Differential and Integral Equations, Volume 3, Number 1, January 1990, pp. 155–160.

ON THE ZEROS OF SOLUTIONS OF HYPERBOLIC EQUATIONS OF NEUTRAL TYPE

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(Submitted by: A.R. Aftabizadeh)

Abstract. Hyperbolic equations of neutral type are studied and sufficient conditions are given that every solution of certain boundary value problems has a zero in bounded domains. The results are based on the condition for the non-existence of positive solutions of ordinary differential inequalities.

Recently there has been an increasing interest in studying the oscillatory behavior of solutions of partial differential equations of neutral type (see [1–3]). To the author's knowledge, the first attempt in this direction was made by Mishev and Bainov [1] who studied the hyperbolic equation of neutral type.

Let G be a bounded domain in $\mathbb{R}^n$ with smooth boundary ∂G, and let $\Omega = G \times (0,\infty)$. We are concerned with the oscillatory behavior of solutions of the hyperbolic equation of neutral type

$$u_{tt}(x,t) - [\Delta u(x,t) + \alpha \Delta u(x,t-\tau)] + c(x,t,u(x,t),u(x,t-\sigma)) = f(x,t), \quad (1)$$

$(x,t) \in \Omega$, where Δ is the Laplacian in $\mathbb{R}^n$. We consider three kinds of boundary conditions:

$$u = \psi \qquad \text{on } \partial G \times (0,\infty); \qquad (\mathrm{B}_1)$$

$$\frac{\partial u}{\partial \nu} = \tilde{\psi} \qquad \text{on } \partial G \times (0,\infty); \qquad (\mathrm{B}_2)$$

$$\frac{\partial u}{\partial \nu} + \mu u = 0 \quad \text{on } \partial G \times (0,\infty), \qquad (\mathrm{B}_3)$$

where ψ, $\tilde{\psi}$ are continuous functions on $\partial G \times (0,\infty)$, μ is a nonnegative continuous function on $\partial G \times (0,\infty)$ and ν denotes the unit exterior normal vector to ∂G. In [1], Mishev and Bainov obtained sufficient conditions for the existence of arbitrarily large zeros of solutions of the problem (1), (B_2). The purpose of this paper is to

Received September 22, 1988.
Partially supported by Grant-in-Aid for Scientific Research (No. 63540156), Ministry of Education, Japan.
AMS Subject Classifications: 35B05.

obtain sufficient conditions guaranteeing that there are bounded domains in which every solution of the boundary value problem has a zero.

We assume throughout this paper that

(A_1) $c(x,t,\xi,\eta)$ is a real-valued continuous function in $\overline{\Omega} \times \mathbb{R}^1 \times \mathbb{R}^1$;

(A_2) $c(x,t,\xi,\eta) \geq k_1^2\xi + k_2^2\eta$ for all $(x,t,\xi,\eta) \in \Omega \times [0,\infty) \times [0,\infty)$, where k_i $(i=1,2)$ are nonnegative constants;

(A_3) $c(x,t,\xi,\eta) \leq k_1^2\xi + k_2^2\eta$ for all $(x,t,\xi,\eta) \in \Omega \times (-\infty,0) \times (-\infty,0)$;

(A_4) $f(x,t)$ is a real-valued continuous function in $\overline{\Omega}$ and α, σ, τ are positive constants.

It is known that the first eigenvalue λ_1 of the eigenvalue problem

$$\Delta w + \lambda w = 0 \quad \text{in } G, \qquad w = 0 \quad \text{on } \partial G$$

is positive and the corresponding eigenfunction $\Phi(x)$ is positive in G. Associated with every function $u \in D(\Omega) \equiv C^2(\Omega) \cap C^1(\overline{\Omega})$, we define the functions $U(t)$ and $\tilde{U}(t)$ by

$$U(t) = \int_G u(x,t)\Phi(x)\,dx, \quad \tilde{U}(t) = \int_G u(x,t)\,dx, \quad t > 0.$$

We use the notation

$$F(t) = \int_G f(x,t)\Phi(x)\,dx, \qquad \tilde{F}(t) = \int_G f(x,t)\,dx, \quad t > 0,$$

$$\Psi(t) = \int_{\partial G} \psi(x,t)\frac{\partial \Phi}{\partial \nu}(x)\,dS, \quad \tilde{\Psi}(t) = \int_{\partial G} \psi(x,t)\,dS, \quad t > 0.$$

Theorem 1. *Assume that (A_1)–(A_4) hold. If there is a number $s \geq T$ such that*

$$H(s) \equiv \int_s^{s+\pi/L} (F(t) - \Psi(t) - \alpha\Psi(t-\tau))\sin L(t-s)\,dt = 0, \tag{2}$$

then every solution $u \in D(\Omega)$ of the problem (1), (B_1) has a zero in $G \times (s-T, s+\pi/L)$, where $T = \max\{\sigma,\tau\}$ and $L = (\lambda_1 + k_1^2)^{1/2}$.

Proof: Suppose to the contrary that there is a solution u of the problem (1), (B_1) which has no zero in $G \times (s-T, s+\pi/L)$. Let $u > 0$ in $G \times (s-T, s+\pi/L)$. Multiplying (1) by $\Phi(x)$ and integrating over G, we obtain

$$\begin{aligned} U''(t) &- \int_G [\Delta u(x,t) + \alpha\Delta u(x,t-\tau)]\Phi(x)\,dx \\ &+ \int_G c(x,t,u(x,t),u(x,t-\sigma))\Phi(x)\,dx = F(t). \end{aligned} \tag{3}$$

It follows from Green's formula that

$$\begin{aligned} \int_G (\Delta u(x,t))\Phi(x)\,dx &= \int_{\partial G}\left(\frac{\partial u}{\partial \nu}\Phi - u\frac{\partial \Phi}{\partial \nu}\right)dS + \int_G u\Delta\Phi\,dx \\ &= -\int_{\partial G}\psi\frac{\partial \Phi}{\partial \nu}\,dS - \lambda_1\int_G u\Phi\,dx = -\Psi(t) - \lambda_1 U(t). \end{aligned} \tag{4}$$

Analogously, we have

$$\int_G (\Delta u(x,t-\tau))\Phi(x)\,dx = -\Psi(t-\tau) - \lambda_1 U(t-\tau). \tag{5}$$

Combining (3)–(5) yields

$$\begin{aligned} &U''(t) + \lambda_1 U(t) + \lambda_1 \alpha U(t-\tau) + \int_G c(x,t,u(x,t),u(x,t-\sigma))\Phi(x)\,dx \\ &\quad = F(t) - \Psi(t) - \alpha\Psi(t-\tau). \end{aligned} \tag{6}$$

Since $T \geq \sigma$, we see that $u(x,t-\sigma) \geq 0$ in $G \times [s, s+\pi/L)$. Assumption (A_2) implies that

$$\int_G c(x,t,u(x,t),u(x,t-\sigma))\Phi(x)\,dx \geq k_1^2 U(t) + k_2^2 U(t-\sigma). \tag{7}$$

Combining (6) with (7), we obtain

$$U''(t) + (\lambda_1 + k_1^2)U(t) + \lambda_1\alpha U(t-\tau) + k_2^2 U(t-\sigma) \leq F(t) - \Psi(t) - \alpha\Psi(t-\tau),$$

$t \in [s, s+\pi/L)$. Since $u > 0$ in $G \times (s-T, s+\pi/L)$, we observe that $U(t-\tau) \geq 0$ and $U(t-\sigma) \geq 0$ in $[s, s+\pi/L)$. Hence, $U(t)$ is a positive solution of

$$U''(t) + (\lambda_1 + k_1^2)U(t) \leq F(t) - \Psi(t) - \alpha\Psi(t-\tau), \quad t \in [s, s+\pi/L). \tag{8}$$

The hypothesis implies that (8) has no positive solution in $[s, s+\pi/L)$ (see Yoshida [4, Remark 2]). This is a contradiction. If $u < 0$ in $G \times (s-T, s+\pi/L)$, $V(t) \equiv \int_G -u(x,t)\Phi(x)\,dx$ satisfies

$$\begin{aligned} &V''(t) + \lambda_1 V(t) + \lambda_1\alpha V(t-\tau) - \int_G c(x,t,u(x,t),u(x,t-\sigma))\Phi(x)\,dx \\ &\quad = -(F(t) - \Psi(t) - \alpha\Psi(t-\tau)). \end{aligned}$$

Proceeding as in the case where $u > 0$, we are led to a contradiction. The proof is complete.

Corollary 1. *Assume that (A_1)–(A_4) hold. If the function $H(s)$ defined by* (2) *is oscillatory at $s = \infty$, then every solution $u \in D(\Omega)$ of the problem* (1), (B_1) *is oscillatory in Ω.*

Proof: The hypothesis implies that for any $t > 0$ there exists a number $s > t + T$ such that (2) holds. From Theorem 1 we see that every solution u of the problem (1), (B_1) has a zero in $G \times (s-T, s+\pi/L)$. Hence, every solution u is oscillatory in Ω.

Example 1. We consider the problem

$$\begin{aligned} &u_{tt}(x,t) - [u_{xx}(x,t) + u_{xx}(x,t-(\pi/2))] + u(x,t) = (\sin t - \cos t)\sin x, \\ &\quad (x,t) \in (0,\pi) \times (0,\infty), \end{aligned} \tag{9}$$

$$u(0,t) = u(\pi,t) = 0, \qquad t > 0. \tag{10}$$

Here $n = 1$, $\alpha = 1$, $\tau = \pi/2$, $G = (0,\pi)$, $k_1 = 1$, $k_2 = 0$ and $f(x,t) = (\sin t - \cos t)\sin x$. It is easy to see that $\lambda_1 = 1$, $L = 2^{1/2}$, $\Phi(x) = \sin x$, $\Psi(t) \equiv 0$ and $F(t) = (\pi/2)(\sin t - \cos t)$. Since

$$H(s) = \int_s^{s+\pi/2^{1/2}} F(t)\sin 2^{1/2}(t-s)\,dt = 2\pi\sin(s-(\pi/4)+(\pi/2^{3/2}))\cos\pi/2^{3/2},$$

we observe that $H(s) = 0$ for $s = s_n = (5/4 - 1/2^{3/2})\pi + n\pi$ ($> \pi/2$), $n = 0,1,2,\ldots$. Theorem 1 implies that every solution $u \in D(\Omega)$ of (9), (10) has a zero in $G \times (s_n - \pi/2, s_n + \pi/2^{1/2})$. One such solution is $u = \sin t \cdot \sin x$.

Theorem 2. *Assume that (A_1)–(A_4) hold, and that $k_1 > 0$. If there is a number $s \geq T$ such that*

$$\tilde{H}(s) \equiv \int_s^{s+\pi/k_1} (\tilde{F}(t) + \tilde{\Psi}(t) + \alpha\tilde{\Psi}(t-\tau))\sin k_1(t-s)\,dt = 0, \tag{11}$$

then every solution $u \in D(\Omega)$ of the problem (1), (B_2) has a zero in $G \times (s-T, s+\pi/k_1)$.

Proof: Suppose that there is a solution u of the problem (1), (B_2) which has no zero in $G \times (s-T, s+\pi/k_1)$. We may assume that $u > 0$ in $G \times (s-T, s+\pi/k_1)$. Integration of (1) over G gives

$$\begin{aligned} \tilde{U}''(t) - \left[\int_{\partial G}\tilde{\psi}(x,t)\,dS + \alpha\int_{\partial G}\tilde{\psi}(x,t-\tau)\,dS\right] \\ + \int_G c(x,t,u(x,t),u(x,t-\sigma))\,dx = \tilde{F}(t), \end{aligned} \tag{12}$$

(cf. Mishev and Bainov [1]). Since $T \geq \sigma$, it holds that $u(x,t-\sigma) \geq 0$ in $G \times [s, s+\pi/k_1)$. In view of assumption (A_2) we obtain

$$\int_G c(x,t,u(x,t),u(x,t-\sigma))\,dx \geq k_1^2\tilde{U}(t) + k_2^2\tilde{U}(t-\sigma) \geq k_1^2\tilde{U}(t), \quad t \in [s, s+\pi/k_1). \tag{13}$$

Combining (12) with (13), we find that $U(t)$ is a positive solution of

$$\tilde{U}''(t) + k_1^2\tilde{U}(t) \leq \tilde{F}(t) + \tilde{\Psi}(t) + \alpha\tilde{\Psi}(t-\tau), \quad t \in [s, s+\pi/k_1). \tag{14}$$

From the hypothesis it follows that (14) has no positive solution in $[s, s+\pi/k_1)$ (see [4, Remark 2]). This is a contradiction and the proof is complete.

Corollary 2. *Assume that (A_1)–(A_4) hold, and that $k_1 > 0$. If the function $H(s)$ defined by (11) is oscillatory at $s = \infty$, then every solution $u \in D(\Omega)$ of the problem (1), (B_2) is oscillatory in Ω.*

Proof: By Theorem 2 we see that every solution u of the problem (1), (B_2) has arbitrarily large zeros in Ω.

Example 2. We consider the problem

$$u_{tt}(x,t) - [u_{xx}(x,t) + u_{xx}(x,t-(\pi/2))] + u(x,t) = (\sin t + \cos t)\sin x, \quad (x,t) \in (0,\pi)\times(0,\infty), \tag{15}$$

$$-u_x(0,t) = u_x(\pi,t) = -\cos t, \quad t > 0. \tag{16}$$

Here $n = 1$, $\alpha = 1$, $\tau = \pi/2$, $G = (0,\pi)$, $k_1 = 1$, $k_2 = 0$ and $f(x,t) = (\sin t + \cos t)\sin x$. An easy calculation shows that $\widetilde{F}(t) = 2(\sin t + \cos t)$, $\widetilde{\Psi}(t) = -2\cos t$ and $\widetilde{\Psi}(t-(\pi/2)) = -2\sin t$. Hence, $\widetilde{F}(t) + \widetilde{\Psi}(t) + \widetilde{\Psi}(t-(\pi/2)) = 0$, $t > 0$. It follows from Theorem 2 that every solution $u \in D(\Omega)$ of (15), (16) has a zero in $G \times (s-(\pi/2), s+\pi)$ for any $s \geq T = \pi/2$. For example, $u = \cos t \cdot \sin x$ is such a solution.

Theorem 3. *Assume that (A_1)–(A_4) hold, and that $k_1 > 0$. If there is a number $s \geq T$ such that*

$$\widetilde{H}_0(s) \equiv \int_s^{s+\pi/k_1} \widetilde{F}(t)\sin k_1(t-s)\,dt = 0, \tag{17}$$

then every solution $u \in D(\Omega)$ of the problem (1), (B_3) had a zero in $G \times (s-T, s+\pi/k_1)$.

Proof: Suppose that there is a solution u of the problem (1), (B_3) such that $u > 0$ in $G \times (s-T, s+\pi/k_1)$. Integrating (1) over G, we obtain

$$\widetilde{U}''(t) - \left[\int_{\partial G} \frac{\partial u}{\partial \nu}(x,t)\,dS + \alpha \int_{\partial G} \frac{\partial u}{\partial \nu}(x,t-\tau)\,dS\right] + \int_G c(x,t,u(x,t),u(x,t-\sigma))\,dx = \widetilde{F}(t). \tag{18}$$

Since $T \geq \sigma$ and $T \geq \tau$, we observe that

$$\int_{\partial G} \frac{\partial u}{\partial \nu}(x,t)\,dS = -\int_{\partial G} \mu(x,t)u(x,t)\,dS \leq 0, \quad t \in [s, s+\pi/k_1), \tag{19}$$

$$\int_{\partial G} \frac{\partial u}{\partial \nu}(x,t-\tau)\,dS = -\int_{\partial G} \mu(x,t-\tau)u(x,t-\tau)\,dS \leq 0, \quad t \in [s, s+\pi/k_1), \tag{20}$$

$$\int_G c(x,t,u(x,t),u(x,t-\sigma))\,dx \geq k_1^2\widetilde{U}(t) + k_2^2\widetilde{U}(t-\sigma) \geq k_1^2\widetilde{U}(t), \quad t \in [s, s+\pi/k_1). \tag{21}$$

Combining (18)–(21) yields

$$\widetilde{U}''(t) + k_1^2\widetilde{U}(t) \leq \widetilde{F}(t), \quad t \in [s, s+\pi/k_1).$$

Arguing as in the proof of Theorem 2, we are led to a contradiction. The proof is complete.

The following corollary is an immediate consequence of Theorem 3.

Corollary 3. *Assume that* (A_1)–(A_4) *hold, and that* $k_1 > 0$. *If the function* $\widetilde{H}_0(s)$ *defined by* (17) *is oscillatory at* $s = \infty$, *then every solution* $u \in D(\Omega)$ *of the problem* (1), (B_3) *is oscillatory in* Ω.

Example 3. We consider the problem

$$u_{tt}(x,t) - [u_{xx}(x,t) + u_{xx}(x,t-\pi)] + 4u(x,t) = 3\sin t \cdot \sin(\frac{x}{2} + \frac{\pi}{4}),$$
$$(x,t) \in (0,\pi) \times (0,\infty), \tag{22}$$

$$-u_x(0,t) + 2^{-1}u(0,t) = u_x(\pi,t) + 2^{-1}u(\pi,t) = 0, \quad t > 0. \tag{23}$$

Here $n = 1$, $\alpha = 1$, $\tau = \pi$, $G = (0,\pi)$, $k_1 = 2$, $k_2 = 0$, $\mu = 2^{-1}$ and $f(x,t) = 3\sin t \cdot \sin(\frac{x}{2} + \frac{\pi}{4})$. It is easily seen that $\widetilde{F}(t) = 6 \cdot 2^{1/2} \sin t$ and

$$\int_s^{s+\pi/2} 6 \cdot 2^{1/2} \sin t \sin 2(t-s)\, dt = 8\sin(s + (\pi/4)).$$

Theorem 3 implies that every solution $u \in D(\Omega)$ of (22), (23) has a zero in $G \times (s-\pi, s+(\pi/2))$ for $s = s_n = (3/4)\pi + n\pi$ $(n = 1, 2, \ldots)$. One such solution is $u = \sin t \cdot \sin(\frac{x}{2} + \frac{\pi}{4})$.

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