - T_{B,\phi'(0)}(\tau)f\|_E \, d\tau + \int_0^t \epsilon\|T_{B,\phi}(\tau)f\|_E \, d\tau \\
&\leq a(t) + M\int_0^t \|T_{B,\phi}(\tau)f - T_{B,\phi'(0)}(\tau)f\|_E \, d\tau,
\end{aligned}
$$

where $a(t) = \epsilon t e^{(w+L)t}\|f\|_E$. By applying Gronwall's inequality (cf. [6, Lemma 3.1]) we obtain $\|T_{B,\phi}(t)f - T_{B,\phi'(0)}(t)f\|_E \leq a(t)e^{Mt} = \epsilon C\|f\|_E$, where $C = C(t) = te^{(w+L+M)t}$. Thus, if $\|f\|_E < \delta_0$, $\|T_{B,\phi}(t)f - T_{B,\phi'(0)}(t)f\|_E \leq \epsilon C\|f\|_E$.

From the proof of Proposition 4.2 we see that if $\bar{x} \in X$ satisfies $B\bar{x} + \phi(1 \otimes \bar{x}) = 0$, then $\bar{f} = 1 \otimes \bar{x}$ is an equilibrium of $T(t)$ in E, i.e., $T(x)\bar{f} = \bar{f}$, $t \geq 0$. Define $R(t)$ by $R(t)h = T(t)(h + \bar{f}) - \bar{f}$, $h \in E$. Clearly, $(R(t))_{t\geq 0}$ is a strongly continuous semigroup on E and $R(t)0 = 0$. Let $\bar{\phi}$ be defined by $\bar{\phi}h = \phi(h + \bar{f}) + B\bar{x}$. Then $\bar{\phi}(0) = 0$. For $s \in [-r_0, 0]$,

$$
\begin{aligned}
R(t)h(s) &= T(t)(h+\bar{f})(s) - \bar{f}(s) \\
\text{(by (4))} \quad &= \begin{cases} (h+\bar{f})(t+s) - \bar{f}(s) & \text{if } t+s \leq 0 \\ T(t+s)(h+\bar{f})(0) - \bar{x} & \text{if } t+s \geq 0 \end{cases} \qquad (13)\\
&= \begin{cases} h(t+s) & \text{if } t+s \leq 0 \\ R(t+s)h(0) & \text{if } t+s \geq 0. \end{cases}
\end{aligned}
$$

By using (3) we also see that

$$\begin{aligned} R(t)h(0) &= T(t)(h+\bar{f})(0) - \bar{f}(0) \\ &= S(t)(h+\bar{f})(0) + \int_0^t S(t-s)\phi(T(s)(h+\bar{f}))\,ds - \bar{f}(0) \\ &= S(t)h(0) + S(t)\bar{f}(0) - \bar{f}(0) + \int_0^t S(t-s)\phi(R(s)h+\bar{f})\,ds \\ &= S(t)h(0) + S(t)\bar{f}(0) - \bar{f}(0) + \int_0^t S(t-s)(\bar{\phi}(R(s)h) - B\bar{x})\,ds \\ &= S(t)h(0) + \int_0^t S(t-s)\bar{\phi}(R(s)h)\,ds + S(t)\bar{f}(0) - \bar{f}(0) - \int_0^t S(t-s)B\bar{x}\,ds \\ &= S(t)h(0) + \int_0^t S(t-s)\bar{\phi}(R(s)h)\,ds. \end{aligned} \tag{14}$$

Here we have used some basic facts from linear semigroup theory. From (13) and (14) we see that $(R(t))_{t\geq 0} = (T_{B,\bar{\phi}}(t))_{t\geq 0}$. From the definition of $\bar{\phi}$ we see that $\bar{\phi}$ is Lipschitz continuous, respectively, Fréchet- differentiable, if ϕ has these properties. Also, the Fréchet derivative of ϕ at $\bar{f}$ is identical to the Fréchet derivative of $\bar{\phi}$ at zero. Thus we can apply Proposition 4.2 to the semigroup $R(t)$ to obtain the Fréchet derivative of $R(t)$ under the appropriate conditions on ϕ. From these observations, Proposition 4.2, and Theorem 4.1 we obtain the following local stability result for (FDE).

Theorem 4.3. *Let X be a Banach lattice and $E = C([-r_0, 0], X)$. Let B and ϕ satisfy (H1) and (H2). Let $\bar{x} \in X$ be such that $B\bar{x} + \phi(1 \otimes \bar{x}) = 0$, and let ϕ be Fréchet- differentiable at $1 \otimes \bar{x}$ with Fréchet derivative $\psi := \phi'(1 \otimes \bar{x}) \in \mathcal{L}(E, X)$. If the semigroup $(T_{B,\psi}(t))_{t\geq 0}$ is uniformly exponentially stable (that is, $\omega(T_{B,\psi}(t) < 0)$, then the equilibrium $1 \otimes \bar{x}$ of the nonlinear semigroup $(T_{B,\phi}(t))_{t\geq 0}$ is exponentially stable.*

In applying Theorem 4.3 to conclude the exponential stability of an equilibrium of the nonlinear semigroup associated with (FDE), one must show that the linearized semigroup is uniformly exponentially stable. This is equivalent to showing that the growth bound of the linearized semigroup is less than zero (i.e., $\omega(T_{B,\psi}(t)) < 0$). If one can show that the linearized semigroup is positive (or dominated by a (positive) semigroup in the sense of Definition 3.7) then Theorem 1.8, Theorem 3.3 and Theorem 3.8 give sufficient conditions under which one can conclude that $\omega(T_{B,\psi}(t)) < 0$. For example, if $X = C(M)$, M compact, then $E = C([-r_0, 0] \times M)$, and if $T_{B,\psi}(t)$ is a positive semigroup on E ($= C(K)$, $K = [-r_0, 0] \times M$), then $s(B+\psi_0) < 0$ implies $\omega(T_{B,\psi}(t)) < 0$. If $S(t)$ (the semigroup generated by B) is norm continuous for $t > 0$, then $T_{B,\psi}(t)$ is eventually norm continuous ([7, Proposition 2.1]) and thus $s(A_{B,\psi}) = \omega(T_{B,\psi}(t))$ ([9, A-IV, (1.7)]). Therefore it is also true in this case that if B generates a positive semigroup and ψ is a positive operator, then $s(B + \psi_0) < 0$ implies $\omega(T_{B,\psi}(t)) < 0$. Another condition which guarantees that $s(A_{B,\psi}) = \omega(T_{B,\psi}(t))$ is if $T_{B,\psi}(t)$ is a positive semigroup on $E = C(K)$ which is "quasi-compact" (that is, $T_{B,\psi}(t)$ approaches the compact operators as $t \to \infty$ [9, B -IV, Definition 2.7]) and r is an eigenvalue of $A_{B,\psi}$ admitting a strictly positive eigenfunction and satisfying Re $r \geq 0$ ([9, B-IV, Corollary 2.11]). This last property is related to a property of positive semigroups called "irreducibility" (see [9, B-III, Section 3]). We will investigate the consequences of irreducibility for the stability study of (FDE) in a future paper.

V. An example. We consider the following equation, which is a modification of an example of a population equation from [9, B-IV, Example 3.11].

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) - d(x)u(x,t) + b(x)\int_{-1}^{0} g(u(x,t+r(s)))\,d\eta(s), \quad t \geq 0,\ x \in [0,1] \tag{Ex}$$

$$\frac{\partial}{\partial x}u(x,t)\big|_{x=0} = 0 = \frac{\partial}{\partial x}u(x,t)\big|_{x=1}, \quad t \geq 0$$

$$u(x,t) = f(x,t), \quad t \in [-r_0, 0],\ x \in [0,1]$$

We assume that g is a Lipschitz continuous, differentiable function, η is a positive measure, $d(x) \geq 0$, $b(x) \geq 0$, $x \in [0,1]$, and r is a delay function on $[-r_0, 0]$. Let $X = C[0,1]$, and $E = C([-r_0,0],X) = C([-r_0,0] \times [0,1])$. Let B_0 be defined by $B_0 u = (d^2u)/(dx^2)$ for $u \in D(B_0) = \{f \in C^2[0,1] : u'(0) = u'(1) = 0\}$. Let M_g denote the bounded multiplication operator for $0 \leq g \in X$. Then (Ex) takes the abstract form

$$\begin{aligned} \dot{u}(t) &= B_0 u(t) - M_d u(t) + M_b \int_{-1}^{0} g(u_t(r(s)))\,d\eta(s), \qquad t \geq 0 \\ u_0 &= f. \end{aligned} \tag{15}$$

Let $B = B_0 - M_d$ and $\phi\tilde{f} = M_b \int_{-1}^{0} g((\tilde{f} \circ r)(s))\,d\eta(s)$ for $\tilde{f} \in E$. It is well-known that B_0 generates a positive contraction semigroup on X and has compact resolvent (see [9, A-I, 2.7]). The same is true for the operator B (see [9, A-II, Theorems 1.29 and 1.30]). It is clear that ϕ is a Lipschitz continuous operator from E to X since g is assumed to be a Lipschitz continuous function. Thus (15) has the form of (FDE) with B and ϕ satisfying (H1) and (H2), and we are guaranteed the existence of a nonlinear semigroup $(T_{B,\phi}(t))_{t\geq 0}$ on E by Theorem 2.1. Assume that $\bar{h} \in X_+$ satisfies $B\bar{h} + \phi\bar{f} = 0$ where $\bar{f} = 1 \otimes \bar{h}$, i.e., that $u(t) \equiv \bar{h}$ is an equilibrium solution of (15). For example, if $g(0) = 0$, then $\bar{h} = 0$ is such an equilibrium of $(T_{B,\phi}(t))_{t\geq 0}$. The Fréchet derivative of ϕ at $\bar{f}$ is given by

$$\psi_r f = \phi'(\bar{f})f = M_b \int_{-1}^{0} g'(\bar{h})f(r(s))\,d\eta(s) = M_b g'(\bar{h}) \int_{-1}^{0} f(r(s))\,d\eta(s),$$

where $g'(\bar{h}) \in \mathcal{L}(X)$.

From the theory developed in Section IV we know that the linearization of (15) has the form

$$\begin{aligned} \dot{u}(t) &= Bu(t) + \psi_r u_t \qquad t \geq 0 \\ u_0 &= f. \end{aligned} \tag{16}$$

From Theorem 3.8 we know that if $s(B + |\psi|_0) < 0$ then the linear semigroup corresponding to (16) will be uniformly exponentially stable independently of the delay function r. For $\psi f = M_b g'(\bar{h}) \int_{-1}^{0} f(s)\,d\eta(s)$, and $x \in X$,

$$\psi_0 x = \psi(1 \otimes x) = M_b g'(\bar{h}) \int_{-1}^{0} x\,d\eta(s) = M_{b\|\eta\|} g'(\bar{h})x.$$

Thus, $\psi_0 = M_{b\|\eta\|} g'(\bar{h})$ and $|\psi|_0 = M_{b\|\eta\|}|g'(\bar{h})|$. Hence, if $s(B_0 - M_d + M_{b\|\eta\|}|g'(\bar{h})|) < 0$ we can conclude from Theorem 4.3 that the equilibrium $\bar{f} = 1 \otimes \bar{h}$ of the nonlinear semigroup corresponding to (15) is exponentially stable independently of the delay function r.

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