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PARABOLIC INTEGRODIFFERENTIAL EQUATIONS WITH NONHOMOGENEOUS BOUNDARY CONDITIONS

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Abstract. We consider a parabolic partial integrodifferential Volterra equation with nonhomogeneous boundary conditions

$$\begin{cases} u_t(t,x) = \Delta u(t,x) + \int_0^t k(t-s)\Delta u(s,x)\,ds + f(t,x), & t\in[0,T],\ x\in\Omega \\ u(0,x) = u_0(x), & x\in\Omega \\ u(t,x) = \varphi(t,x), & t\in[0,T],\ x\in\partial\Omega \end{cases} \tag{*}$$

and a similar problem with infinite delay, where Δ is the Laplace operator and $k:[0,+\infty[\to\mathbb{R}$. Under suitable assumptions on the kernel k, we state some results about the existence, uniqueness and regularity of the solutions of (*) and of the equation with infinite delay.

0. Introduction. This paper deals with a class of parabolic partial integrodifferential Volterra equations with nonhomogeneous boundary conditions

$$\begin{cases} u_t(t,x) = \Delta u(t,x) + \displaystyle\int_0^t k(t-s)\Delta u(s,x)\,ds + f(t,x), & t\in[0,T],\ x\in\Omega \\ u(0,x) = u_0(x), & x\in\Omega \\ u(t,x) = \varphi(t,x), & t\in[0,T],\ x\in\partial\Omega \end{cases} \tag{0.1}$$

and the similar problem with infinite delay

$$\begin{cases} u_t(t,x) = \Delta u(t,x) + \displaystyle\int_{-\infty}^t k(t-s)\Delta u(s,x)\,ds + f(t,x), & t\in\mathbb{R},\ x\in\Omega \\ u(t,x) = \varphi(t,x), & t\in\mathbb{R},\ x\in\partial\Omega \end{cases} \tag{0.2}$$

where $T>0$, Ω is a bounded open set in $\mathbb{R}^n$, $n\in\mathbb{N}$, with regular boundary $\partial\Omega$, Δ is the Laplace operator and the Laplace transform of $k:[0,+\infty[\to\mathbb{R}$ verifies suitable assumptions.

Many authors have studied problems (0.1) and (0.2) when $\varphi\equiv 0$ using Laplace transform and other methods; see for instance [1]-[5], [8]-[10].

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To solve (0.1) and (0.2), we need to introduce Dirichlet map $D: C(\partial\Omega) \to C(\overline{\Omega}) \cap C^2(\Omega)$, where for any $\psi \in C(\partial\Omega)$, $D\psi$ is the solution of the Dirichlet problem having ψ as boundary value.

The function $v(t,x) = u(t,x) - D\varphi(t,x)$ is the solution of an integrodifferential problem with homogeneous boundary conditions. Using the known results about this problem (see [1], [2] and [8]), we establish some theorems about the solutions of problems (0.1) and (0.2).

Our work is organized as follows. For the sake of completeness, §1 is devoted to a review of some results about the existence, uniqueness and Hölder regularity of the solutions of abstract integrodifferential equations with homogeneous boundary conditions (see [1], [2] and [8]). In addition, we list some notations that we shall use throughout the paper. In §2 we state and prove our main results about the existence and regularity of the strict and strong solutions of (0.1) (see Theorems 2.3, 2.5 and 2.7). Section 3 is finally devoted to the study of the bounded solutions of (0.2); our main results are stated in Theorems 3.3 and 3.6.

1. Preliminaries. In this section we recall some known results about abstract integrodifferential equations (see [1], [2], [8]). In addition, we list the notations, which we shall use in the sequel.

Let X be a complex Banach space with norm $\|\cdot\|$. If Y is another Banach space, we denote by $L(X;Y)$ the Banach space of all linear bounded operators $T: X \to Y$, endowed with the norm $\|T\| = \sup\{\|T(x)\|, \|x\| \le 1\}$. We set $L(X) = L(X;X)$.

Let $A: D(A) \subset X \to X$ be a linear operator such that

$$\begin{cases} \text{there exist } M > 0,\ \omega \in \mathbb{R} \text{ and } \theta \in]\pi/2, \pi] \text{ such that} \\ \text{(i) the resolvent set } \rho(A) \text{ of } A \text{ contains the sector} \\ \qquad S = \{\lambda \in \mathbb{C};\ \lambda \ne \omega,\ |\arg(\lambda - \omega)| < \theta\}; \\ \text{(ii) for any } \lambda \in S\ \|(\lambda - A)^{-1}\|_{L(X)} \le M|\lambda - \omega|^{-1}. \end{cases} \tag{1.1}$$

Assumption (1.1) means that A generates an analytic semigroup in X. Since A is a closed operator, $D(A)$ is a Banach space, endowed with the graph norm $\|x\|_{D(A)} = \|x\| + \|Ax\|$, $x \in D(A)$.

Let $k: [0, +\infty[\to \mathbb{R}$ be a locally integrable function, verifying the following properties:

$$\begin{cases} \text{(i) } k \text{ is absolutely Laplace transformable and its Laplace transform } \hat{k} \\ \quad \text{is analytically extendible to } S \text{ (the analytic extension will be still} \\ \quad \text{denoted by } \hat{k}); \\ \text{(ii) there are } \beta \in]0,1],\ N > 0 \text{ such that } |\lambda^\beta \hat{k}(\lambda)| \le N \text{ for any } \lambda \in S. \end{cases} \tag{1.2}$$

Under assumptions (1.1) and (1.2) it is possible to construct a resolvent operator $R: [0, +\infty[\to L(X)$ for the problem

$$\begin{cases} v'(t) = Av(t) + \int_0^t k(t-s)Av(s)\,ds, \quad t > 0 \\ v(0) = v_0. \end{cases} \tag{1.3}$$

This was done in [1] when $D(A)$ is dense in X and in [8] in the general case. (In [1] and [8] a more general kernel $K(t) \in L(D(A);X)$ was considered.) In particular, it was shown that there are $r_0 > 0$, $\theta_0 \in]\pi/2, \theta[$ such that, for every $\lambda \in S$, $|\lambda| \geq r_0$ and $|\arg \lambda| \leq \theta_0$, the linear operator $\lambda - A - \hat{k}(\lambda)A : D(A) \to X$ is invertible. Setting $F(\lambda) = (\lambda - A - \hat{k}(\lambda)A)^{-1}$ then $F(\lambda)$ belongs to $L(X; D(A))$ and the resolvent operator for problem (1.3) is given by

$$\begin{cases} R(t) = \dfrac{1}{2\pi i} \displaystyle\int_\gamma e^{\lambda t} F(\lambda)\, d\lambda, \ t > 0 \\ R(0) = 1 \end{cases} \tag{1.4}$$

where γ is the path $\gamma = \gamma^+ + \gamma^0 + \gamma^-$; $\gamma^\pm = \{\lambda \in \mathbb{C};\ \lambda = re^{\pm i\theta_0},\ r \geq r_0\}$ and $\gamma^0 = \{\lambda \in \mathbb{C}; \lambda = r_0 e^{i\eta},\ -\theta_0 \leq \eta \leq \theta_0\}$ are oriented counterclockwise.

The resolvent operator has the following properties.

Proposition 1.1. ([1], [8]). *Let* (1.1), (1.2) *hold and let* $R(\cdot)$ *be defined by* (1.4). *Then*

(i) *there is a constant* $C_1 > 0$ *such that*

$$\|R(t)\|_{L(X)} + \|tR'(t)\|_{L(X)} + \|t^2 R''(t)\|_{L(X)} \leq C_1 e^{r_0 t},\ t \geq 0; \tag{1.5}$$

(ii) *setting* $\overline{kA * R}(t) = \dfrac{1}{2\pi i} \displaystyle\int_\gamma e^{\lambda t} \hat{k}(\lambda) A F(\lambda)\, d\lambda$, $t \geq 0$ *we have*

$$R'(t)x = AR(t)x + \overline{kA * R}(t)x, \quad t > 0,\ x \in X; \tag{1.6}$$

(iii) *for every* $x \in X$, $t \geq 0$, $\int_0^t R(s)x\, ds$ *belongs to* $D(A)$, *and*

$$A \int_0^t R(s)x\, ds = R(t)x - x - \int_0^t \overline{kA * R}(s)x\, ds. \tag{1.7}$$

Remark 1.2. Under the assumptions of Prop. 1.1, for any $x \in X$ and $t \geq 0$ we have

$$\int_0^t R'(s)x\, ds = R(t)x - x. \tag{1.8}$$

Proof: Integrating (1.6) over $[0, t]$, by (1.7) we get

$$\int_0^t R'(s)x\, ds = A \int_0^t R(s)x\, ds + \int_0^t \overline{kA * R}(s)x\, ds = R(t)x - x,$$

that is (1.8).

The existence of the resolvent operator for problem (1.3) allows one to solve in a strict sense the inhomogeneous problem

$$\begin{cases} v'(t) = Av(t) + \displaystyle\int_0^t k(t-s)Av(s)\, ds + g(t), \ \ 0 \leq t \leq T,\ T > 0, \\ v(0) = v_0 \end{cases} \tag{1.9}$$

where A, k satisfy assumptions (1.1), (1.2), and $g : [0, T] \to X$ is Hölder continuous.

We recall that a function $v : [0, T] \to D(A)$ is said to be a strict solution of (1.9) in $[0, T]$ if $v \in C([0, T]; D(A)) \cap C^1([0, T]; X)$ and satisfies (1.9).

We now introduce the functional spaces which will be used throughout the paper.

Let I be any interval in $\mathbb{R}$. We denote by $C_b(I;X)$ the space of all continuous and bounded functions $v : I \to X$, endowed with the norm $\|v\|_\infty = \{\sup \|v(x)\|,\ x \in I\}$. Given $\alpha \in]0,1[$, $C_b^\alpha(I;X)$ is the subspace of $C_b(I;X)$ consisting of the α-Hölder continuous functions v; that is, $[v]_\alpha \doteq \sup\{|t-s|^{-\alpha}\|v(t)-v(s)\|;\ t,s \in I,\ t \neq s\} < +\infty$. It is endowed with the norm $\|v\|_{C_b^\alpha(I;X)} = \|v\|_\infty + [v]_\alpha$. $C_b^1(I;X)$ (resp. $C_b^{1,\alpha}(I;X)$) is the space of all differentiable functions $v : I \to X$ such that both v and v' belong to $C_b(I;X)$ (resp. $C_b^\alpha(I;X)$). $h_b^\alpha(I;X)$ is the subspace of $C_b^\alpha(I;X)$ consisting of the functions v such that $\lim_{\tau\to 0}\sup_{x,y\in I,\ |x-y|\le\tau} \tau^{-\alpha}\|v(x)-v(y)\| = 0$. If the interval I is compact we omit, of course, subscript "b".

The following results about the existence, uniqueness and regularity of the strict solution of problem (1.9) have been proved in [8], Prop. 2.2-2.4.

Theorem 1.3. *Let $g \in C^\alpha([0,T];X)$, $0<\alpha<1$, and let $v_0 \in D(A)$ be such that*

$$Av_0 + g(0) \in \overline{D(A)}. \tag{1.10}$$

Then the function

$$v(t) = R(t)v_0 + \int_0^t R(t-s)g(s)\,ds, \quad 0 \le t \le T \tag{1.11}$$

is the unique strict solution of (1.9) and belongs to $C^\alpha([\epsilon,T];D(A)) \cap C^{1,\alpha}([\epsilon,T];X)$ for any $\epsilon \in]0,T[$.

To study the Hölder regularity of the solution of (1.9) up to $t=0$, we need to introduce a class of intermediate spaces between $D(A)$ and X, which are defined for $0<\alpha<1$ by (see [6], [11])

$$D_A(\alpha,\infty) = \{x \in X;\ [x]_\alpha \doteq \sup_{t>0} \|t^\alpha (A-\omega)(t-A+\omega)^{-1}x\| < +\infty\}. \tag{1.12}$$

$D_A(\alpha,\infty)$ is a Banach space under the norm $\|x\|_{D_A(\alpha,\infty)} = \|x\| + [x]_\alpha$. Other properties and characterizations of $D_A(\alpha,\infty)$ can be found in [11].

Theorem 1.4. *Let $g \in C^\alpha([0,T];X)$, $0<\alpha<1$, and $v_0 \in D(A)$ be such that*

$$Av_0 + g(0) \in D_A(\alpha,\infty). \tag{1.13}$$

Then the strict solution v of (1.9) belongs to $C^\alpha([0,T];D(A)) \cap C^{1,\alpha}([0,T];X)$ and $v'(t)$ belongs to $D_A(\alpha,\infty)$ for any $t \in [0,T]$.

The following result has been proved in [1] when $D(A)$ is dense in X, but it can be easily stated in the general case.

Proposition 1.5. *If $\alpha \in]0,1[$, $v_0 \in D_A(\alpha,\infty)$ and $g \in C([0,T];X)$, then v defined by (1.11) belongs to $C^\alpha([0,T];X)$.*

We now review some known results about the bounded solutions of the abstract parabolic integrodifferential equation with infinite delay

$$v'(t) = Av(t) + \int_{-\infty}^t k(t-s)Av(s)ds + g(t),\ t \in \mathbb{R}, \tag{1.14}$$

(see [2]). Here A and k verify respectively (1.1) and (1.2). In addition, it holds

$$\int_0^{+\infty} |k(t)|\, dt < +\infty. \tag{1.15}$$

We fix once and for all a maximal analytic extension of $\hat{k}(\cdot)$ (which we still call $\hat{k}(\cdot)$) and denote by Λ its domain of definition. We set $\rho_0(A,k) = \{\lambda \in \Lambda :\ \lambda - A - \hat{k}(\lambda)A$ is invertible and $(\lambda - A - \hat{k}(\lambda)A)^{-1} \in L(X; D(A))\}$. $\rho_0(A,k)$ is an open set in $\mathbb{C}$.

We now define an analytic extension of $F(\cdot)$ on the set $\rho(A,k) = \rho_0(A,k) \cup \{\lambda_0 \in \mathbb{C};\ \lambda_0$ is an isolated removable singularity of $F(\cdot)\}$ setting, for any $\lambda_0 \in \rho(A,k)\backslash\rho_0(A,k)$, $F(\lambda_0) = \lim_{\lambda\to\lambda_0} F(\lambda)$. Moreover, we denote by $\sigma(A,k)$ the complementary set $\mathbb{C}\backslash\rho(A,k)$.

In order to solve equation (1.14) for any g in $C_b(\mathbb{R}; X)$ we need one more hypothesis:

$$i\mathbb{R} \subset \rho(A,k). \tag{1.16}$$

Since $\sigma(A,k)$ is closed, then (1.16) implies that $\sigma(A,k) = \sigma_-(A,k) \cup \sigma_+(A,k)$, and

$$\begin{cases} -\omega_1 \doteq \sup\{\mathrm{Re}\lambda; \lambda \in \sigma_-(A,k)\} < 0 \\ \omega_2 \doteq \inf\{\mathrm{Re}\lambda; \lambda \in \sigma_+(A,k)\} > 0. \end{cases} \tag{1.17}$$

We define the operators $R_+(t)$ and $R_-(t)$ by

$$\begin{cases} R_+(t) = \dfrac{1}{2\pi i}\displaystyle\int_{\gamma^+} e^{\lambda t} F(\lambda)\, d\lambda, & t \in \mathbb{R} \\ R_-(t) = \dfrac{1}{2\pi i}\displaystyle\int_{\gamma^-} e^{\lambda t} F(\lambda)\, d\lambda, & t > 0 \\ R_-(0) = 1 - R_+(0) \end{cases} \tag{1.18}$$

where γ^+ is any closed Jordan curve surrounding $\sigma_+(A,k)$ (oriented in the counterclockwise sense) contained in the half plane $\{\lambda \in \mathbb{C};\ \mathrm{Re}\lambda > 0\}$, and γ^- is any Jordan curve surrounding $\sigma_-(A,k)$, contained in the half plane $\{\lambda \in \mathbb{C};\ \mathrm{Re}\lambda < 0\}$ and joining $\infty e^{-i\theta_0}$ with $\infty e^{i\theta_0}$. Obviously, $R(t) = R_+(t) + R_-(t)$, for every $t \geq 0$.

We now list some estimates about the operators $R_\pm(t)$ defined by (1.18), which will be used in the sequel.

Lemma 1.6. ([2]). *Assume* (1.1), (1.2), (1.15) *and* (1.16). *If* ω_1 *and* ω_2 *are the real numbers defined by* (1.17), *then for every positive* $\epsilon < \min\{\omega_1, \omega_2\}$ *there are* $H_-(\epsilon) > 0$, $H_+(\epsilon) > 0$ *such that*

$$\|R_-(t)\|_{L(X)} + \|tR'_-(t)\|_{L(X)} + \|t^2 R''_-(t)\|_{L(X)} \leq H_-(\epsilon)e^{-(\omega_1-\epsilon)t},\ t > 0 \tag{1.19}$$

$$\|R_+(t)\|_{L(X)} + \|R'_+(t)\|_{L(X)} + \|R''_+(t)\|_{L(X)} \leq H_+(\epsilon)e^{(\omega_2-\epsilon)t}, \qquad t \leq 0 \tag{1.20}$$

We recall that a function v is a strict solution of (1.14) if $v \in C_b^1(\mathbb{R}; X) \cap C_b(\mathbb{R}; D(A))$ and fulfills (1.14). A function v belonging to $C_b(\mathbb{R}; X)$ is said to be a strong solution of (1.14) if there is a sequence $\{v_n\}$ of functions in $C_b^1(\mathbb{R}; X) \cap C_b(\mathbb{R}; D(A))$ such that for any compact interval $I \subset \mathbb{R}$, $v_n \to v$ and

$$v'_n - Av_n - \int_{-\infty}^t k(t-s)Av_n(s)ds \to g \quad \text{in } C(I; X).$$

The following result about the existence, uniqueness and regularity of the strict and strong solutions for (1.14) has been proved in [2].

Theorem 1.7. *Assume* (1.1), (1.2), (1.15) *and* (1.16).

(i) *Let* $g \in C_b^\alpha(\mathbb{R}; X)$, $0 < \alpha < 1$. *Then the function*

$$v(t) = \int_{-\infty}^{t} R_-(t-s)g(s)\,ds - \int_{t}^{+\infty} R_+(t-s)g(s)\,ds, \quad t \in \mathbb{R} \tag{1.21}$$

is the unique strict solution of (1.14) *and* v *belongs to* $C_b^\alpha(\mathbb{R}; D(A)) \cap C_b^{1,\alpha}(\mathbb{R}; X)$.

(ii) *For any* $g \in C_b(\mathbb{R}; X)$, (1.14) *has a unique strong solution given by* (1.21). *In addition,* v *belongs to* $C_b^\alpha(\mathbb{R}; X)$ *for any* $\alpha \in]0,1[$.

2. Parabolic integrodifferential equations with nonhomogeneous boundary conditions. Let Ω be a bounded open set in $\mathbb{R}^n$, $n \in \mathbb{N}$, with regular boundary $\partial\Omega$ of class C^1.

In this section we shall study several properties (existence, uniqueness and regularity) of the solutions of the parabolic integrodifferential equation with nonhomogeneous boundary conditions

$$\begin{cases} u_t(t,x) = \Delta u(t,x) + \displaystyle\int_0^t k(t-s)\Delta u(s,x)\,ds + f(t,x), & t \in [0,T],\ x \in \Omega \\ u(0,x) = u_0(x), \quad x \in \Omega \\ u(t,x) = \varphi(t,x), \quad t \in [0,T],\ x \in \partial\Omega \end{cases} \tag{2.1}$$

where $T > 0$, k verifies (1.2) and f, u_0, φ are known functions.

We set

$$\begin{cases} X = C(\overline{\Omega});\ \|v\| = \sup\limits_{x\in\overline{\Omega}} |v(x)|, \quad v \in X \\ D(A) = \{v \in X;\ \Delta v \in X,\ v(x) = 0 \text{ for any } x \in \partial\Omega\} \\ Av = \Delta v, \quad v \in D(A) \end{cases} \tag{2.2}$$

where Δ is the Laplace operator in the distributional sense. Then A satisfies (1.1) (see [12], [13]). In addition, we recall that $\overline{D(A)} = \{v \in X;\ v(x) = 0 \text{ for any } x \in \partial\Omega\}$.

To solve problem (2.1), we introduce Dirichlet map $D : C(\partial\Omega) \to C(\overline{\Omega}) \cap C^2(\Omega)$, defined setting for any $\psi \in C(\partial\Omega)$ $D\psi = z$, where z is the solution of Dirichlet problem

$$\begin{cases} \Delta z(x) = 0, & x \in \Omega \\ z(x) = \psi(x), & x \in \partial\Omega. \end{cases} \tag{2.3}$$

Thanks to the regularity of $\partial\Omega$, for any $\psi \in C(\partial\Omega)$ there exists a unique solution of problem (2.3).

We now prove a result, which will be useful in the following.

Lemma 2.1. *If* ψ *belongs to* $C^{1,\alpha}([0,T]; C(\partial\Omega))$, $0 < \alpha < 1$ *and* $T > 0$, *(resp.* $h^\alpha([0,T]; C(\partial\Omega))$*), then* $D\psi$ *belongs to* $C^{1,\alpha}([0,T]; C(\overline{\Omega}))$ *(resp.* $h^\alpha([0,T]; C(\overline{\Omega}))$*), where* D *is Dirichlet map.*

Proof: For $t, s \in [0,T]$, $D\psi(t,\cdot) - D\psi(s,\cdot)$ is a harmonic function and $D\psi(t,x) - D\psi(s,x) = \psi(t,x) - \psi(s,x)$ for any $x \in \partial\Omega$. Thanks to the maximum principle we have

$$\sup_{x\in\overline{\Omega}} |D\psi(t,x) - D\psi(s,x)| \le \sup_{x\in\partial\Omega} |\psi(t,x) - \psi(s,x)|,$$

hence, $D\psi \in C^{\alpha}([0,T];C(\overline{\Omega}))$.

For any $t \in [0,T]$, let $z(t,\cdot)$ be the solution of the problem

$$\begin{cases} \Delta z(t,x) = 0, & x \in \Omega \\ z(t,x) = \psi_t(t,x), & x \in \partial\Omega. \end{cases} \tag{2.4}$$

Let $t_0 \in [0,T]$ be fixed, the function $\big(D\psi(t,\cdot) - D\psi(t_0,\cdot)\big)/(t-t_0) - z(t_0,\cdot)$ is harmonic and

$$\frac{D\psi(t,x) - D\psi(t_0,x)}{t-t_0} - z(t_0,x) = \frac{\psi(t,x) - \psi(t_0,x)}{t-t_0} - \psi_t(t_0,x),$$

for any $x \in \partial\Omega$. Again, by the maximum principle, we have

$$\sup_{x\in\overline{\Omega}} \left| \frac{D\psi(t,x) - D\psi(t_0,x)}{t-t_0} - z(t_0,x) \right| \leq \sup_{x\in\partial\Omega} \left| \frac{\psi(t,x) - \psi(t_0,x)}{t-t_0} - \psi_t(t_0,x) \right| .$$

Therefore, $D\psi$ is differentiable in $[0,T]$ and for any $t \in [0,T]$ $(D\psi)_t(t,\cdot)$ is the solution of problem (2.4). By the maximum principle we get $(D\psi)_t \in C^{\alpha}([0,T];C(\overline{\Omega}))$. The other case follows using a similar argument.

Using Theorems 1.3-1.4, we shall prove some results about the existence, uniqueness and regularity of the solution of problem (2.1).

We first define the strict solution of (2.1).

Definition 2.2. *A function $u : [0,T] \times \overline{\Omega} \to \mathbb{R}$ is said to be a strict solution of* (2.1) *if $u \in C^1([0,T];C(\overline{\Omega}))$, $\Delta u \in C([0,T];C(\overline{\Omega}))$ and satisfies* (2.1).

Theorem 2.3. *Let $f \in C^{\alpha}([0,T];C(\overline{\Omega}))$, $\varphi \in C^{1,\alpha}([0,T];C(\partial\Omega))$, $0 < \alpha < 1$ and $u_0 \in C(\overline{\Omega})$ be such that $\Delta u_0 \in C(\overline{\Omega})$ and verify the compatibility conditions*

$$u_0(x) = \varphi(0,x), \quad \text{for every } x \in \partial\Omega; \tag{2.5}$$

$$\Delta u_0(x) + f(0,x) = \varphi_t(0,x), \quad \text{for every } x \in \partial\Omega. \tag{2.6}$$

Then there exists a unique strict solution $u(t,x)$ of (2.1)*, such that u belongs to $C^{1,\alpha}\big([\epsilon,T]; C(\overline{\Omega})\big)$ and $\Delta u \in C^{\alpha}([\epsilon,T];C(\overline{\Omega}))$ for any $\epsilon \in]0,T[$.*

Proof: First of all, we observe that by Lemma 2.1 we have $D\varphi \in C^{1,\alpha}\big([0,T]; C(\overline{\Omega})\big)$, where D is Dirichlet map. We set

$$g(t,x) = f(t,x) - D\varphi_t(t,x), \quad t \in [0,T], \ x \in \overline{\Omega}; \tag{2.7}$$

$$v_0(x) = u_0(x) - D\varphi(0,x), \quad x \in \overline{\Omega}. \tag{2.8}$$

With the convention $g(t) = g(t,\cdot)$, we consider the abstract problem

$$\begin{cases} v'(t) = Av(t) + \displaystyle\int_0^t k(t-s)Av(s)\,ds + g(t), \quad t \in [0,T] \\ v(0) = v_0. \end{cases} \tag{2.9}$$

Thanks to the assumptions, g belongs to $C^{\alpha}([0,T];X)$, where $X = C(\overline{\Omega})$. By (2.8), we get $\Delta v_0 = \Delta u_0$, because $D\varphi(0,\cdot)$ is a harmonic function. In addition, by (2.5) we have $v_0(x) = 0$ for any $x \in \partial\Omega$, so that $v_0 \in D(A)$.

Let us show that (1.10) holds. In fact, from (2.7) and (2.8) we gain

$$Av_0 + g(0) = \Delta u_0 + f(0,\cdot) - D\varphi_t(0,\cdot), \tag{2.10}$$

hence, by (2.6), $Av_0 + g(0) \in \overline{D(A)}$. We may now apply Theorem 1.3, the function

$$v(t,\cdot) = R(t)v_0 + \int_0^t R(t-s)g(s,\cdot)\,ds, \quad 0 \le t \le T \tag{2.11}$$

is the unique strict solution of (2.9) and v belongs to $C^\alpha([\epsilon,T];D(A)) \cap C^{1,\alpha}([\epsilon,T];X)$ for every $\epsilon \in]0,T[$.

Let us check that the function

$$u(t,x) = v(t,x) + D\varphi(t,x), \quad t \in [0,T], \ x \in \overline{\Omega} \tag{2.12}$$

is a strict solution of (2.1). Since v is the solution of (2.9), u is differentiable in $[0,T]$, and we have

$$u_t(t,x) = \Delta u(t,x) + \int_0^t k(t-s)\Delta u(s,x)\,ds + f(t,x),$$

$t \in [0,T]$, $x \in \Omega$. From (2.8), it follows $u(0,x) = u_0(x)$ for any $x \in \Omega$. Because of $v(t,\cdot) \in D(A)$ for any $t \in [0,T]$, we have $v(t,x) = 0$, $x \in \partial\Omega$, so that $u(t,x) = \varphi(t,x)$ for any $t \in [0,T]$, $x \in \partial\Omega$.

The uniqueness follows from (2.12) and from the uniqueness of the solution of (2.9). Finally, (2.12) and the regularity of φ and v yield $u \in C^{1,\alpha}([\epsilon,T];C(\overline{\Omega}))$ and $\Delta u \in C^\alpha([\epsilon,T]; C(\overline{\Omega}))$ for any $\epsilon \in]0,T[$.

We now give a representation formula for the strict solution of (2.1).

Corollary 2.4. *Under the assumptions of Theorem 2.3, the strict solution of* (2.1) *is given by*

$$u(t,\cdot) = R(t)u_0 + \int_0^t R(t-s)f(s,\cdot)\,ds - \int_0^t R'(t-s)D\varphi(s,\cdot)\,ds, \quad t \in [0,T]. \tag{2.13}$$

Proof: First of all we observe that the integral $\int_0^t R'(t-s)D\varphi(s)\,ds$ is well defined, where $D\varphi(s) = D\varphi(s,\cdot)$. In fact,

$$\int_0^t R'(t-s)D\varphi(s)\,ds = \int_0^t R'(t-s)(D\varphi(s) - D\varphi(t))\,ds + \int_0^t R'(s)D\varphi(t)\,ds,$$

from which, by (1.8), it follows

$$\int_0^t R'(t-s)D\varphi(s)\,ds = \int_0^t R'(t-s)(D\varphi(s) - D\varphi(t))\,ds + R(t)D\varphi(t) - D\varphi(t). \tag{2.14}$$

The integral on the right hand side of (2.14) is convergent, because by (1.5) and Lemma 2.1 we have

$$\begin{aligned} &\int_0^t \|R'(t-s)(D\varphi(s) - D\varphi(t))\|\,ds \le \\ &C_1[D\varphi]_\alpha \int_0^t e^{r_0(t-s)}(t-s)^{\alpha-1}\,ds \le C_1[D\varphi]_\alpha e^{r_0 T}\alpha^{-1}T^\alpha, \quad t \in [0,T]. \end{aligned} \tag{2.15}$$

Moreover, by (2.14), (2.15) and (1.5) we get

$$\Big\| \int_0^t R'(t-s)D\varphi(s)\,ds\Big\| \le \big(C_1 e^{r_0 T} T^{\alpha}\alpha^{-1} + C_1 e^{r_0 T} + 1\big)\|D\varphi\|_{\alpha}, \ \ t \in [0,T]. \tag{2.16}$$

We remark, for later convenience, that up to this point it was enough to assume $\varphi \in C^{\alpha}([0,T];C(\partial\Omega))$.

Let us show now that

$$\int_0^t R(t-s)(D\varphi)'(s)\,ds = -R(t)D\varphi(0) + D\varphi(t) + \int_0^t R'(t-s)D\varphi(s)\,ds, \ \ 0 \le t \le T. \tag{2.17}$$

Fix $\delta \in]0,T[$ and, for any $\epsilon \in]0,\delta[$ set

$$w_\epsilon(t) = \int_0^{t-\epsilon} R(t-s)(D\varphi)'(s)\,ds, \ \ \delta \le t \le T.$$

Integrating by parts, we get

$$\begin{aligned} w_\epsilon(t) &= \int_0^{t-\epsilon} R(t-s)\frac{\partial}{\partial s}\,(D\varphi(s) - D\varphi(t))\,ds = R(\epsilon)(D\varphi(t-\epsilon) - D\varphi(t)) \\ &\quad - R(t)(D\varphi(0) - D\varphi(t)) + \int_0^{t-\epsilon} R'(t-s)(D\varphi(s) - D\varphi(t))ds, \ \ \delta \le t \le T. \end{aligned} \tag{2.18}$$

We observe that w_ϵ converges to $\int_0^t R(t-s)(D\varphi)'(s)\,ds$ in $C([\delta,T];X)$ as $\epsilon \to 0^+$, since by (1.5), we have

$$\Big\|w_\epsilon(t) - \int_0^t R(t-s)(D\varphi)'(s)\,ds\Big\| \le \int_{t-\epsilon}^t \|R(t-s)(D\varphi)'(s)\|\,ds \le C_1\|(D\varphi)'\|_\infty e^{r_0 T}\epsilon,$$

$\delta \le t \le T$. Moreover, $R(\epsilon)(D\varphi(t-\epsilon)-D\varphi(t))$ converges to 0 in $C([\delta,T];X)$ as $\epsilon \to 0^+$, since again by (1.5) we gain $\|R(\epsilon)(D\varphi(t-\epsilon) - D\varphi(t))\| \le C_1 e^{\epsilon r_0}[D\varphi]_\alpha \epsilon^\alpha$, $\delta \le t \le T$. Finally, $\int_0^{t-\epsilon} R'(t-s)(D\varphi(s) - D\varphi(t))\,ds$ converges to $\int_0^t R'(t-s)(D\varphi(s)-D\varphi(t))\,ds$ in $C([\delta,T];X)$ as $\epsilon \to 0^+$, since by (1.5) we get

$$\Big\|\int_0^{t-\epsilon} R'(t-s)(D\varphi(s)-D\varphi(t))\,ds - \int_0^t R'(t-s)(D\varphi(s)-D\varphi(t))\,ds\Big\| \le C_1[D\varphi]_\alpha e^{r_0 T}\epsilon^\alpha \alpha^{-1},$$

$\delta \le t \le T$. Therefore, letting $\epsilon \to 0^+$, by (2.18) we have

$$\int_0^t R(t-s)(D\varphi)'(s)\,ds = -R(t)(D\varphi(0) - D\varphi(t)) + \int_0^t R'(t-s)(D\varphi(s) - D\varphi(t))\,ds,$$

$0 \le t \le T$, from which, in virtue of (2.14), (2.17) follows.

On the other hand, by (2.12), (2.11), (2.7) and (2.8), we have

$$u(t,\cdot) = R(t)u_0 - R(t)D\varphi(0) + \int_0^t R(t-s)f(s,\cdot)\,ds - \int_0^t R(t-s)(D\varphi)'(s)\,ds + D\varphi(t)$$

from which, taking into account (2.17), we get (2.13).

To study the Hölder regularity of the strict solution of (2.1) up to $t=0$, we recall the characterization of the spaces $D_A(\alpha,\infty)$ (see (1.12)) when A is the operator defined by (2.2). For any $\alpha \in]0,1[$, $\alpha \ne 1/2$, we have [7]

$$D_A(\alpha,\infty) = C_0^{2\alpha}(\overline{\Omega}) = \{v \in C^{2\alpha}(\overline{\Omega});\ v(x) = 0,\ x \in \partial\Omega\}. \tag{2.19}$$

We now prove the following

Theorem 2.5. *Assume $\partial\Omega$ of class $C^{2+\alpha}$, $0<\alpha<1/2$. Let $f \in C^{\alpha}([0,T];C(\overline{\Omega}))$, $\varphi \in C^{1,\alpha}([0,T];C(\partial\Omega))$ and $u_0 \in C(\overline{\Omega})$ be such that $\Delta u_0 \in C(\overline{\Omega})$ and verify the compatibility conditions* (2.5) *and* (2.6). *Moreover, we suppose*

$$\Delta u_0 + f(0,\cdot) \in C^{2\alpha}(\overline{\Omega}), \tag{2.20}$$

$$\varphi_t(t,\cdot) \in C^{2\alpha}(\partial\Omega), \quad \textit{for any } t \in [0,T]. \tag{2.21}$$

Then the strict solution u of (2.1) *belongs to $C^{1,\alpha}([0,T];C(\overline{\Omega}))$, $\Delta u \in C^{\alpha}([0,T];C(\overline{\Omega}))$ and $u_t(t,\cdot) \in C^{2\alpha}(\overline{\Omega})$ for any $t \in [0,T]$.*

Proof: Let $v(t,x)$ be the strict solution of problem (2.9). To apply Theorem 1.4, we must show that (1.13) holds.

First of all, we observe that, due to (2.21) and Schauder theorem, we have

$$D\varphi_t(t,\cdot) \in C^{2\alpha}(\overline{\Omega}), \quad \text{for any } t \in [0,T]. \tag{2.22}$$

Therefore, by (2.6), (2.10) and (2.20) we get $Av_0 + g(0,\cdot) \in C_0^{2\alpha}(\overline{\Omega})$, that is, taking into account (2.19), (1.13).

By Theorem 1.4, v belongs to $C^{\alpha}([0,T];D(A)) \cap C^{1,\alpha}([0,T];X)$ and $v'(t) \in D_A(\alpha,\infty)$ for any $t \in [0,T]$. Therefore, thanks to (2.12) the strict solution u of (2.1) belongs to $C^{1,\alpha}([0,T];C(\overline{\Omega}))$ and $\Delta u \in C^{\alpha}([0,T];C(\overline{\Omega}))$.

Moreover, again by (2.12), we have $u_t(t,\cdot) = v_t(t,\cdot) + D\varphi_t(t,\cdot)$, for any $t \in [0,T]$; hence, by (2.22) and $v_t(t,\cdot) \in C^{2\alpha}(\overline{\Omega})$ (see (2.19)), we get $u_t(t,\cdot) \in C^{2\alpha}(\overline{\Omega})$.

We now define the strong solution of problem (2.1).

Definition 2.6. *A function $u(t,x) \in C([0,T]\times\overline{\Omega})$ is said to be a strong solution for problem* (2.1) *if*

$$u(0,x) = u_0(x), \quad x \in \overline{\Omega}, \tag{2.23}$$

$$u(t,x) = \varphi(t,x), \quad t \in [0,T],\ x \in \partial\Omega, \tag{2.24}$$

and there exists a sequence of functions $\{u_n(t,x)\}_{n\in\mathbb{N}}$ such that for any $n \in \mathbb{N}$, $u_n \in C^1([0,T];C(\overline{\Omega}))$, $\Delta u_n \in C([0,T];C(\overline{\Omega}))$ and fulfills

i) $\lim_{n\to+\infty} u_n = u$ *in* $C([0,T]\times\overline{\Omega})$;

ii) $\lim_{n\to+\infty} \left(\frac{\partial u_n}{\partial t}(t,x) - \Delta u_n(t,x) - \int_0^t k(t-s)\Delta u_n(s,x)\,ds\right) = f(t,x)$ *in* $C([0,T]\times\overline{\Omega})$.

Finally, we shall prove a result about the existence and regularity of the strong solution of (2.1).

Theorem 2.7. *Let $f \in C([0,T]\times\overline{\Omega})$, $\varphi \in h^{\alpha}([0,T];C(\partial\Omega))$, $0<\alpha<1$ and $u_0 \in C(\overline{\Omega})$ verify the compatibility condition* (2.5). *Then the function $u(t,x)$ given by* (2.13) *is the unique strong solution of* (2.1).

In addition, if $\partial\Omega$ is of class $C^{2+\alpha}$, $0<\alpha<1/2$, $u_0 \in C^{2\alpha}(\overline{\Omega})$ and $\varphi(0,\cdot) \in C^{2\alpha}(\partial\Omega)$, then u belongs to $C^{\alpha}([0,T];C(\overline{\Omega}))$.

Proof: First of all, we observe that u given by (2.13) is well defined by the same argument used in the proof of Corollary 2.4 and (2.14)-(2.16) hold.

Set $v_0(x) = u_0(x) - D\varphi(0,x)$, $x \in \overline{\Omega}$, by (2.8) we have $v_0 \in \overline{D(A)}$, where A is the operator defined by (2.2). Therefore, there exists a sequence $\{v_{0n}\}$ in $D(A)$ such that $\{v_{0n}\}$ converges *to v_0 in $C(\overline{\Omega})$* as $n \to +\infty$.

Let $\{f_n\}$ be a sequence in $C^1([0,T];C(\overline{\Omega}))$ such that f_n converges uniformly to f. Moreover, one can easily see that there exists a sequence $\{\varphi_n\}$ in $C^2([0,T];C(\partial\Omega))$ such that φ_n converges to φ in $C^\alpha([0,T];C(\partial\Omega))$ and for any $n \in \mathbb{N}$

$$\varphi_{nt}(0,x) = \Delta v_{0n}(x) + f_n(0,x), \quad x \in \partial\Omega. \tag{2.25}$$

Set

$$u_{0n}(x) = v_{0n}(x) + D\varphi_n(0,x), \quad x \in \overline{\Omega}, \tag{2.26}$$

it is easy to prove that u_{0n} converges to u_0. In addition, by (2.25) and (2.26) any u_{0n} fulfills the compatibility conditions (2.6) and (2.5), when φ and f are replaced, respectively, by φ_n and f_n.

In view of Corollary 2.4, there exists a unique strict solution $u_n(t,x)$ of the problem

$$\begin{cases} \dfrac{\partial u_n}{\partial t}(t,x) = \Delta u_n(t,x) + \displaystyle\int_0^t k(t-s)\Delta u_n(s,x)\,ds + f_n(t,x), & t \in [0,T], \ x \in \Omega \\ u_n(0,x) = u_{0n}(x), \quad x \in \Omega \\ u_n(t,x) = \varphi_n(t,x), \quad t \in [0,T], \ x \in \partial\Omega \end{cases} \tag{2.27}$$

given by

$$u_n(t,\cdot) = R(t)u_{0n} + \int_0^t R(t-s)f_n(s,\cdot)\,ds - \int_0^t R'(t-s)D\varphi_n(s,\cdot)\,ds, \quad t \in [0,T].$$

Let us show now that u_n and u verify the conditions of Definition 2.6. By (2.16), we get

$$\left\| \int_0^t R'(t-s)D(\varphi_n - \varphi)(s)\,ds \right\| \le (C_1 e^{r_0 T} T^\alpha \alpha^{-1} + C_1 e^{r_0 T} + 1)\|D\varphi_n - D\varphi\|_\alpha.$$

Therefore, thanks to the properties of the resolvent operator $R(t)$ (see §1), u_n converges to u in $C([0,T];C(\overline{\Omega}))$. The point (ii) of Definition 2.6 follows from (2.27). Moreover, from the uniform convergence of $u_n(t,x) = \varphi_n(t,x)$ to $\varphi(t,x)$ on $[0,T] \times \partial\Omega$ and from (i) of Definition 2.6, we have $u(t,x) = \varphi(t,x)$, $t \in [0,T]$, $x \in \partial\Omega$.

Finally, we must prove the last statement of the theorem. First of all, we observe that from (2.13), taking into account (2.14), we get

$$\begin{aligned} u(t,\cdot) = R(t)(u_0 - D\varphi(0,\cdot)) + \int_0^t R(t-s)f(s,\cdot)\,ds - R(t)(D\varphi(t,\cdot) - D\varphi(0,\cdot)) \\ + D\varphi(t,\cdot) - \int_0^t R'(t-s)(D\varphi(s,\cdot) - D\varphi(t,\cdot))\,ds, \quad t \in [0,T]. \end{aligned} \tag{2.28}$$

Since $\varphi(0,\cdot) \in C^{2\alpha}(\partial\Omega)$, by Schauder theorem we have $D\varphi(0,\cdot) \in C^{2\alpha}(\overline{\Omega})$. Therefore, by (2.5) and (2.19) $u_0 - D\varphi(0,\cdot) \in D_A(\alpha,\infty)$. We may apply Proposition 1.5, the function $R(t)(u_0 - D\varphi(0,\cdot)) + \int_0^t R(t-s)f(s,\cdot)\,ds$ belongs to $C^\alpha([0,T];C(\overline{\Omega}))$.

Let us check now that the function $R(t)\psi(t)$ belongs to $C^\alpha([0,T];C(\overline{\Omega}))$, where $\psi(t) = D\varphi(t,\cdot) - D\varphi(0,\cdot)$ is α-Hölder continuous (see Lemma 2.1). Indeed, for any $t,\tau \in [0,T]$, $t > \tau$ we have

$$R(t)\psi(t) - R(\tau)\psi(\tau) = R(t)(\psi(t) - \psi(\tau)) + \int_\tau^t R'(\sigma)\psi(\tau)\,d\sigma,$$

from which, taking into account (1.5) and $\psi(0) = 0$, it follows that

$$\begin{aligned}\|R(t)\psi(t) - R(\tau)\psi(\tau)\| &\le C_1 e^{r_0 t}[\psi]_\alpha (t-\tau)^\alpha + C_1 e^{r_0 t}\|\psi(\tau)\|\tau^{-\alpha}\int_\tau^t \sigma^{\alpha-1}\,d\sigma \\ &\le C_1 e^{r_0 t}[\psi]_\alpha (1+\alpha^{-1})(t-\tau)^\alpha.\end{aligned}$$

If we prove that the function $\int_0^t R'(t-s)(D\varphi(s)-D\varphi(t))\,ds$ belongs to $C^\alpha([0,T];C(\overline{\Omega}))$, then from (2.28) our statement follows. (Here we use again the convention $D\varphi(t) = D\varphi(t,\cdot)$.)

Let t, $\tau \in [0,T]$, $t > \tau$, be fixed; we have

$$\begin{aligned}&\int_0^t R'(t-s)(D\varphi(s) - D\varphi(t))\,ds - \int_0^\tau R'(\tau-s)(D\varphi(s) - D\varphi(\tau))\,ds \\ &= \int_\tau^t R'(t-s)(D\varphi(s) - D\varphi(t))\,ds + \int_0^\tau \Big(\int_{\tau-s}^{t-s} R''(\sigma)\,d\sigma\Big)(D\varphi(s) - D\varphi(\tau))\,ds \\ &\quad + (R(t) - R(t-\tau))(D\varphi(\tau) - D\varphi(t)).\end{aligned} \tag{2.29}$$

Using (1.5), we get

$$\Big\|\int_\tau^t R'(t-s)(D\varphi(s) - D\varphi(t))\,ds\Big\| \le C_1 e^{r_0 T}[D\varphi]_\alpha \alpha^{-1}(t-\tau)^\alpha \tag{2.30}$$

$$\begin{aligned}&\Big\|\int_0^\tau \Big(\int_{\tau-s}^{t-s} R''(\sigma)\,d\sigma\Big)(D\varphi(s) - D\varphi(\tau))\,ds\Big\| \\ &\le C_1 e^{r_0 T}[D\varphi]_\alpha \int_0^\tau \Big(\int_{\tau-s}^{t-s} \sigma^{-2}\,d\sigma\Big)(\tau-s)^\alpha\,ds \\ &\le C_1 e^{r_0 T}[D\varphi]_\alpha \int_0^\tau (\tau-s)^{\alpha-1}(t-s)^{-1}\,ds(t-\tau) \\ &= C_1 e^{r_0 T}[D\varphi]_\alpha \int_0^\tau (\tau-s)^{\alpha-1}\Big(1+\frac{\tau-s}{t-\tau}\Big)^{-1}\,ds \\ &\le \Big(C_1 e^{r_0 T}[D\varphi]_\alpha \int_0^{+\infty} y^{\alpha-1}(1+y)^{-1}\,dy\Big)(t-\tau)^\alpha\end{aligned} \tag{2.31}$$

Therefore, by (2.29), (2.30), (2.31) and (1.5) we gain

$$\begin{aligned}&\Big\|\int_0^t R'(t-s)(D\varphi(s) - D\varphi(t))\,ds - \int_0^\tau R'(\tau-s)(D\varphi(s) - D\varphi(\tau))\,ds\Big\| \\ &\le C_1 e^{r_0 T}[D\varphi]_\alpha\Big(\alpha^{-1} + \int_0^{+\infty} y^{\alpha-1}(1+y)^{-1}\,dy + 2\Big)(t-\tau)^\alpha.\end{aligned}$$

3. Bounded solutions of parabolic integrodifferential equation with infinite delay and nonhomogeneous boundary condition.

Let Ω be a bounded open set in $\mathbb{R}^n$, $n \in \mathbb{N}$, with regular boundary $\partial\Omega$ of class C^1.

This section deals with a class of parabolic integrodifferential equations with nonhomogeneous boundary conditions and infinite delay

$$\begin{cases} u_t(t,x) = \Delta u(t,x) + \int_{-\infty}^{t} k(t-s)\Delta u(s,x)\,ds + f(t,x), \quad t \in \mathbb{R},\ x \in \Omega \\ u(t,x) = \varphi(t,x), \quad t \in \mathbb{R},\ x \in \partial\Omega \end{cases} \tag{3.1}$$

where Δ is the Laplace operator, k verifies (1.2) and (1.15) and f, φ are known functions.

In order to solve problem (3.1) we need one more hypothesis. If A is the operator defined by (2.2), we must assume with the same notations of Section 1

$$i\mathbb{R} \subset \rho(A,k). \tag{3.2}$$

Repeating the same steps as in the proof of Lemma 2.1, it is easy to prove the following.

Lemma 3.1. *If ψ belongs to $C_b^{1,\alpha}(\mathbb{R};C(\partial\Omega))$ (resp. $h_b^\alpha(\mathbb{R};C(\partial\Omega))$), then $D\psi$ belongs to $C_b^{1,\alpha}(\mathbb{R};C(\overline{\Omega}))$ (resp. $h_b^\alpha(\mathbb{R};C(\overline{\Omega}))$), where D is Dirichlet map.*

Using Theorem 1.7, we shall prove some results about the existence, uniqueness and regularity of the bounded solutions of (3.1).

We first introduce the notion of strict solution of (3.1).

Definition 3.2. *A function $u : \mathbb{R}\times\overline{\Omega} \to \mathbb{R}$ is said to be a strict solution of (3.1) if u belongs to $C_b^1(\mathbb{R};C(\overline{\Omega}))$, Δu belongs to $C_b(\mathbb{R};C(\overline{\Omega}))$ and u fulfills (3.1).*

Theorem 3.3. *Assume (1.2), (1.15) and (3.2). Let $f \in C_b^\alpha(\mathbb{R};C(\overline{\Omega}))$ and $\varphi \in C_b^{1,\alpha}(\mathbb{R};C(\partial\Omega))$, $0<\alpha<1$. Then there exists a unique strict solution $u(t,x)$ of (3.1) such that $u \in C_b^{1,\alpha}(\mathbb{R};C(\overline{\Omega}))$ and $\Delta u \in C_b^\alpha(\mathbb{R};C(\overline{\Omega}))$.*

Proof: First of all, we observe that by Lemma 3.1, we have $D\varphi \in C_b^{1,\alpha}(\mathbb{R};C(\overline{\Omega}))$. We set

$$g(t,x) = f(t,x) - D\varphi_t(t,x), \quad t\in\mathbb{R},\ x\in\overline{\Omega}. \tag{3.3}$$

With the convention $g(t) = g(t,\cdot)$, we consider the abstract equation

$$v'(t) = Av(t) + \int_{-\infty}^{t} k(t-s)Av(s)\,ds + g(t), \quad t\in\mathbb{R}. \tag{3.4}$$

Thanks to our assumptions, g belongs to $C_b^\alpha(\mathbb{R};X)$, where $X = C(\overline{\Omega})$.

In view of Theorem 1.7-(i), the function

$$v(t,\cdot) = \int_{-\infty}^{t} R_-(t-s)g(s,\cdot)\,ds - \int_{t}^{+\infty} R_+(t-s)g(s,\cdot)\,ds, \quad t\in\mathbb{R}, \tag{3.5}$$

is the unique strict solution of (3.4) and v belongs to $C_b^\alpha(\mathbb{R};D(A)) \cap C_b^{1,\alpha}(\mathbb{R};X)$.

Let us check now that the function

$$u(t,x) = v(t,x) + D\varphi(t,x), \quad t\in\mathbb{R},\ x\in\overline{\Omega} \tag{3.6}$$

is a strict solution of equation (3.1). Since v is the solution of (3.4), u is differentiable in $\mathbb{R}$, and we get

$$u_t(t,x) = \Delta u(t,x) + \int_{-\infty}^{t} k(t-s)\Delta u(s,x)\,ds + f(t,x), \quad t\in\mathbb{R},\ x\in\Omega.$$

Because of $v(t,\cdot)\in D(A)$ for any $t\in\mathbb{R}$, we have $v(t,x)=0$, $x\in\partial\Omega$, so that $u(t,x)=\varphi(t,x)$ for every $t\in\mathbb{R}$, $x\in\partial\Omega$. The uniqueness follows from (3.6) and from the uniqueness of the solution of (3.4). Finally, (3.6) and the regularity of φ and v yield $u\in C_b^{1,\alpha}(\mathbb{R};C(\overline{\Omega}))$ and $\Delta u \in C_b^\alpha(\mathbb{R};C(\overline{\Omega}))$.

We now give a representational formula for the strict solution of (3.1).

Corollary 3.4. *Under the assumptions of* Theorem 3.3, *if, the strict solution of* (3.1) *is given by*

$$\begin{aligned} u(t,\cdot) = \int_{-\infty}^{t} R_-(t-s)f(s,\cdot)\,ds - \int_{-\infty}^{t} R'_-(t-s)D\varphi(s,\cdot)\,ds \\ - \int_{t}^{+\infty} R_+(t-s)f(s,\cdot)\,ds + \int_{t}^{+\infty} R'_+(t-s)D\varphi(s,\cdot)\,ds, \quad t \in \mathbb{R}. \end{aligned} \tag{3.7}$$

Proof: First of all, we show that the integrals $\int_{-\infty}^{t} R'_-(t-s)D\varphi(s)\,ds$ and $\int_{t}^{+\infty} R'_+(t-s)D\varphi(s)\,ds$ are well defined, with the convention $D\varphi(s) = D\varphi(s,\cdot)$. In fact,

$$\int_{-\infty}^{t} R'_-(t-s)D\varphi(s)\,ds = \int_{-\infty}^{t} R'_-(t-s)(D\varphi(s) - D\varphi(t))\,ds - R_-(0)D\varphi(t), \quad t \in \mathbb{R} \tag{3.8}$$

since, by (1.19), one has

$$\|R_-(t-s)D\varphi(t)\| \leq H_-(\epsilon)\|D\varphi\|_\infty e^{-(\omega_1-\epsilon)(t-s)}, \quad s < t,\ 0 < \epsilon < \omega_1.$$

The integral on the right hand side of (3.8) is convergent, because by (1.19), we have

$$\int_{-\infty}^{t} \|R'_-(t-s)(D\varphi(s) - D\varphi(t))\|\,ds \leq H_-(\epsilon)\|D\varphi\|_\alpha \int_{0}^{+\infty} e^{-(\omega_1-\epsilon)s}s^{\alpha-1}\,ds, \tag{3.9}$$

$0 < \epsilon < \omega_1$. Moreover, by (1.20) we obtain

$$\int_{t}^{+\infty} \|R'_+(t-s)D\varphi(s)\|\,ds \leq H_+(\epsilon)\|D\varphi\|_\infty \int_{t}^{+\infty} e^{(\omega_2-\epsilon)(t-s)}\,ds, \quad 0 < \epsilon < \omega_2.$$

We remark, for later convenience, that up to this point it was enough to assume $\varphi \in C_b^\alpha(\mathbb{R}; C(\partial\Omega))$. By (3.6), (3.5) and (3.3) we have

$$\begin{aligned} u(t,\cdot) = \int_{-\infty}^{t} R_-(t-s)f(s,\cdot)\,ds - \int_{-\infty}^{t} R_-(t-s)(D\varphi)'(s,\cdot)\,ds \\ - \int_{t}^{+\infty} R_+(t-s)f(s,\cdot)\,ds + \int_{t}^{+\infty} R_+(t-s)(D\varphi)'(s,\cdot)\,ds + D\varphi(t,\cdot), \quad t \in \mathbb{R}. \end{aligned} \tag{3.10}$$

Integrating by parts and using respectively (1.19) and (1.20) we get

$$\int_{-\infty}^{t} R_-(t-s)(D\varphi)'(s,\cdot)\,ds = R_-(0)D\varphi(t,\cdot) + \int_{-\infty}^{t} R'_-(t-s)D\varphi(s,\cdot)\,ds \tag{3.11}$$

$$\int_{t}^{+\infty} R_+(t-s)(D\varphi)'(s,\cdot)\,ds = -R_+(0)D\varphi(t,\cdot) + \int_{t}^{+\infty} R'_+(t-s)D\varphi(s,\cdot)\,ds. \tag{3.12}$$

Substituting (3.11) and (3.12) into (3.10) and using (1.18), one gets (3.7).

We *now define* the strong solution of equation (3.1).

Definition 3.5. *A function $u \in C_b(\mathbb{R}; C(\overline{\Omega}))$ is said to be a strong solution for* (3.1) *if*

$$u(t,x) = \varphi(t,x), \quad t \in \mathbb{R},\ x \in \partial\Omega \tag{3.13}$$

and there exists a sequence $\{u_n(t,x)\}_{n\in\mathbb{N}}$ of functions in $C_b^1(\mathbb{R}; C(\overline{\Omega}))$ such that $\Delta u_n \in C_b(\mathbb{R}; C(\overline{\Omega}))$ and for any compact interval I of $\mathbb{R}$ we have

(i) $\lim_{n\to+\infty} u_n = u$ *in* $C(I \times \overline{\Omega})$,

(ii) $\lim_{n\to+\infty} \left(\frac{\partial u_n}{\partial t}(t,x) - \Delta u_n(t,x) - \int_{-\infty}^t k(t-s)\Delta u_n(s,x)\,ds\right) = f(t,x)$ *in* $C(I \times \overline{\Omega})$.

Finally, we shall prove a theorem about the existence, uniqueness and regularity of the strong solution of (3.1).

Theorem 3.6. *Assume* (1.2), (1.15) *and* (3.2). *Let $f \in C_b(\mathbb{R}; C(\overline{\Omega}))$ and $\varphi \in h_b^\alpha(\mathbb{R}; C(\partial\Omega))$, $0 < \alpha < 1$. Then the function $u(t,x)$ given by* (3.7) *is the unique strong solution of* (3.1). *In addition, u belongs to $C_b^\alpha(\mathbb{R}; C(\overline{\Omega}))$.*

Proof: First of all, we observe that u given by (3.7) is well defined by the same argument used in the proof of Corollary 3.4, and (3.8)-(3.9) hold.

Let $\{f_n\}$ be a sequence in $C_b^1(\mathbb{R}; C(\overline{\Omega}))$ such that f_n converges uniformly to f on any compact interval of $\mathbb{R}$ and $\|f_n\|_\infty \le \|f\|_\infty$, $n \in \mathbb{N}$. Moreover, let $\{\varphi_n\}$ be a sequence in $C_b^2(\mathbb{R}; C(\partial\Omega))$ such that φ_n converges to φ in $C^\alpha(I; C(\partial\Omega))$ for any compact interval $I \subset \mathbb{R}$ and $\|\varphi_n\|_\alpha \le \|\varphi\|_\alpha$, $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, we consider the problem

$$\begin{cases} \dfrac{\partial u_n}{\partial t}(t,x) = \Delta u_n(t,x) + \displaystyle\int_{-\infty}^t k(t-s)\Delta u_n(s,x)\,ds + f_n(t,x), & t \in \mathbb{R},\ x \in \Omega \\ u_n(t,x) = \varphi_n(t,x), \quad t \in \mathbb{R},\ x \in \partial\Omega. \end{cases} \tag{3.14}$$

In view of Corollary 3.4, there exists a unique strict solution $u_n(t,x)$ of (3.14) given by

$$\begin{aligned} u_n(t,\cdot) = &\int_{-\infty}^t R_-(t-s) f_n(s,\cdot)\,ds - \int_{-\infty}^t R'_-(t-s) D\varphi_n(s,\cdot)\,ds \\ &- \int_t^{+\infty} R_+(t-s) f_n(s,\cdot)\,ds + \int_t^{+\infty} R'_+(t-s) D\varphi_n(s,\cdot)\,ds. \end{aligned} \tag{3.15}$$

Let us show now that u_n and u verify the conditions of Definition 3.5.

First of all, we prove that $\int_{-\infty}^t R'_-(t-s)D\varphi_n(s)\,ds$ converges uniformly to $\int_{-\infty}^t R'_-(t-s)D\varphi(s)\,ds$ on any compact interval $I \subset \mathbb{R}$ (here we use the convention $D\varphi_n(s) = D\varphi_n(s,\cdot)$ and $D\varphi(s) = D\varphi(s,\cdot)$). Let $0 < \epsilon < \omega_1$ be fixed and $t_0,\ t_1 \in \mathbb{R}$, $t_0 < t_1$, it is clear that $\int_{-\infty}^T e^{-(\omega_1-\epsilon)(t_0-s)}(t_0-s)^{\alpha-1}\,ds$ converges to 0 as $T \to -\infty$. Therefore, for any $\eta > 0$ there exists $T < t_0$ such that

$$\int_{-\infty}^T e^{-(\omega_1-\epsilon)(t_0-s)}(t_0-s)^{\alpha-1}\,ds < \eta. \tag{3.16}$$

Let $t \in [t_0, t_1]$; by (3.8), (1.19) and Lemma 3.1, we get

$$\begin{aligned} &\Big\| \int_{-\infty}^t R'_-(t-s) D(\varphi_n - \varphi)(s)\,ds \Big\| \le \\ &\le \int_{-\infty}^t \|R'_-(t-s)(D(\varphi_n-\varphi)(s) - D(\varphi_n-\varphi)(t))\|\,ds + \|R_-(0)D(\varphi_n-\varphi)(t)\| \\ &\le H_-(\epsilon)\int_{-\infty}^t e^{-(\omega_1-\epsilon)(t-s)}(t-s)^{-1}\|D(\varphi_n-\varphi)(s) - D(\varphi_n-\varphi)(t)\|\,ds \\ &\quad + \|R_-(0)\|\,\|\varphi_n-\varphi\|_{C^\alpha([t_0,t_1];C(\partial\Omega))}. \end{aligned} \tag{3.17}$$

In view of (3.16) and Lemma 3.1, one has

$$\int_{-\infty}^{t} e^{-(\omega_1-\epsilon)(t-s)}(t-s)^{-1}\|D(\varphi_n-\varphi)(s)-D(\varphi_n-\varphi)(t)\|\,ds$$
$$=\int_{-\infty}^{T} e^{-(\omega_1-\epsilon)(t-s)}(t-s)^{-1}\|D(\varphi_n-\varphi)(s)-D(\varphi_n-\varphi)(t)\|\,ds$$
$$+\int_{T}^{t} e^{-(\omega_1-\epsilon)(t-s)}(t-s)^{-1}\|D(\varphi_n-\varphi)(s)-D(\varphi_n-\varphi)(t)\|\,ds$$
$$\leq 2\|\varphi\|_\alpha\int_{-\infty}^{T} e^{-(\omega_1-\epsilon)(t_0-s)}(t_0-s)^{\alpha-1}\,ds$$
$$+\|\varphi_n-\varphi\|_{C^\alpha([T,t_1];C(\partial\Omega))}\int_{T}^{t} e^{-(\omega_1-\epsilon)(t-s)}(t-s)^{\alpha-1}\,ds$$
$$\leq 2\|\varphi\|_\alpha\eta+\|\varphi_n-\varphi\|_{C^\alpha([T,t_1];C(\partial\Omega))}\alpha^{-1}(t_1-T)^\alpha,$$

and substituting into (3.17), we gain

$$\Big\|\int_{-\infty}^{t} R'_-(t-s)D(\varphi_n-\varphi)(s)\,ds\Big\|$$
$$\leq 2H_-(\epsilon)\eta\|\varphi\|_\alpha+(H_-(\epsilon)\alpha^{-1}(t_1-T)^\alpha+\|R_-(0)\|)\|\varphi_n-\varphi\|_{C^\alpha([T,t_1];C(\partial\Omega))}.$$

Since φ_n converges to φ in $C^\alpha([T,t_1];C(\partial\Omega))$, our statement follows.

We show now that $\int_t^{+\infty} R'_+(t-s)D\varphi_n(s)\,ds$ converges uniformly to $\int_t^{+\infty} R'_+(t-s)D\varphi(s)\,ds$ on any compact interval of $\mathbb{R}$. Let $t_0,\ t_1\in\mathbb{R}$, $t_0<t_1$ and $0<\epsilon<\omega_2$. We note that $\int_T^{+\infty} e^{(\omega_2-\epsilon)(t_1-s)}\,ds$ converges to 0 as $T\to+\infty$. Therefore, for any $\eta>0$ there exists $T>t_1$ such that

$$\int_{T}^{+\infty} e^{(\omega_2-\epsilon)(t_1-s)}\,ds<\eta. \tag{3.18}$$

By (1.20) and (3.18), for any $t\in[t_0,t_1]$ we have

$$\Big\|\int_{t}^{+\infty} R'_+(t-s)D(\varphi_n-\varphi)(s)\,ds\Big\|\leq H_+(\epsilon)\int_{t}^{+\infty} e^{(\omega_2-\epsilon)(t-s)}\big\|D(\varphi_n-\varphi)(s)\big\|\,ds$$
$$=H_+(\epsilon)\int_{t}^{T} e^{(\omega_2-\epsilon)(t-s)}\|D(\varphi_n-\varphi)(s)\|\,ds+H_+(\epsilon)\int_{T}^{+\infty} e^{(\omega_2-\epsilon)(t-s)}\|D(\varphi_n-\varphi)(s)\|\,ds$$
$$\leq H_+(\epsilon)\|\varphi_n-\varphi\|_{C([t_0,T];C(\partial\Omega))}(T-t_0)+2H_+(\epsilon)\|\varphi\|_\infty\eta.$$

Similar arguments show that, $\int_{-\infty}^{t} R_-(t-s)f_n(s,\cdot)\,ds$ and $\int_t^{+\infty} R_+(t-s)f_n(s,\cdot)\,ds$ converge uniformly, respectively, to $\int_{-\infty}^{t} R_-(t-s)f(s,\cdot)\,ds$ and $\int_t^{+\infty} R_+(t-s)f(s,\cdot)\,ds$ on any compact interval of $\mathbb{R}$. Point (i) of Definition 3.5 is thus proved; (ii) follows from (3.14). Moreover, from the uniform convergence of $u_n(t,x)=\varphi_n(t,x)$ to $\varphi(t,x)$ on $I\times\partial\Omega$, for any compact interval $I\subset\mathbb{R}$, and from (i) of Definition 3.5, (3.13) follows.

Finally, it remains to be proved that u is α-Hölder continuous. By Theorem 1.7-(ii) the function $\int_{-\infty}^{t} R_-(t-s)f(s,\cdot)\,ds-\int_t^{+\infty} R_+(t-s)f(s,\cdot)\,ds$ is α-Hölder continuous. Let

$t, \tau \in \mathbb{R}$, $t > \tau$ and $0 < \epsilon < \omega_1$; by (3.8) we get

$$\begin{aligned}
&\int_{-\infty}^{t} R'_-(t-s)D\varphi(s)\,ds - \int_{-\infty}^{\tau} R'_-(\tau-s)D\varphi(s)\,ds \\
&= \int_{-\infty}^{t} R'_-(t-s)(D\varphi(s)-D\varphi(t))\,ds - \int_{-\infty}^{\tau} R'_-(\tau-s)(D\varphi(s)-D\varphi(\tau))\,ds \\
&\quad - R_-(0)(D\varphi(t)-D\varphi(\tau)) \\
&= \int_{\tau}^{t} R'_-(t-s)(D\varphi(s)-D\varphi(t))\,ds + \int_{-\infty}^{\tau} \Big(\int_{\tau-s}^{t-s} R''_-(\sigma)\,d\sigma\Big)(D\varphi(s)-D\varphi(\tau))\,ds \\
&\quad + (R_-(t-\tau)-R_-(0))(D\varphi(t)-D\varphi(\tau)).
\end{aligned} \tag{3.19}$$

Using (1.19) and repeating the same steps that led us to formulas (2.30) and (2.31), we have

$$\Big\| \int_{\tau}^{t} R'_-(t-s)(D\varphi(s)-D\varphi(t))\,ds \Big\| \le H_-(\epsilon)[D\varphi]_\alpha \alpha^{-1}(t-\tau)^\alpha; \tag{3.20}$$

$$\Big\| \int_{-\infty}^{\tau} \Big(\int_{\tau-s}^{t-s} R''_-(\sigma)\,d\sigma\Big)(D\varphi(s)-D\varphi(\tau))\,ds \Big\| \le \Big(H_-(\epsilon)[D\varphi]_\alpha \int_0^{+\infty} y^{\alpha-1}(1+y)^{-1}\,dy\Big)(t-\tau)^\alpha. \tag{3.21}$$

Therefore, by (3.19), (3.20), (3.21) and (1.19) we gain

$$\Big\| \int_{-\infty}^{t} R'_-(t-s)D\varphi(s)\,ds - \int_{-\infty}^{\tau} R'_-(\tau-s)D\varphi(s)\,ds \Big\| \le C(\epsilon,\alpha,\varphi)(t-\tau)^\alpha,$$

where $C(\epsilon,\alpha,\varphi)$ is a constant depending only on ϵ, α and φ.

On the other hand, the function $\int_t^{+\infty} R'_+(t-s)D\varphi(s)\,ds$ is Lipschitz continuous. Indeed, for any t, $\tau \in \mathbb{R}$, $t < \tau$ we have

$$\begin{aligned}
&\int_{t}^{+\infty} R'_+(t-s)D\varphi(s)\,ds - \int_{\tau}^{+\infty} R'_+(\tau-s)D\varphi(s)\,ds \\
&\qquad = \int_{t}^{\tau} R'_+(t-s)D\varphi(s)\,ds - \int_{\tau}^{+\infty} \Big(\int_{t-s}^{\tau-s} R''_+(\sigma)\,d\sigma\Big)D\varphi(s)\,ds.
\end{aligned} \tag{3.22}$$

For any $0 < \epsilon < \omega_2$, by (1.20) one gets

$$\Big\| \int_{t}^{\tau} R'_+(t-s)D\varphi(s)\,ds \Big\| \le H_+(\epsilon)\|D\varphi\|_\infty(\tau-t), \tag{3.23}$$

$$\Big\| \int_{\tau}^{+\infty} \Big(\int_{t-s}^{\tau-s} R''_+(\sigma)\,d\sigma\Big)D\varphi(s)\,ds \Big\| \le H_+(\epsilon)\|D\varphi\|_\infty(\omega_2-\epsilon)^{-1}(\tau-t). \tag{3.24}$$

Finally, from (3.22), (3.23) and (3.24), it follows that

$$\Big\| \int_{t}^{+\infty} R'_+(t-s)D\varphi(s)\,ds - \int_{\tau}^{+\infty} R'_+(\tau-s)D\varphi(s)\,ds \Big\| \le C(\epsilon,\varphi,\omega_2)(\tau-t),$$

where $C(\epsilon,\varphi,\omega_2)$ is a constant depending only on ϵ, φ and ω_2.

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