Differential and Integral Equations, Volume 3, Number 2, March 1990, pp. 295–303.

A THEOREM OF THE KREIN-RUTMAN TYPE FOR AN INTEGRO-DIFFERENTIAL OPERATOR

FRANCISCO JULIO SOBREIRA DE ARAUJO CORREA
Departamento de Matemática e Estatística, Universidade Federal da Paraíba
Centro de Ciências e Tecnologia, 58.100 – Campina Grande – PB – Brazil

(Submitted by: Peter Hess)

Abstract. It is proved a theorem of the Krein-Rutman type for the problem $-\Delta u + Bu = \lambda mu$ in Ω, $u = 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $B = \delta(-\Delta + \gamma)^{-1}$ under Dirichlet boundary conditions, (δ, γ) belongs to some unbounded region of $(\mathbb{R}^+ - \{0\}) \times (\mathbb{R}^+ - \{0\})$, λ is a real parameter and m is a continuous indefinite weight function assuming a positive value at some point $x_0 \in \Omega$.

1. Introduction. During recent years, many authors have studied reaction-diffusion systems derived from several applications, such as mathematical biology, chemical reactions and combustion theory, among other physical phenomena. See, for example [4], [6] and [9].

In this work, we study the elliptic system

$$\begin{aligned} &-\Delta u = \lambda f(x,u) - v \quad \text{in } \Omega, \quad -\Delta v = \delta u - \gamma v \quad \text{in } \Omega, \\ &u = v = 0 \quad \text{on } \partial\Omega, \quad \lambda \geq 0 \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a given nonlinearity and δ, $\gamma \in \mathbb{R}^+ - \{0\}$. The solutions (u, v) of this problem represent steady-state solutions of a reaction-diffusion system derived from mathematical biology. Problem (1) is equivalent to the integro-differential problem

$$\begin{aligned} -\Delta u + \delta(-\Delta + \gamma)^{-1} u &= \lambda f(x,u) \quad \text{in } \Omega, \\ u = 0 & \qquad\qquad\quad \text{on } \partial\Omega, \ \lambda \geq 0 \end{aligned} \tag{2}$$

where $v = \delta(-\Delta + \gamma)^{-1} u$ and $\delta(-\Delta + \gamma)^{-1}$ is considered under Dirichlet boundary conditions.

To study some questions related to problem (2), such as bifurcation and stability, among other things, very often we need to know some information about eigenvalues and eigenfunctions of the linear eigenvalue problem

$$-\Delta u + \delta(-\Delta + \gamma)^{-1} u = \lambda mu \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega \tag{3}$$

Received December 2, 1988.
Partially supported by CNPq/Brazil.
AMS Subject Classifications: 35J65, 35J50, 47H15.

where $m \in C(\bar{\Omega})$ is, in general, an indefinite weight. In particular, the first positive eigenvalue and its corresponding eigenfunction play a fundamental role in some questions related to (2). From now on we suppose that $m \in C(\bar{\Omega})$ and $m(x_0) > 0$ for some $x_0 \in \Omega$.

It is well known that in the scalar case

$$Lu = \lambda mu \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega \tag{4}$$

where L is a formally self-adjoint elliptic operator for which a maximum principle holds, we have the following result: *"The first positive eigenvalue of* (4) *is simple, its associated eigenfunction φ may be taken positive in Ω and the exterior normal derivative $\partial\varphi/\partial n$ is negative on $\partial\Omega$."* This result is due to Manes-Micheletti [7].

A similar result was proved by Hess-Kato [5] if L is an elliptic non self-adjoint operator for which a maximum principle is valid. Results of this nature are known as the Theorem of the Krein-Rutman type. In [1], a partial result of this type is proved for problem (2). More precisely, we have shown that for each γ there is $\delta_0 = \delta_0(\gamma)$ such that a theorem of the Krein-Rutman type is true for all (δ, γ) with $\delta \in (0, \delta_0)$.

Our goal in this work is to prove a Krein-Rutman type theorem for (3) which is valid for the values of (δ, γ) in some unbounded region of $\mathbb{R}^+ \times \mathbb{R}^+$. The fundamental tools in both proofs of Manes-Micheletti [7] and Hess-Kato [5] is the maximum principle for L. But the nonexistence of a general maximum principle for the operator $-\Delta + \delta(-\Delta + \gamma)^{-1}$, see [2], imposes serious difficulties to prove a Krein-Rutman type theorem for (3) for all $(\delta, \gamma) \in \mathbb{R}^+ \times \mathbb{R}^+$. In view of this, we suspect that a theorem of this sort is not true for all $(\delta, \gamma) \in \mathbb{R}^+ \times \mathbb{R}^+$.

This work has the following structure: in section 2 we establish some notations and state the main results, in section 3 we provide the proof of the Krein-Rutman type theorem while in section 4 we apply this result to a nonlinear eigenvalue problem.

2. Some notations and statement of results. Problem (3) will be studied in its variational formulation

$$(u, v)_{\delta,\gamma} = \lambda \int muv, \quad \text{for all} \quad v \in H_0^1(\Omega) \tag{5}$$

where $(u, v)_{\delta,\gamma} = (u, v)_{H^1} + (\delta(-\Delta + \gamma)^{-1}u, v)_{L^2}$, $(u, v)_{H^1} = \int \nabla u \cdot \nabla v$ and $(u, v)_{L^2}$ is the inner product in $L^2 \equiv L^2(\Omega)$. Hereafter, the integral $\int u\,dx$ will be denoted by $\int u$. The bilinear form on the left side of (5) defines in $H_0^1 \equiv H_0^1(\Omega)$ an inner product equivalent to the usual one $\int \nabla u \cdot \nabla v$, in view of Poincare's inequality and $(\delta(-\Delta + \gamma)^{-1}u, u)_{L^2} \geq 0$ for all $u \in L^2$. We denote by $\|\cdot\|_{\delta,\gamma}$ the norm associated to $(\ ,\ )_{\delta,\gamma}$ and by $\|\cdot\|_{H^1}$ the norm derived from $(\ ,\ )_{H^1}$.

For fixed $u \in H_0^1$, the map $v \mapsto \int muv$ is a bounded linear functional in H_0^1. So, by the Riesz-Frechet representation theorem there is an element in H_0^1, denote it by $T(\delta, \gamma)u$, such that

$$(T(\delta, \gamma)u, v)_{\delta,\gamma} = \int muv \quad \text{for all} \quad v \in H_0^1(\Omega). \tag{6}$$

Thus, problem (5) may be rewritten in the equivalent forms

$$(u,v)_{\delta,\gamma} = \lambda(T(\delta,\gamma)u,v)_{\delta,\gamma} \quad \text{for all} \quad v \in H_0^1 \quad \text{or} \quad \lambda T(\delta,\gamma)u = u.$$

It is easy to show that $T(\delta,\gamma) : H_0^1 \to H_0^1$ is a linear, compact and self-adjoint operator. As in [3], we can use the spectral theory for compact self-adjoint operators to show the existence of a sequence of real eigenvalues of (5) satisfying $0 < \hat{\lambda}_1(\delta,\gamma) \leq \hat{\lambda}_2(\delta,\gamma) \leq \dots \hat{\lambda}_n(\delta,\gamma) \to +\infty$. In fact, $\hat{\lambda}_n(\delta,\gamma)$ also depends on m, but for the sake of simplicity, we adopt the above notation. If there is $\bar{x}_0 \in \Omega$ such that $m(\bar{x}_0) < 0$, we obtain also a sequence of eigenvalues $(\hat{\lambda}_{-n}(\delta,\gamma))_{n\in\mathbb{N}}$ satisfying $\cdots \leq \hat{\lambda}_{-2}(\delta,\gamma) \leq \hat{\lambda}_{-1}(\delta,\gamma) < 0$ and $\hat{\lambda}_{-n}(\delta,\gamma) \to -\infty$. We denote by $\varphi_1(\delta,\gamma)$ any eigenfunction of (5) associated with $\hat{\lambda}_1(\delta,\gamma)$. By a standard bootstrap, argument it is proved that $\varphi_1(\delta,\gamma) \in C^1(\bar{\Omega})$. Let us denote by λ_1 the first eigenvalue of $(-\Delta, H_0^1)$ and by $\lambda_1(m)$ the first positive eigenvalue of

$$-\Delta u = \lambda m u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{7}$$

Using spectral theory for linear compact self-adjoint operators, we obtain the following characterizations:

$$\frac{1}{\lambda_1} = \sup\left\{\frac{\int u^2}{\|u\|_{H^1}^2}; \quad 0 \neq u \in H_0^1\right\}$$

and

$$\frac{1}{\lambda_1(m)} = \sup\left\{\frac{\int mu^2}{\|u\|_{H^1}^2}; \quad 0 \neq u \in H_0^1\right\}.$$

Let us denote by E_M the set

$$E_M = \left\{(\delta,\gamma);\ \delta,\gamma > 0 \text{ and } (1 + \delta\lambda_1^{-1}(\lambda_1+\gamma)^{-1})\lambda_1(m)M \leq \gamma - 2\sqrt{\delta}\right\}$$

where M is a fixed number strictly greater than $\|m\|_\infty$.

Remark 1. The set E_M has the following geometrical interpretation. Note that the inequality

$$(1 + \delta\lambda_1^{-1}(\lambda_1+\gamma)^{-1})\lambda_1(m)M \leq \gamma - 2\sqrt{\delta}$$

is equivalent to

$$\lambda_1\gamma^2 + \lambda_1(\lambda_1 - 2\sqrt{\delta} - \lambda_1(m)M)\gamma - \lambda_1^2\lambda_1(m)M - \lambda_1(m)M\delta - 2\lambda_1^2\sqrt{\delta} \geq 0. \tag{8}$$

For each $\delta > 0$, the expression on the left side of (8) is a quadratic function of γ for which the independent term is negative. So this function has exactly one positive root which will be denoted by $h(\delta)$. A simple computation shows that

$$h(\delta) = \frac{2\sqrt{\delta} + \lambda_1(m)M - \lambda_1 + (\lambda_1(\lambda_1 + 2\sqrt{\delta} + \lambda_1(m)M)^2 + 4\lambda_1(m)M\delta)^{\frac{1}{2}}}{2\lambda_1^{\frac{1}{2}}}$$

and

$$h'(\delta) = \frac{1}{2\sqrt{\delta}} + \frac{1}{\sqrt{\delta}} \cdot \frac{\lambda_1(\lambda_1 + 2\sqrt{\delta} + \lambda_1(m)M) + \lambda_1(m)M\delta}{\lambda_1^{\frac{1}{2}}((\lambda_1 + 2\sqrt{\delta} + \lambda_1(m)M)^2 + 4\lambda_1(m)M\delta))^{\frac{1}{2}}}.$$

It is easy to see that $(\delta,\gamma) \in E_M$, if and only if (δ,γ) belongs to the epigraph of the function $h : (0,+\infty) \to (0,+\infty)$. That is, $(\delta,\gamma) \in E_M$ if and only if $\gamma \geq h(\delta)$. The main result of this work is the following.

Theorem 1. *If $(\delta,\gamma) \in E_M$, the first positive eigenvalue $\hat{\lambda}_1(\delta,\gamma)$ of* (5) *is simple, $\varphi_1(\delta,\gamma)$ can be taken positive in Ω and the exterior normal derivative $\frac{\partial\varphi_1(\delta,\gamma)}{\partial n}$ is negative on $\partial\Omega$. A similar statement holds if $m(\bar{x}_0) < 0$ for some $\bar{x}_0 \in \Omega$*

Remark 2. Let us consider the linear eigenvalue problem

$$-\Delta u + \delta(-\Delta+\gamma)^{-1}u = \lambda u \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega. \tag{9}$$

As a consequence of the maximum principle due to de Figueiredo-Mitidieri [4], we have that $(-\Delta + \delta(-\Delta+\gamma)^{-1})^{-1} : C_0^1(\bar{\Omega}) \to C_0^1(\bar{\Omega})$ is a strongly positive operator if $\gamma \geq 2\sqrt{\delta}$ and, consequently, a theorem as before holds for (9) if (δ,γ) is in the epigraph of the function $\delta \mapsto 2\sqrt{\delta}$ for all $\delta > 0$. We also observe that the functions $\delta \mapsto 2\sqrt{\delta}$ and $h(\delta)$ have some similar behaviour. In fact, these functions are both increasing, differentiable and their derivatives go to infinity if $\delta \to 0^+$, among other similarities. Moreover, if $\gamma \geq h(\delta)$ then $\gamma > 2\sqrt{\delta}$.

As an application of the above result is studied the nonlinear eigenvalue problem

$$\begin{aligned} &-\Delta u = \lambda f(x,u) - v, \quad -\Delta v = \delta u - \gamma v \quad \text{in} \quad \Omega, \\ &u = v = 0 \quad \text{on} \quad \partial\Omega, \ \lambda \geq 0 \end{aligned} \tag{10}$$

in its equivalent integro-differential form

$$\begin{aligned} &-\Delta u + \delta(-\Delta+\gamma)^{-1}u = \lambda f(x,u) \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega, \lambda \geq 0 \end{aligned} \tag{11}$$

where $v = \delta(-\Delta+\gamma)^{-1}u$ and $\delta(-\Delta+\gamma)^{-1}$ is considered under Dirichlet boundary conditions. Here, we suppose that $f : \bar{\Omega} \times \mathbb{R}^+ \mapsto \mathbb{R}$ satisfies

(f) $f(x,t) = m(x)t + h(t)$, $h(0) = 0$, h is continuously differentiable and $h'(0) = 0$.

In section 4 is proved the following result:

Theorem 2. *If $f : \bar{\Omega} \times \mathbb{R}^+ \mapsto \mathbb{R}$ satisfies (f) and $(\delta,\gamma) \in E_M$, then there exists a continuum $\Sigma \subset \mathbb{R}^+ \times C_0^1(\bar{\Omega})$ containing $(\hat{\lambda}_1(\delta,\gamma), 0)$ and consisting (except for this point) of positive solutions of* (11).

3. Proof of Theorem 1.

Let A be the following set:

$$A = \left\{(\delta,\gamma) \in E_M;\ \varphi_1(\delta,\gamma) > 0 \text{ in } \Omega \text{ and } \tfrac{\partial\varphi_1(\delta,\gamma)}{\partial n} < 0 \text{ on } \partial\Omega\right\}.$$

First we shall prove that $A = E_M$. Since E_M is a connected set, it is sufficient to show that A is nonempty and A is open and closed in E_M. These facts will follow from the lemmas below.

Lemma 1. *A is nonempty.*

Proof: Let δ be a positive fixed number and consider a sequence (γ_n) satisfying $0 < \gamma_n \to +\infty$ if $n \to +\infty$. Hence,

$$\left| \|u\|^2_{\delta,\gamma_n} - \|u\|^2_{H^1} \right| \leq \delta(\lambda_1 + \gamma_n)^{-1}\|u\|^2_{L^2} \to 0 \ \text{ if } n \to +\infty$$

for each $u \in H_0^1$. Thus, $\|u\|_{\delta,\gamma_n}^2 \to \|u\|_{H^1}^2$ if $n \to +\infty$ which implies that

$$\frac{\int mu^2}{\|u\|_{\delta,\gamma_n}^2} \to \frac{\int mu^2}{\|u\|_{H^1}^2} \quad \text{for each } 0 \neq u \in H_0^1.$$

On the other hand, we have $\|u\|_{\delta,\gamma_n}^2 \geq \|u\|_{H^1}^2$ and so

$$\frac{\int mu^2}{\|u\|_{H^1}^2} \geq \frac{\int mu^2}{\|u\|_{\delta,\gamma_n}^2} \quad \text{for all } u \in H_0^1 \text{ such that } \int mu^2 > 0.$$

Thus, $\lambda_1(m) \leq \hat{\lambda}_1(\delta,\gamma_n)$. Suppose, by contradiction, that $(1/\hat{\lambda}_1(\delta,\gamma_n)) \not\to (1/\lambda_1(m))$. Thus, there exists $\varepsilon > 0$ such that $(1/\hat{\lambda}_1(\delta,\gamma_n)) + \varepsilon \leq (1/\lambda_1(m))$ for infinitely many values of n. Let $u_0 \in H_0^1$ be satisfying $(1/\lambda_1(m)) = (\int mu_0^2/\|u_0\|_{H^1}^2)$. Hence,

$$\frac{\int mu_0^2}{\|u_0\|_{\delta,\gamma_n}^2} + \varepsilon \leq \frac{\int mu_0^2}{\|u_0\|_{H^1}^2}$$

which implies, by making $n \to +\infty$, that $\varepsilon \leq 0$, which is a contradiction. Consequently $\hat{\lambda}_1(\delta,\gamma_n) \to \lambda_1(m)$. Now we consider arbitrary eigenfunctions $\varphi_1(\delta,\gamma_n)$ associated to $\hat{\lambda}_1(\delta,\gamma_n)$ and normalized in $C^1(\bar{\Omega})$, that is, $\|\varphi_1(\delta,\gamma_n)\|_{C^1} = 1$. In view of this and using elliptic regularity, we obtain $\|\varphi_1(\delta,\gamma_n)\|_{W^{2,p}} \leq C$ for all $n \in \mathbb{N}$. Choosing $p > N$ we find $\varphi \in C^1(\bar{\Omega})$ such that $\varphi_1(\delta,\gamma_n) \to \varphi$ in $C^1(\bar{\Omega})$, eventually for a subsequence. Since

$$\int \nabla\varphi_1(\delta,\gamma_n)\cdot\nabla v + \int \varphi_1(\delta,\gamma_n)\cdot\delta(-\Delta+\gamma_n)^{-1}v = \hat{\lambda}_1(\delta,\gamma_n)\int m\varphi_1(\delta,\gamma_n)v,$$

for all $v \in H_0^1$, $\varphi_1(\delta,\gamma_n) \to \varphi$ in $C^1(\bar{\Omega})$, $\delta(-\Delta+\gamma_n)^{-1}v \to 0$ in $L^2(\Omega)$ and $\hat{\lambda}_1(\delta,\gamma_n) \to \lambda_1(m)$, we have

$$\int \nabla\varphi\cdot\nabla v = \lambda_1(m)\int m\varphi v, \quad \text{for all } v \in H_0^1$$

Thus, we may take $\varphi > 0$ in Ω and $\partial\varphi/\partial n < 0$ on $\partial\Omega$. Consequently, we can find $n_0 \in \mathbb{N}$ such that $\varphi_1(\delta,\gamma_n) > 0$ in Ω and $\partial\varphi_1(\delta,\gamma_n)/\partial n < 0$ on $\partial\Omega$ because $\varphi_1(\delta,\gamma_n) \to \varphi$ in $C^1(\bar{\Omega})$. So, if n_0 is sufficiently large, $(\delta,\gamma_n) \in E_M$ if $n \geq n_0$, which implies that A is nonempty.

Lemma 2. *A is open in E_M.*

Proof: We shall prove that E_M/A is closed in E_M. For this purpose, consider the sequence $(\delta_n,\gamma_n) \in E_M/A$ converging to $(\delta_0,\gamma_0) \in E_M$. We must show that $(\delta_0,\gamma_0) \in E_M/A$. Suppose, by contradiction, that $(\delta_0,\gamma_0) \in A$. First of all we shall show that $\|u\|_{\delta_n,\gamma_n} \to \|u\|_{\delta_0,\gamma_0}$ for all $u \in H_0^1$. For this we estimate the right member of the identity below

$$\left|\|u\|_{\delta_n,\gamma_n}^2 - \|u\|_{\delta_0,\gamma_0}^2\right| = \left|\delta_n((-\Delta+\gamma_n)^{-1}u,u)_{L^2} - \delta_0((-\Delta+\gamma_0)^{-1}u,u)_{L^2}\right|.$$

Setting $u_n = (-\Delta + \gamma_n)^{-1}u$ and $u_0 = (-\Delta + \gamma_0)^{-1}u$, we have

$$\begin{aligned} -\Delta u_n + \gamma_n u_n = u, \quad -\Delta u_0 + \gamma_0 u_0 = u \ \text{ in } \ \Omega, \\ u_n = u_0 = 0 \ \text{ on } \ \partial\Omega, \end{aligned}$$

for all $n \in \mathbb{N}$. Subtracting the above equations, we obtain

$$\begin{aligned} -\Delta(u_n - u_0) + \gamma_0(u_n - u_0) = (\gamma_0 - \gamma_n)u_n \ \text{ in } \ \Omega, \\ u_n - u_0 = 0 \ \text{ on } \ \partial\Omega \end{aligned}$$

and using elliptic regularity

$$\|u_n - u_0\|_{L^2} \leq C|\gamma_0 - \gamma_n| \|u_n\|_{L^2} \leq C'|\gamma_0 - \gamma_n| \|u\|_{L^2}$$

which implies that $u_n \to u_0$ in L^2. Hence,

$$\begin{aligned} &\left| \|u\|^2_{\delta_n,\gamma_n} - \|u\|^2_{\delta_0,\gamma_0} \right| \leq \\ &|\delta_n - \delta_0| ((-\Delta + \gamma_n)^{-1}u, u)_{L^2} + \delta_0 |(u_n - u_0, u)_{L^2}| \to 0 \quad (n \to +\infty). \end{aligned}$$

Since $\lambda_1(\delta_n, \gamma_n) \leq (\|u\|^2_{\delta_n,\gamma_n})/(\int mu^2)$ for all $u \in H^1_0$ satisfying $\int mu^2 > 0$, we have

$$\limsup_{n\to+\infty} \hat{\lambda}_1(\delta_n, \gamma_n) \leq \frac{\|u\|^2_{\delta_0,\gamma_0}}{\int mu^2}$$

for all u as before. Consequently, $\limsup_{n\to+\infty} \hat{\lambda}_1(\delta_n, \gamma_n) \leq \hat{\lambda}_1(\delta_0, \gamma_0)$ and we may choose a subsequence of $(\hat{\lambda}_1(\delta_n, \gamma_n))$ which converges to some $\lambda \geq 0$. But

$$\begin{aligned} -\Delta\varphi_1(\delta_n, \gamma_n) = \hat{\lambda}_1(\delta_n, \gamma_n) m\varphi_1(\delta_n, \gamma_n) - \delta_n(-\Delta + \gamma_n)^{-1}\varphi_1(\delta_n, \gamma_n) \ \text{ in } \ \Omega, \\ \varphi_1(\delta_n, \gamma_n) = 0 \ \text{ on } \ \partial\Omega \end{aligned}$$

where $\varphi_1(\delta_n, \gamma_n)$ is normalized in $C^1(\bar{\Omega})$. This implies that $\|\varphi_1(\delta_n, \gamma_n)\|_{W^{2,p}} \leq C$ and $\varphi_1(\delta_n, \gamma_n) \to \varphi$ in $C^1(\bar{\Omega})$, eventually for a subsequence, for some $\varphi \in C^1(\bar{\Omega})$ if $p > N$. Proceeding as in the proof of Lemma 1 we obtain

$$\int \nabla\varphi \cdot \nabla v + \int \delta_0(-\Delta + \gamma_0)^{-1}\varphi \cdot v = \lambda \int m\varphi v, \ \text{for all } v \in H^1_0.$$

Since $\varphi \not\equiv 0$, λ cannot be equal to zero. Because $0 < \lambda \leq \hat{\lambda}_1(\delta_0, \gamma_0)$ and $\hat{\lambda}_1(\delta_0, \gamma_0)$ is the first positive eigenvalue of the above problem, then $\lambda = \hat{\lambda}_1(\delta_0, \gamma_0)$. Furthermore, we may suppose $\varphi > 0$ in Ω and $(\partial\varphi/\partial n) < 0$ on $\partial\Omega$ because $(\delta_0, \gamma_0) \in A$. Thus, there is $n_0 \in \mathbb{N}$ such that $\varphi_1(\delta_n, \gamma_n) > 0$ in Ω and $(\partial\varphi_1(\delta_n, \gamma_n)/\partial n) < 0$ on $\partial\Omega$ for all $n \geq n_0$. Then, $(\delta_n, \gamma_n) \in A$ if $n \geq n_0$, which is a contradiction. Consequently, E_M/A is closed in E_M.

Lemma 3. *A is closed in* E_M.

Proof: Let (δ_n,γ_n) be a sequence in A which converges to $(\delta,\gamma)\in E_M$. Thus, $\varphi_1(\delta_n,\gamma_n)>0$ in Ω, $(\partial\varphi_1(\delta_n,\gamma_n)/\partial n)<0$ on $\partial\Omega$ and $\varphi_1(\delta_n,\gamma_n)\to\varphi_1(\delta,\gamma)$ in $C^1(\bar\Omega)$. Here we consider $\|\varphi_1(\delta_n,\gamma_n)\|_{C^1}=1$ for all $n=1,2,\dots$. Hence, $\varphi_1(\delta,\gamma)\geq 0$ in Ω and $(\partial\varphi_1(\delta,\gamma)/\partial n)\leq 0$ on $\partial\Omega$. Thus,

$$\begin{aligned}-\Delta\varphi_1(\delta,\gamma)+\delta(-\Delta+\gamma)^{-1}\varphi_1(\delta,\gamma)+\hat\lambda_1(\delta,\gamma)M\varphi_1(\delta,\gamma)&\\ =\hat\lambda_1(\delta,\gamma)(m+M)\varphi_1(\delta,\gamma)\ \text{ in }\Omega,&\qquad(12)\\ \varphi_1(\delta,\gamma)=0\ \text{ on }\partial\Omega.&\end{aligned}$$

Note that $\hat\lambda_1(\delta,\gamma)=\hat\lambda_1(\delta,\gamma,m)$ and

$$\hat\lambda_1(\delta,\gamma,m)\leq\frac{\|u\|^2_{H^1}+\delta((-\Delta+\gamma)^{-1}u,u)_{L^2}}{\int mu^2}$$

for all $u\in H^1_0$ satisfying $\int mu^2>0$. But,

$$|\delta((-\Delta+\gamma)^{-1}u,u)|_{L^2}\leq\delta\|(-\Delta+\gamma)^{-1}u\|_{L^2}\cdot\|u\|_{L^2}\leq\delta(\lambda_1+\gamma)^{-1}\|u\|^2_{L^2}$$

which implies that

$$\hat\lambda_1(\delta,\gamma,m)\leq(1+\delta{\lambda_1}^{-1}(\lambda_1+\gamma)^{-1})\frac{\|u\|^2_{H_1}}{\int mu^2}.$$

Taking the inf on the right hand side over all $u\in H^1_0$ with $\int mu^2>0$ we get

$$\hat\lambda_1(\delta,\gamma,m)M\leq(1+\delta{\lambda_1}^{-1}(\lambda_1+\gamma)^{-1})\lambda_1(m)M.$$

Since $(\delta_n,\gamma_n)\in A\subset E_M$ and $(\delta_n,\gamma_n)\to(\delta,\gamma)$ we have that

$$(1+\delta\lambda_1^{-1}(\lambda_1+\gamma)^{-1})\lambda_1(m)M\leq\gamma-2\sqrt{\delta}.$$

This implies that $\hat\lambda_1(\delta,\gamma,m)M\leq\gamma-2\sqrt{\delta}$. So we can apply the maximum principle due to de Fiqueiredo-Mitidieri [4] to problem (12) to conclude that $\varphi_1(\delta,\gamma)>0$ in Ω and $(\partial\varphi_1(\delta,\gamma)/\partial n)<0$ on $\partial\Omega$. So $(\delta,\gamma)\in A$ which implies that A is closed in E_M.

These three lemmas imply that $A=E_M$.

Lemma 4. *If* $(\delta,\gamma)\in E_M$, *then* $\hat\lambda_1(\delta,\gamma)$ *is simple.*

The proof of this lemma is performed in the same way as it is done by de Figueiredo [3] for the scalar case. In view of this we will omit it.

Thus, Theorem 1 follows immediately from the above lemmas.

4. A nonlinear eigenvalue problem. In this section, we shall study problem (1) in its equivalent form (2). In order to use Rabinowitz's bifurcation theorem, see [8], we rewrite problem (2) in the form of a functional equation in the space

$C_0^1(\bar\Omega) = \{u \in C^1(\bar\Omega); u = 0 \text{ on } \partial\Omega\}$ equipped with the cone P of positive functions. For this, consider $\tilde f : \bar\Omega\times\mathbb{R} \to \mathbb{R}$ the odd extension in t of the function $f : \bar\Omega\times\mathbb{R}^+ \to \mathbb{R}$ and denote by $\tilde F$ the Nemytskii operator associated to $\tilde f$. We now consider the functional equation

$$u = \lambda(-\Delta + B)^{-1}\tilde F(u) \quad \text{for } (\lambda, u) \in \mathbb{R}^+ \times C_0^1(\bar\Omega) \tag{13}$$

where $B = \delta(-\Delta+\gamma)^{-1}$, $v = Bu$ and (δ,γ) is a fixed element in E. Note that if u is a positive solution of (13) then u is a positive solution of (2).

Remark 3. If $m \le 0$ on $\bar\Omega$ we may show, as in Lemma 9 of [5], that there is no bifurcation point $(\lambda, 0) \in \mathbb{R}^+ \times C_0^1(\bar\Omega)$ for positive solutions of (11).

Remark 4. If we denote by M the operator $Mu = mu$ we have that $(-\Delta+B)^{-1}M : C_0^1(\bar\Omega) \to C_0^1(\bar\Omega)$ is a well defined, linear and compact operator. By a bootstrap argument we may show that this operator and $T(\delta,\gamma) : H_0^1 \to H_0^1$ have coincident eigenvalues associated to the same eigenfunctions. Also, it is possible to show that $\hat\lambda_1(\delta,\gamma)$ is a simple characteristic value of $(-\Delta+B)^{-1}M : C_0^1(\bar\Omega) \to C_0^1(\bar\Omega)$.

Proof of Theorem 2: The proof of this result is a straightforward adaptation of the one performed by Hess-Kato [5] to the scalar case. Let us denote by $\tilde H : C_0^1(\bar\Omega) \to C_0^1(\bar\Omega)$ the Nemytskii operator associated to an odd extension of h. So equation (13) is equivalent to

$$u = \lambda(-\Delta+B)^{-1}Mu + \lambda(-\Delta+B)^{-1}\tilde H(u), \quad (\lambda, u) \in \mathbb{R}^+ \times C_0^1(\bar\Omega). \tag{14}$$

Since $\hat\lambda_1(\delta,\gamma)$ has algebraic multiplicity one and $(-\Delta+B)^{-1}\tilde H(u) = o(\|u\|_{C^1})$ for $u \to 0$, the global bifurcation theorem due to Rabinowitz [8] guarantees the existence of a continuum C of solutions of (14), bifurcating from $(\hat\lambda_1(\delta,\gamma), 0)$, which is unbounded or contains another point $(\lambda, 0)$, where $(\lambda, 0)$ is a charateristic value of $(-\Delta+B)^{-1}M$.

Next, we shall show the existence of a neighborhood U of $(\hat\lambda_1(\delta,\gamma),0)$ in $\mathbb{R}\times C_0^1(\bar\Omega)$ such that $(C/\mathbb{R}\times\{0\}) \cap U \subset \mathbb{R}^+ \times \operatorname{int} P$. In fact, suppose that there is a sequence $(\lambda_n, u_n) \in C/\mathbb{R}\times\{0\}$ with $\lambda_n \to \hat\lambda_1(\delta,\gamma)$, $u_n \to 0$ in $C_0^1(\bar\Omega)$ and $u_n \notin \operatorname{int} P$. Setting $v_n = \|u_n\|_{C^1}^{-1} u_n$ we obtain from (14) the following equation:

$$v_n = \lambda_n(-\Delta+B)^{-1}Mv_n + \lambda(-\Delta+B)^{-1}\frac{\tilde H(u_n)}{\|u_n\|_{C^1}}.$$

Using the compactness of $(-\Delta+B)^{-1}$ and the closedness of $-\Delta+B$, we conclude that $v_n \to v$ in $C_0^1(\bar\Omega)$, eventually for a subsequence, $\|v\|_{C^1} = 1$, $v \notin \operatorname{int} P$ and $\hat\lambda_1(\delta,\gamma)v = (-\Delta+B)^{-1}Mv$ which implies that $\hat\lambda_1(\delta,\gamma)T(\delta,\gamma)v = v$. Thus, $v \in \operatorname{int} P$ which is a contradiction. Hence, there is a local continuum $\Sigma \subset (C/\mathbb{R}\times\{0\}) \cap U \subset \mathbb{R}^+ \times \operatorname{int} P$ consisting of positive solution of (14) bifurcating from $(\hat\lambda_1(\delta,\gamma), 0)$.

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