Differential and Integral Equations, Volume 3, Number 3, May 1990, pp. 503–522.

FINITE RANK, RELATIVELY BOUNDED PERTURBATIONS OF SEMI-GROUPS GENERATORS. PART III: A SHARP RESULT ON THE LACK OF UNIFORM STABILIZATION

R. TRIGGIANI

Department of Applied Mathematics, University of Virginia
Charlottesville, Virginia 22903 USA

(Submitted by: G. Da Prato)

Abstract. We provide a result of negative character on the lack of uniform stabilization when a group generator A on a Hilbert space Y is additively perturbed by a (typically nondissipative) perturbation operator which is A-bounded and of finite rank (range). Applications include conservative systems (waves and plate equations) with feedback operators on the boundary of the spatial domain. Such result is sharp in two directions: (i) within the class of finite range perturbations, and (ii) within the class of A-bounded perturbations. Indeed, uniform (exponential) stabilization may indeed occur in the first case (i), provided that the perturbation is just "ε worse" than A-bounded (constructive example of an hyperbolic mixed problem with boundary damping provided); and in the second case (ii), provided that the perturbation is A-compact rather than just finite rank (constructive example of an elastic system provided).

1. Introduction, summary of results, literature. Let Y be a (separable) Hilbert space with inner product (,). Let $A : Y \supset \mathcal{D}(A) \to Y$ be a (closed, densely defined) linear operator, which is assumed to be the generator of a s.c. (strongly continuous) semigroup or group of operators on Y, conveniently denoted by $\exp[At]$. Let a and b be two arbitrary vectors in Y and consider the abstract dynamics

$$\dot{y} = Ay + (Ay, a)b, \quad y(0) = y_0 \in Y \tag{1.1}$$

$$A_F \equiv A + (A\cdot, a)b, \quad \mathcal{D}(A_F) \supset \mathcal{D}(A) \tag{1.2}$$

$$P \equiv (A\cdot, a)b. \tag{1.3}$$

We note that the perturbation operator P is A-bounded (or relatively bounded with respect to A, [15], one-dimensional range[1], and, typically, non-dissipative. Thus,

Received Feburary 13, 1989.
AMS Subject Classifications: 35, 47.
Research partially supported by the Air Force Office of Scientific Research under Grant AFOSR 84-0365.
Presented at International Conference in honor of A.V. Balakrishnan, held at Washington D.C., June 1987, and at International Conference, Analysis and Optimization of Systems, held at INRIA, Antibes, France, June 1988.

[1]For simplicity of writing we take one-dimensional range perturbations, even though the main results here plainly extend to finite rank (range) perturbations.

P is unclosable and has A-bound equal to zero [15, p. 166 and p. 196]. Moreover, P is A-compact [15, p. 195]. The dynamics (1.1) was extensively studied in recent works [16], [17], [30], [31] with motivation coming from and ultimately directed to, boundary feedback partial differential equations defined on an open, bounded domain Ω of $\mathbb{R}^n$ (e.g. parabolic, hyperbolic and fourth order beam or plate-like equations). In these references, it was also documented that it may well be more advantageous to consider, in place of (1.1), its dual version

$$\dot{z} = A^*[z + a(b,z)_Z], \quad z(0) = z_0 \in Z \tag{1.4}$$

$$A_F^* \equiv A^*[\cdot + a(b,\cdot)_Z] \tag{1.5}$$

written on a (separable) Hilbert space Z which in applications may turn out to be a more "desirable" space than Y, smaller and equipped with a stronger topology than Y itself. Thus, in these cases, A_F^* in (1.5) is, technically, the restriction to $Z \subset Y$ of the adjoint operator to A_F in (1.2).

More precisely, the following issues concerning the dynamics (1.1) (or (1.4)) have been recently investigated in these references:

(i) well-posedness of problem (1.1); in particular, generation of a s.c. semigroup by the perturbed operator A_F—or lack thereof—when A is assumed to be the generator of a s.c. group [16], [30];

(ii) spectrum assignment for the operator A_F, given A and $a \in Y$, by means of a suitable vector $b \in Y$; alternatively, given A, by means of a suitable pair of vectors $a, b \in Y$ [17];

(iii) spectrality (in the sense of [7], [9, Vol. III])—or lack thereof—when A is assumed to be spectral; in particular, the existence of a Riesz basis of eigenvectors of A_F [17], [31].

(iv) simultaneous achievement of (ii) and (iii).

These works provide both positive and negative answers (counterexamples) to the above questions, with a rather clearly defined area of transition between these two classes of results. Finally, these works culminate with a study of the boundary feedback *stabilization* problem of (1.1), and thus of the underlying "concrete" dynamics—modelled by (1.1) or (1.4)—with feedback action exercised on the boundary of the spatial domain $\Omega \subset \mathbb{R}^n$.

The present note intends to complement the companion papers [16, 17, 30, 31] by focusing more specifically on the question of *uniform* (exponential) stabilization—or lack thereof—for the dynamics (1.1) (or (1.4)), when A is the generator of a s.c. group which is uniformly bounded for negative times. Applications which we have in mind will include "concrete" boundary feedback equations of *hyperbolic* type, of current interest in boundary control theory. Again, our main results presented here are of both negative and positive character, with a well-identified transition between them.

We begin with A being the generator of a s.c. group, which is uniformly bounded for negative times. Then, on the negative side, we show that precisely under those conditions on the pair of data A and $a \in Y$ which guarantee [16], that for any $b \in Y$ the operator A_F in (1.2) is the generator of a s.c. (feedback) semigroup $S_F(t) \equiv \exp[A_F t]$ on Y, such semigroup satisfies $\|S_F(t)\| \geq 1$, $t > 0$ and thus $S_F(t)$ fails to be uniformly stable (see Theorem 2.1 below). This result on the *lack* of uniform stabilization for the dynamics (1.1) greatly extends the well-known and much quoted result of D. Russell [28] (see also J.S. Gibson [10]) from the case where the perturbation operator is compact on the (infinite dimensional) space Y to the case where the perturbation operator is only assumed to be A-bounded and of finite dimensional range (and hence unclosable); i.e. P as in (1.3); and, moreover, it relaxes the condition on the group by requiring only that it be uniformly bounded, rather than contraction as in [28], for negative times. Our recent improvement [35] on [10] is reported as Theorem 2.2 in the case of compact perturbations.

An illustration of this "negative" situation is provided by the wave equation with "interior observation" of the velocity acting as a feedback applied in the Dirichlet boundary conditions, as in problem (3.1) below. Accordingly, such a feedback system is never *uniformly* stable (on its natural space of well-posedness); yet it may well be *strongly* stable on this space (for appropriate choices of the feedback vectors w and g in (3.1c) below), see [18].

On the positive side, we show in Sections 4 and 5 that our Theorem 2.1 on the lack of uniform stabilization is *sharp* in two directions of potential extension.

(i) First, Theorem 2.1 is sharp within the class of finite range perturbations, with A generator of a s.c. group which is uniformly bounded for negative times, in the sense that the perturbation cannot be "worse" than A-bounded. In fact, in Section 4 we analyze a one-dimensional wave equation as in (4.1) below with "boundary observation" of the velocity acting in the Neumann boundary conditions. This system turns out to possess the following properties: (i) the perturbation operator $\mathcal{P}$ due to the boundary feedback is indeed one-dimensional range, but "barely" fails to be A-bounded, with A being the usual s.c. unitary group generator of the free dynamics; hence such perturbation $\mathcal{P}$ "barely" fails to be of the form P as in (1.3), and the feedback system likewise "barely" fails to admit the abstract representation (1.2); (ii) the overall feedback system may indeed be made *uniformly* (exponentially) stable (on its natural space of well-posedness); indeed with arbitrarily preassigned decay rate (exponential) by appropriate choices of the boundary vectors w, g in (4.1c) below. This uniform stabilization result is obtained, a fortiori, from the precise structure of the feedback solutions, which are displayed explicitly as *series* solutions in terms of the (nonorthogonal) eigenfunctions of the underlying feedback generator A_F and its corresponding eigenvalues. Crucial in this respect is the property that these eigenfunctions form a Riesz basis (in the natural space of well-posedness) and thus location of the spectrum of the group generator A_F does determine the uniform growth or decay of all feedback solutions. By appropriate choices of the boundary vectors w, g in (4.1c) below, the eigenvalues of A_F may be forced to lie on any preassigned vertical axis in Re $\lambda < 0$, thereby achieving an arbitrarily preassigned uniform decay rate (exponential).

(ii) Second, Theorem 2.1 is sharp within the class of A-bounded perturbations, in the sense that the assumption that the perturbation be of finite range cannot be relaxed to read that it be only A-compact. A class of examples of recent origin [6], arising from elastic systems with "gentle" dissipation is given in Section 5 to illustrate this point.

We conclude: in order to hope to obtain *uniform* stabilization with *finite range* perturbations, one needs to have perturbations with a higher degree of unboundedness than the operator of the free dynamics.

2. Lack of uniform stabilization and the feedback semigroup. The setup and results obtained in Part I, [16], permit one to readily obtain the following theorem on the lack of *uniform* stabilization of the feedback semigroup generated by A_F in (1.2) which clarifies [18, Propos. 3.1].

Theorem 2.1. *Let A be the generator of a s.c. group $G(t)$ on Y satisfying*

$$\|G(-t)\| \leq \text{ const for } t \geq 0. \tag{2.1}$$

Assume that for $a \in Y$ we have:

$$\begin{cases} (My)(t) \equiv (AG(t)y, a) \\ M: \text{ initially defined for } y \in \mathcal{D}(A) \\ \qquad \text{can be extended to be continuous } Y \to L_1(0,T), \end{cases} \tag{2.2}$$

see (2.4) of [16] (as it was verified in all applications of Section 3 of [16]). Then, for any $b \in Y$, the s.c. semigroup $S_F(t)$ on Y, $t \geq 0$, generated by the (feedback) operator $A_F = A + (A\cdot, a)b$ satisfies

$$\|S_F(t)\| \geq 1, \quad t \geq 0. \tag{2.3}$$

Proof: Generation by A_F of a s.c. semigroup $S_F(t)$ on Y, $t \geq 0$, is guaranteed by Theorem 2.1 of [16] under assumption (2.2). (In general, i.e. for general $a, b \in Y$, the operator A_F in (1.2) does *not* generate a s.c. semigroup on Y, even in the case $a \in \mathcal{D}(A^{*1-\varepsilon})$, $\varepsilon > 0$ arbitrarily preassigned, and hence with P being only A^ε-bounded; see counterexamples in [16], [30].) Writing $A = A_F - (A\cdot, a)b$, we have for $y_0 \in Y$

$$G(t)y_0 = S_F(t)y_0 = \int_0^t S_F(t-\tau)b(AG(\tau)y_0, a)\, d\tau. \tag{2.4}$$

Now, with $t > 0$ fixed the operator

$$f(\cdot) \in L_1(0,T) \to \int_0^t S_F(t-\tau)bf(\tau)\, d\tau \in Y$$

is compact, since the set $\{S_F(t-\tau)b,\ 0 \leq \tau \leq t\}$ is a compact set in Y [8, p. 369], [9, Vol. I, p. 507]. This fact, combined with assumption (2.2), yields that for $t > 0$ fixed the operator

$$y_0 \in Y \to \int_0^t S_F(t-\tau)b(AG(\tau)y_0, a)\, d\tau \equiv [S_F(t) - G(t)]y_0 \in Y$$

is compact on Y. Thus, applying $G(-t)$, we obtain that the operator $K_t \equiv G(-t)S_F(t) - I$ is compact on Y. Hence

$$1 \leq \|G(-t)S_F(t)\| \tag{2.5}$$

for, otherwise, the compact operator K_t would be boundedly invertible on the infinite dimensional space Y, which is impossible. Then (2.5) implies the desired conclusion (2.3) via (2.1). In fact, if (2.3) fails and thus, as is well-known, by the semigroup properties, we have $\|S_F(t)\| \leq Me^{-\delta t}$, $t \geq 0$, for some $\delta > 0$, then (2.5) implies

$$\|G(-t)\| \geq \frac{1}{M} e^{\delta t} \to +\infty$$

which contradicts (2.1).

Remark 2.1. Since Theorem 2.1 of [16] proved that, in fact, $(AS_F(t)y_0, a) \in L_1(0,T)$, $y_0 \in Y$, we could equally well apply the above argument to

$$S_F(t)y_0 = G(t)y_0 + \int_0^t G(t-\tau)b(AS_F(\tau)y_0, a)\, d\tau \tag{2.6}$$

instead of (2.4).

Remark 2.2. Note that $\|G(-t)\| \leq 1$ implies $\|G(t)\| \geq 1$, $t \in \mathbb{R}$, (from $I = G(-t)G(t)$). Thus, assumption (2.1) implies the obvious precondition that $G(t)$ is *not* uniformly stable to begin with, as in the case of the damped wave equation. (Indeed, [31] and Application 4.4 in [17] provide an analysis of the damped wave equation (with "interior observation" of the position and the velocity acting as a nontrivial feedback in the Dirichlet boundary conditions), where assumption (2.1) fails and uniform stabilization holds true.) The converse implication $\|G(t)\| \geq 1 \to \|G(-t)\| \leq 1$ is *not* always true (see [13, p. 665] for one example where $\|G(t)\| = \exp[(1/2)\pi|t|]$, $t \in \mathbb{R}$); however, it is true when the generator A of $G(t)$ is normal, or more generally, is a scalar type operator [7], [9, III] (in particular, when A has a Riesz basis of eigenvectors).

Remark 2.3. The argument below (2.5) may be used to relax Russell's result [28] from the requirement $\|G(-t)\| \leq 1$, $t \geq 0$ assumed in [28] to the weaker assumption, (2.1), also in the case of a bounded compact perturbation.

For use in the examples below, we quote also a recent result on the lack of uniform stabilization when the perturbation is compact.

Theorem 2.2 [35]. *Let Y be a Hilbert space, or more generally a reflexive Banach space with the "approximation property."*

Assume the following hypotheses.

(i) *Either*[2] *$G(t)$ is strongly stable; i.e.*

$$G(t)y \to 0 \ \text{ as } t \to +\infty, \forall y \in Y, \tag{2.7}$$

or else $G^*(t)$ is strongly stable, i.e.

$$G^*(t)y \to 0 \ \text{ as } t \to +\infty, \forall y \in Y; \tag{2.8}$$

(ii) *B is a bounded compact operator on Y;*

(iii) *the semigroup $S_B(t)$, $t \geq 0$, generated by the operator $A_B = A + B$, is uniformly (exponentially) stable*

$$\|S_B(t)\| \leq Me^{-\delta t}, \quad t \geq 0 \tag{2.9}$$

for some positive constants M, δ.

---

[2]Conditions (1.1) and (1.2) are not equivalent, in general. They are, however, if A (equivalently, A^*) has compact resolvent on X [B.1, p. 245]. Reflexivity is needed only in connection with case (i).

Then, $G(t)$ is uniformly (exponentially) stable.

Theorem 2.2, which improves upon a previous result of Gibson [10], by among other things, removing the assumption that $G(t)$, $t \geq 0$ be contraction is also a sharp result within the class of compact perturbations, in the sense that if assumption (i) regarding (2.7), (2.8) is removed from the Theorem, then its conclusion is false [35].

3. Illustration of Theorem 2.1: a hyperbolic boundary feedback system which may be strongly stable, yet cannot be uniformly stable. We consider the following hyperbolic second order equation in feedback form, "with interior observation of the velocity" acting as a feedback in the Dirichlet boundary conditions

$$\begin{cases} \text{a)} & u_{tt}(t,x) = \Delta u(t,x) & \text{in } (0,\infty)\times\Omega \\ \text{b)} & u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) & \text{in } \Omega \\ \text{c)} & u(t,\sigma) = \langle u_t(t,\cdot), w(\cdot)\rangle_\Omega\, g(\sigma) & \text{in } (0,\infty)\times\Gamma. \end{cases} \tag{3.1}$$

Here Ω is an open bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary Γ, w and g are two fixed functions in $L_2(\Omega)$ and $L_2(\Gamma)$, respectively, while $\langle\ ,\ \rangle$ denotes the $L_2(\Omega)$-inner product. Let $\mathcal{A}$ be the (positive self-adjoint) operator $L_2(\Omega) \supset \mathcal{D}(\mathcal{A}) \to L_2(\Omega)$ defined by $\mathcal{A}f = -\Delta f$, $\mathcal{D}(\mathcal{A}) = H^2(\Omega)\cap H_0^1(\Omega)$. Finally, let D be the Dirichlet map defined by: $Dv = h$, where $\Delta h = 0$ in Ω and $h|_\Gamma = v$ on Γ. We refer to [16]–[18] to substantiate the following claims:

(i) the feedback system (3.1) admits the abstract dual representation as in (1.4), with $Z = L_2(\Omega)\times H^{-1}(\Omega)$ and

$$A^* = \begin{vmatrix} 0 & I \\ -\mathcal{A} & 0 \end{vmatrix}; \quad a = \begin{vmatrix} -Dg \\ 0 \end{vmatrix}; \quad b = \begin{vmatrix} 0 \\ \mathcal{A}w \end{vmatrix}. \tag{3.2}$$

(ii) for any $w \in \mathcal{D}(\mathcal{A}^{1/2}) = H_0^1(\Omega)$ and any $g \in L_2(\Gamma)$, assumption (2.2) is satisfied and thus problem (3.1), i.e. its abstract representation (1.4), (3.2) generates a s.c. semigroup $S_F^*(t) = \exp[A_F^* t]$ on Z (see [16, Theorem 3.2[3]]). (We recall that verification of assumption (2.2) by the vector a and the operator A given by (3.2) is a highly nontrivial issue, for it reduces to a "sharp" (trace) regularity result of the normal derivative for the solution of the corresponding hyperbolic problem with homogeneous boundary conditions, recently established in [23], [20], [24].)

Thus under the present circumstances, Theorem 2.1 applies and yields that for *any* choice of the pair $[w,g] \in H_0^1(\Omega)\times L_2(\Gamma)$, the corresponding feedback semigroup guaranteed by (ii) above *cannot be uniformly stable:* $\|S_F(t)\| = \|S_F^*(t)\| \geq 1$, $t \geq 0$ where $\|\ \|$ denotes the $\mathcal{L}(Z)$-norm.

On the other hand, Theorem 1.1 in [18] furnishes, in particular, a large class of vectors $w \in \mathcal{D}(\mathcal{A}^{3/4+\rho})$, $\rho > 0$ and $g \in L_2(\Gamma)$ such that the corresponding $S_F(t)$ is now a s.c. group on Z and $\|S_F^*(t)z\|_Z \to 0$ for all $z \in Z$; i.e. problem (3.1) is *strongly stable* on $Z = L_2(\Omega)\times H^{-1}(\Omega)$.

[3] A missprint has occurred in the statement of Theorem 3.2 in [16]: The word "group" should be replaced with the word "semigroup".

Uniform stabilization cannot be achieved if the right hand side of (3.1c) contains finitely many boundary terms rather than just one, and if moreover the right hand side of (3.1a) contains a perturbation which is compact in $[u, u_t]$ as in Theorem 2.2.

4. Theorem 2.1 is sharp. A wave equation example where the perturbation barely fails to be $\mathcal{A}$-bounded and where uniform stabilization is achieved with arbitrarily preassigned exponential rate.

4.1. The example. Consider the following second order hyperbolic problem in feedback form over the one-dimensional domain $\Omega = (0,1)$

$$\begin{cases} u_{tt}(t,x) = u_{xx}(t,x) & 0 < x < 1,\ t > 0 \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), & 0 < x < 1 \\ \begin{vmatrix} -u_x(t,0) \\ u_x(t,1) \end{vmatrix} = \left(\begin{vmatrix} u_t(t,0) \\ u_t(t,1) \end{vmatrix}, \begin{vmatrix} w_1 \\ w_2 \end{vmatrix} \right)_{R_2} \begin{vmatrix} g_1 \\ g_2 \end{vmatrix}, & \end{cases} \qquad (4.1\text{ a,b,c})$$

a special case of the purely boundary feedback system in $\Omega \subset \mathbb{R}^n$

$$u_{tt}(t,x) = \Delta u(t,x) \qquad (4.2a)$$

$$u(0,x) = u_1(x),\ u_t(0,x) = u_2(x) \qquad (4.2b)$$

$$\frac{\partial u}{\partial \nu}\Big|_\Gamma = \left(u_t(t,\cdot)\big|_\Gamma, w(\cdot)\right)_\Gamma g(\cdot) \qquad (4.2c)$$

with fixed vectors $w, g \in L_2(\Gamma)$, where $(\ ,\ )_\Gamma$ denotes the $L_2(\Gamma)$-inner product. Thus, in (4.1c), $w_i, g_i \in \mathbb{R}$. Note that the feedback term at the right of (4.1c) or (4.2c) defines a *nonlocal* operator. Thus, model (4.1) or (4.2) differs from, and is not covered by, the second order hyperbolic problem studied in [1-2], [22], [32], with *local* boundary dissipation given by $\partial w/\partial\nu|_\Gamma = -k(x)w_t$, $k(x) \ge k_0 > 0$, in place of (4.2c). The latter, once specialized to $k(x) \equiv 1$ and $\Omega = (0,1)$, becomes:

$$-u_x(t,0) = -u_t(t,0) \quad \text{and} \quad u_x(t,1) = -u_t(t,1) \qquad (4.3)$$

instead of (4.1c). Thus, we cannot invoke these references.

4.2. Abstract model $\dot{y} = (A + \mathcal{P})y$ for system (4.2). $\mathcal{P}$ barely fails to be $\mathcal{A}$-bounded. Let $\mathcal{A} : L_0^2(\Omega) \supset \mathcal{D}(\mathcal{A}) \to L^2(\Omega)$ be the positive self-adjoint operator $\mathcal{A}f = -\Delta f$, $\mathcal{D}(\mathcal{A}) = \{f \in H^2(\Omega) : \partial f/\partial\nu|_\Gamma = 0\}$, which defines an isomorphism from $\mathcal{D}(\mathcal{A})$ onto $L_0^2(\Omega) = \{\psi \in L^2(\Omega) : \int_\Omega \psi\, d\Omega = 0\} = L^2(\Omega)/\mathcal{N}(\mathcal{A})$, with $\mathcal{N}(\mathcal{A})$ being the null space of $\mathcal{A}$ consisting of constant functions. A necessary and sufficient condition for the existence of a generalized solution of the elliptic problem

$$\Delta h = 0 \text{ in } \Omega; \quad \frac{\partial h}{\partial \nu} = v \text{ in } \Gamma \qquad (4.4)$$

is $\int_\Gamma v\, d_\Gamma = 0$, and there is a unique solution $\bar{h}$ orthogonal in $L^2(\Omega)$ to $\mathcal{N}(\mathcal{A})$ [25, p. 199]. We then set

$$\bar{h} = Nv \in L_0^2(\Omega). \qquad (4.5)$$

From elliptic theory we obtain (e.g. [26], [21])

$$N: \text{ continuous } L_0^2(\Gamma) \to \mathcal{D}(\mathcal{A}^{3/4-\rho}) \equiv H^{3/2-2\rho}(\Omega)/\text{const, for every } \rho > 0 \quad (4.6a)$$

where $L_0^2(\Gamma) = \{v : \int_\Gamma v \, d\Gamma = 0\}$, so that

$$\mathcal{A}^{3/4-\rho}N \in \mathcal{L}(L_0^2(\Gamma), L^2(\Omega)); \quad N^*\mathcal{A}^{3/4-\rho} \in \mathcal{L}\big(L_0^2(\Omega, L_0^2(\Gamma)\big). \tag{4.6b}$$

The isomorphism defined by $\mathcal{A}$ from $\mathcal{D}(\mathcal{A})$ onto $L_0^2(\Omega)$ can be extended as an isomorphism from $\mathcal{D}(\mathcal{A}^\alpha)$ onto $[\mathcal{D}(\mathcal{A}^{1-\alpha})]'$ for all $0 \le \alpha \le 1$, which will be denoted by the same symbol $\mathcal{A}$. The topology on these spaces is

$$\|y\|_{\mathcal{D}(\mathcal{A}^\alpha)} = \|\mathcal{A}^\alpha y\|_{L^2(\Omega)}; \quad \|z\|_{[\mathcal{D}(\mathcal{A}^{1-\alpha})]'} = \|\mathcal{A}^{\alpha-1}z\|_{L^2(\Omega)}. \tag{4.7}$$

Henceforth we shall use, in particular, the case $\alpha = 3/4 - \rho$, with $\rho > 0$ fixed once and all by (4.6). By Green's second theorem one obtains (e.g. [21], [32])

$$N^*\mathcal{A}f = f\big|_\Gamma, \quad f \in \mathcal{D}(\mathcal{A}) \tag{4.8}$$

where $(Nv, z)_{L_0^2(\Omega)} = (v, N^*z)_{L_0^2(\Gamma)}$, for $v \in L_0^2(\Gamma)$ and $z \in L_0^2(\Omega)$.

Now let w and g in (4.2c) be chosen in $L_0^2(\Gamma)$. Then the abstract model for problem (4.2) is given in second order form by [16]–[21], [33]

$$\ddot{u} = -\mathcal{A}u + \mathcal{A}Ng(\dot{u}|_\Gamma, w)_\Gamma, \text{ or} \tag{4.9a}$$

$$\ddot{u} = -\mathcal{A}u + \mathcal{A}Ng(N^*\mathcal{A}\dot{u}, w)_\Gamma \in [\mathcal{D}(\mathcal{A}^{3/4-\rho})]', \tag{4.9b}$$

and in first order form, with $y = [y_1, y_2]$, $y_1 = u$, $y_2 = \dot{u}$, by

$$\dot{u} = Ay + \mathcal{P}y \tag{4.10}$$

on

$$Y = Y_1 \times Y_2, \quad Y_1 = \mathcal{D}(\mathcal{A}^{1/4-\rho}); \; Y_2 = [\mathcal{D}(\mathcal{A}^{1/4+\rho})]' \tag{4.11}$$

$$A = \begin{vmatrix} 0 & I \\ -\mathcal{A} & 0 \end{vmatrix}; \quad \mathcal{P} = \begin{vmatrix} 0 & 0 \\ 0 & \mathcal{A}Ng(N^*\mathcal{A}\cdot, w)_\Gamma \end{vmatrix} \tag{4.12a}$$

$$\mathcal{D}(A) = \mathcal{D}(\mathcal{A}^{3/4-\rho}) \times \mathcal{D}(\mathcal{A}^{1/4-\rho}); \quad \mathcal{D}(\mathcal{P}) = Y_1 \times \mathcal{D}(\mathcal{A}^{1/4+\varepsilon}). \tag{4.12b}$$

The skew adjoint operator $A = -A^*$ generates a s.c. unitary group on Y and Re $(Ay, y)_Y = 0$. From

$$\begin{aligned} \|\mathcal{P}y\|_Y^2 &= \|\mathcal{A}Ng\|_{Y_2} |(N^*\mathcal{A}y_2, w|_\Gamma)| \\ &= \|\mathcal{A}^{3/4-\rho}Ng\|_{L^2(\Omega)} |(\mathcal{A}^{1/4+\varepsilon}y_2, \mathcal{A}^{3/4-\varepsilon}Nw)|_{L^2(\Omega)} \end{aligned} \tag{4.13}$$

with $\varepsilon > 0$ arbitrary

$$\|Ay\|_Y^2 = \|y_2\|_{Y_1}^2 + \|\mathcal{A}y_1\|_{Y_2}^2 = \|\mathcal{A}^{1/4-\rho}y_2\|_{L^2(\Omega)}^2 + \|\mathcal{A}^{3/4-\rho}y_1\|_{L^2(\Omega)}^2. \tag{4.14}$$

We see that $\mathcal{P}$ barely fails to be A-bounded on Y (we would need $\varepsilon = \rho = 0$ in (4.13) to have that $\mathcal{P}$ is A-bounded on Y, while in these equations ε and ρ are arbitrary positive numbers instead). We may also rewrite (4.10) in $E = \mathcal{D}(\mathcal{A}^{1/2}) \times L_0^2(\Omega)$ in factor form as

$$\dot{z} = A[I + \Pi]z, \quad \Pi = \begin{vmatrix} 0 & -Ng(\cdot|_\Gamma, w)_\Gamma \\ 0 & 0 \end{vmatrix} \tag{4.15}$$

$$\mathcal{D}(A[I+\Pi]) = \{(z_1, z_2) \in \mathcal{D}(\mathcal{A}^{3/4-\rho}) \times \mathcal{D}(\mathcal{A}^{1/2}) \,:\, z_1 - Ng(z_2|_\Gamma, w)_\Gamma \in \mathcal{D}(\mathcal{A})\}.$$

It is in this space E that the solution of the one-dimensional problem (4.1) will be given explicitly in the subsequent sections, where $A(I + \Pi)$ will be rewritten as A_F.

4.3. Separation of variables and eigenvalues of system (4.1). We seek solutions of (4.1) by separation of variables in the form $u(t, x) = T(t)\phi(x)$. It will suffice for our purposes here to specialize the situation by taking $w_1 = g_1 = 0$; moreover, we shall take both w_2, g_2 *real* in (4.1d). Setting henceforth $b \equiv w_2 g_2$ (real $\neq 0$), the boundary condition (4.1c) becomes: $u_x(t, 0) \equiv 0$, $u_x(t, 1) \equiv b u_t(t, 1)$, $t > 0$. With $T''(t)/T(t) \equiv \phi''(x)/\phi(x) \equiv \lambda$ and $\lambda^2 = \mu$, we obtain the following eigenvalue-eigenfunction problem

$$\begin{aligned} &\phi''(x) - \mu^2\phi(x) = 0 \quad 0 < x < 1 \\ &\phi'(0) = 0; \ \mu b \phi(1) = \phi'(1) \end{aligned} \tag{4.16}$$

on the space $L^2(\Omega)$, $\Omega = (0, 1)$. The eigenvalues are obtained from $\tanh \mu = b$, with corresponding eigenfunction $\phi(x)$ and companion $T(t)$ given by

$$\phi(x) = \cosh \mu x \quad T(t) = e^{\mu t}. \tag{4.17}$$

Setting $\mu = \alpha + i\beta$ and using the fact that b is real, we obtain the solution

$$\mu_n = \alpha_n + i\beta_n, \quad \beta_n = \pm\frac{\pi}{2}n, \quad n = 0, 1, 2, \ldots \tag{4.18}$$

$$\frac{\sinh 2\alpha_n}{\cosh 2\alpha_n + (-1)^n} \equiv b \tag{4.19}$$

from which we conclude as follows.

Case 1. Let $0 < b \equiv w_2 g_2 < 1$, or else let $-1 < b < 0$. The function

$$f(x) = \frac{\sinh x}{\cosh x + 1} \tag{4.20}$$

is strictly increasing, with $f(0) = 0$, $\lim_{x\to+\infty} f(x) = 1$, and for $0 < x < \infty$, satisfies $(1/2)\tanh x < f(x) < \tanh x$ and $f(-x) = -f(x)$, $x > 0$. Let $f^{-1}(x)$ be its inverse. Then, given the parameter b as above, we first let

$$2\alpha_1 = f^{-1}(b)\begin{cases} > 0, & \text{if } 0 < b < 1 \\ < 0, & \text{if } -1 < b < 0 \\ = 0, & \text{if } b = 0 \end{cases} \tag{4.21}$$

and then the eigenvalues are given in this case by

$$\mu_n^{+,-}(b) = \alpha_1 \pm i\frac{\pi}{2}n = \frac{f^{-1}(b) \pm i\pi n}{2}, \quad n = 0, 2, 4, \ldots. \tag{4.22}$$

The series solution of system (4.1) (so far formal, to be properly analyzed below) is given for $n = 0, 2, 4, \dots$ by:

$$u(t,x) = \sum_n C_n^{+,-} e^{(\alpha_1 \pm i(\pi/2)n)t} \cosh(\alpha_1 \pm i\frac{\pi}{2}n)x \tag{4.23a}$$

$$u_t(t,x) = \sum_n C_n^{+,-} e^{(\alpha_1 \pm (\pi/2)n)t} [\alpha_1 \pm i\frac{\pi}{2}n] \cosh(\alpha_1 \pm i\frac{\pi}{2}n)x. \tag{4.23b}$$

Case 2. Let $1 < b < \infty$, or else let $-\infty < b < 1$. The function

$$h(x) = \frac{\sinh x}{\cosh x - 1} \tag{4.24}$$

is strictly decreasing for $0 < x \to +\infty$, with $h(x) > 1$ and $\lim_{x \to +\infty} h(x) = 1$, $\lim_{x \to 0^+} h(x) = +\infty$; moreover $h(-x) = -h(x)$, $x > 0$. Let $h^{-1}(x)$ be its inverse. Then, given the parameter b in this case, we first let

$$2\alpha_2 = h^{-1}(b) \begin{cases} > 0 & \text{if } b > 1 \\ < 0 & \text{if } b < -1 \end{cases} \tag{4.25}$$

and then the eigenvalues are given in this case by

$$\mu_n^{+,-}(b) \equiv \alpha_2 \pm i\frac{\pi}{2}n = \frac{h^{-1}(b) \pm i\pi n}{2}, \quad n = 1, 3, 5, \dots \tag{4.26}$$

with corresponding series solution as in (4.23), except that (4.22) is now replaced by (4.26).

Remark 4.1. The cases $b = \pm 1$ where no eigenvalues exist are special. Indeed when $b = -1$ the semigroup is nilpotent, while when $b = 1$ problem (4.1) is not well-posed. We shall omit these two cases.

4.4. Spectral properties of the operator $\mathcal{A}_F$ on the energy space E. Riesz basis of its eigenfunctions. Uniform stability of (4.21) on E. Henceforth we take initial data $[u_0, u_1]$ in the space $E = \mathcal{D}(\mathcal{A}^{1/2}) \times L^2(\Omega)$, (energy space) topologized by

$$(y, z)_E = (y_1, z_1)_{\mathcal{D}(\mathcal{A}^{1/2})} + (y_2, z_2)_{L^2(\Omega)} \tag{4.27}$$

$$(y_1, z_1)_{\mathcal{D}(\mathcal{A}^{1/2})} = (\mathcal{A} y_1, z_1)_{L^2(\Omega)} = \int_\Omega \nabla y_1 \cdot \nabla \bar{z}_1 \, d\Omega. \tag{4.28}$$

We define the operator $A_F : E \supset \mathcal{D}(A_F) \to E$ (candidate for generation for the feedback system (4.1)) by

$$A_F f = \begin{vmatrix} 0 & I \\ \Delta & 0 \end{vmatrix} \begin{vmatrix} f_1 \\ f_2 \end{vmatrix} \tag{4.29a}$$

$$\mathcal{D}(A_F) = \{ f = [f_1, f_2] \in E : f_1 \in H^2(\Omega),\ f_2 \in \mathcal{D}(\mathcal{A}^{1/2}); \\ f_{1x}|_{x=0} = 0;\ f_{1x}|_{x=1} = b f_2|_{x=1} \}. \tag{4.29b}$$

For future convenience we shall indicate its dependence on the boundary parameter b by writing $\mathcal{A}_F = \mathcal{A}_F(b)$.

The adjoint $A_F^*(b)$ of $\mathcal{A}_F(b)$ in $E : (\mathcal{A}_F y, z)_E = (y, \mathcal{A}_F^* z)_E$ is computed from here via (4.27)-(4.28) and is given by

$$\mathcal{A}_F^*(b)z = -\begin{vmatrix} 0 & I \\ \Delta & 0 \end{vmatrix} \begin{vmatrix} z_1 \\ z_2 \end{vmatrix} \tag{4.30a}$$

$$\mathcal{D}(\mathcal{A}_F^*(b)) = \{z = [z_1, z_2] \in E : z_1 \in H^2(\Omega),\ z_2 \in \mathcal{D}(\mathcal{A}^{1/2}); \\ z_{1x}|_{x=0} = 0;\ z_{1x}|_{x=1} = -bz_2|_{x=1}\}. \tag{4.30b}$$

By comparing (4.29) and (4.30), we deduce

$$\mathcal{A}_F^*(b) = -\mathcal{A}_F(-b). \tag{4.31}$$

Lemma 4.1 spectral properties of $\mathcal{A}_F(b)$ and $\mathcal{A}_F^*(b)$).

(i) The eigenvalues of the operator $\mathcal{A}_F(b)$ in (4.29) are given by

$$\mu_n^{+,-}(b) = \begin{cases} \alpha_1 \pm i\frac{\pi}{2}n \text{ as in (4.22), } n = 0, 2, 4, \ldots & \text{if } |b| < 1 \\ \alpha_2 \pm i\frac{\pi}{2}n \text{ as in (4.26), } n = 1, 3, 5, \ldots & \text{if } |b| > 1 \end{cases} \tag{4.32 a, b}$$

with corresponding eigenvectors $\Phi_n^{+,-}(b)$

$$\Phi_n^{+,-}(b) = \begin{vmatrix} \Phi_{n1}^{+,-}(b) \\ \Phi_{n2}^{+,-}(b) \end{vmatrix} = \begin{vmatrix} K_\alpha \frac{\cosh \mu_n^{+,-}(b)x}{\mu_n^{+,-}(b)} \\ K_\alpha \cosh \mu_n^{+,-}(b)x \end{vmatrix} \\ = K_\alpha \begin{vmatrix} \frac{\cosh[\alpha_k \pm i(\pi/2)n]x}{\alpha_k \pm i(\pi/2)n} \\ \cosh[\alpha_k \pm i(\pi/2)n]x \end{vmatrix} \tag{4.33}$$

$$K_\alpha \sqrt{\frac{2\alpha_k}{\sinh 2\alpha_k}} \quad \begin{cases} k = 1; & n = 0, 2, 4, \ldots \\ k = 2; & n = 1, 3, 5, \ldots \end{cases} \tag{4.34}$$

normalized in E in the sense that, recalling (4.27)-(4.28)

$$\|\Phi_n^{+,-}(b)\|_E^2 = \|\Phi_{n1}^{+,-}(b)\|_{\mathcal{D}(\mathcal{A}^{1/2})}^2 + \|\Phi_{n2}^{+,-}(b)\|_{L^2(\Omega)}^2 \\ = \|\frac{d}{dx}\Phi_{n1}^{+,-}(b)\|_{L^2(\Omega)}^2 + \|\Phi_{n2}^{+,-}(b)\|_{L^2(\Omega}^2 = 1. \tag{4.35}$$

(ii) The eigenvalues of the adjoint operator $A_F^*(b)$ defined by (4.30) are given by

$$\bar{\mu}_n^{+,-}(b) = \mu_n^{-,+}(b) = -\mu_n^{+,-}(-b) = \alpha_k \pm i\frac{\pi}{2}n, \quad \begin{cases} k = 1; & n = 0, 2, 4, \ldots \\ k = 2; & n = 1, 3, 5, \ldots \end{cases} \tag{4.36}$$

with corresponding eigenvectors, denoted by $\Phi_n^{+,-*}(b)$,

$$\mathcal{A}_F^*(b)\Phi_n^{+,-*}(b) = \bar{\mu}_n^{+,-}(b)\Phi_n^{+,-*}(b) \tag{4.37}$$

given by

$$\Phi_n^{+,-*}(b) = \begin{vmatrix} \Phi_{n1}^{+,-*}(b) \\ \Phi_{n2}^{+,-*}(b) \end{vmatrix} = \frac{1}{K_\alpha} \begin{vmatrix} & \cosh \mu_n^{+,-}(-b)x \\ \mu_n^{+,-}(-b) & \cosh \mu_n^{+,-}(-b)x \end{vmatrix} = \frac{1}{K_\alpha} \begin{vmatrix} & \cosh[-\alpha_k \pm i(\pi/2)n]x \\ [-\alpha_k \pm i(\pi/2)n] & \cosh[-\alpha_k \pm i(\pi/2)n]x \end{vmatrix} \tag{4.38}$$

which are normalized by the following biorthogonality condition

$$\begin{aligned} &\left(\Phi_n^{+,-}(b), \Phi_m^{+,-*}(b)\right)_E \\ &= \left(\frac{d}{dx}\Phi_{n1}^{+,-}(b), \frac{d}{dx}\Phi_{m1}^{+,-*}(b)\right)_{L^2(\Omega)} + \left(\Phi_{n2}^{+,-}(b), \Phi_{m2}^{+,-*}(b)\right)_{L^2(\Omega)} \\ &= (\text{Kronecker})\delta_{nm} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}. \end{aligned} \tag{4.39}$$

Proof: Part (i) was proved in Section 4.3, see (4.22), (4.26) and (4.17) except for normalization (4.35) whose verification is omitted.

Part (ii) may, of course, be verified. To prove it, however, it is expedient to use (4.31). The eigenvectors of $\mathcal{A}_F^*(b) = -\mathcal{A}_F(-b)$ corresponding to its eigenvalues $\bar{\mu}_n^{+,-}(b) = \alpha_k \mp i(\pi/2)n$ (i.e. eigenvalues $[-\alpha_k \pm i(\pi/2)n]$ of $\mathcal{A}_F(-b)$) are the same as the eigenvectors of $\mathcal{A}_F(-b)$ corresponding to its eigenvalues $[-\alpha_k \pm i(\pi/2)n]$. Then Part (ii) follows from Part (i), except for the biorthogonality condition, whose verification with the help of [11, §352, 2a), 2b), p. 148] is omitted.

The following result is crucial in order to deduce proper convergence of the solution series (4.23) in E and, moreover, to establish the sought-after uniform (exponential) growth or decay of $[u(t), u_t(t)]$ in E from the location of the spectrum of $\mathcal{A}_F$ as given by (4.32). Such solution will define the s.c. *group* on E by:

$$\begin{vmatrix} u(t) \\ u_t(t) \end{vmatrix} = e^{\mathcal{A}_F t} \begin{vmatrix} u_1 \\ u_2 \end{vmatrix}, \text{ with generator } \mathcal{A}_F \text{ in (4.29)},$$

and the "spectrum determined growth condition" [34] does not generally hold for group generators, as is well-known through counterexamples ([13, p. 665], [27, p. 117], [36]). We overcome this difficulty by showing directly that the eigenvectors of $\mathcal{A}_F$ form a Riesz basis (see [N-F.1], [12], [29] for this notion) on E.

Theorem 4.2. *The sequence $\{\Phi_n^{+,-}(b)\}$ in (4.33) of normalized eigenvectors of $\mathcal{A}_F(b)$ forms a Riesz basis in E, in the sense that: any (real) element $u = [u_0, u_1] \in E$ admits the following (unique) expansion*

$$u = \sum_n U_n^{+,-}(u)\Phi_n^{+,-}(b) \tag{4.40}$$

with (recall (4.27)-(4.28))

$$U_n^{+,-}(u) = (u, \Phi_n^{+,-*})_E = \left(\frac{d}{dx}u_0, \frac{d}{dx}\Phi_{n1}^{+,-*}(b)\right)_{L^2(\Omega)} + \left(u_1, \Phi_{n2}^{+,-*}(b)\right)_{L^2(\Omega)} \tag{4.41}$$

(consistently with (4.39)) and there are two constants $C_{1\alpha}, C_{2\alpha} > 0$, depending on α_k, $k = 1, 2$, but not on $u \in E$, such that

$$C_{1\alpha} \sum_n |U_n^{+,-}(u)|^2 \leq \|u\|_E^2 \leq C_{2\alpha} \sum |U_n^{+,-}(u)|^2. \tag{4.42}$$

Proof (sketch): We take real $[u_0, u_1] \in E$ and compute from (4.41), (4.38) (all inner products and norms below are in $L^2(0,1)$):

$$\begin{aligned} K_\alpha U_n^{+,-} &= \{(u_{0x}, \sinh(-\alpha_k x) \cos \tfrac{\pi}{2} nx) + (u_1, \cosh(-\alpha_k x) \cos \tfrac{\pi}{2} nx)\} \\ &\quad \pm i\{(u_{0x}, \cosh(-\alpha_k x) \sin \tfrac{\pi}{2} nx) + (u_1, \sinh(-\alpha_k x) \sin \tfrac{\pi}{2} nx)\} \\ |K_\alpha U_n^{+,-}|^2 &= |(u_{0x} \sinh(-\alpha_k x) + u_1 \cosh(-\alpha_k x), \cos \tfrac{\pi}{2} nx)|^2 \\ &\quad + |(u_{0x} \cosh(-\alpha_k x) + u_1 \sinh(-\alpha_k x), \sin \tfrac{\pi}{2} nx)|^2. \end{aligned} \tag{4.43}$$

Since $\{\sqrt{2} \cos n\frac{\pi}{2}x\}$ and $\{\sqrt{2} \sin n\frac{\pi}{2}x\}$ for either $n = 0, 2, 4, \ldots$ and $k = 1$, or else for $n = 1, 3, 5, \ldots$ and $k = 2$ form an orthonormal basis on $L^2(0,1)$, we obtain from (4.43)

$$2K_\alpha^2 \sum_n |U_n^{+,-}|^2 = \|u_1 \cosh \alpha_k x - u_{0x} \sinh \alpha_k x\|^2 + \|u_{0x} \cosh \alpha_k x - u_1 \sinh \alpha_k x\|^2 \tag{4.44}$$

where $\| \ \|$ is the $L^2(0,1)$-norm. From (4.44) we plainly obtain

$$\text{Right Hand Side of (4.44)} = \text{const}_\alpha \{\|u_{0x}\|^2 + \|u_1\|^2\} = \text{const}_\alpha \left\| \begin{vmatrix} u_0 \\ u_1 \end{vmatrix} \right\|_E^2 \tag{4.45}$$

and the left hand side inequality of (4.42) is proved. To show the right hand side inequality of (4.42) we use the inequality

$$4\{\|y\|^2 + \|z\|^2\} \geq \|y + z\|^2 + \|y - z\|^2 \tag{4.46}$$

on the right of (4.44) thus obtaining (since $|\cosh \alpha_k x - \sinh \alpha_k x| \geq \text{const}\, \alpha_k > 0$ on $0 \leq x \leq 1$):

$$\begin{aligned} 8K^2 \sum_n |U_n^{+,-}|^2 &\geq \|(\cosh \alpha_k x - \sinh \alpha_k x)[u_1 + u_{0x}]\|^2 \\ &\quad + \| \cosh \alpha_k x + \sinh \alpha_k x)[u_1 - u_{0x}]\|^2 \\ &\geq \text{const}_\alpha \{\|u_1 + u_{0x}\|^2 + \|u_1 - u_{0x}\|^2\} \end{aligned}$$

(using again (4.46))

$$\begin{aligned} &\geq \text{const}_\alpha \{\|u_1\|^2 + \|u_{0x}\|^2\} \\ &= \text{const}_\alpha \left\| \begin{vmatrix} u_0 \\ u_1 \end{vmatrix} \right\|_E^2 \end{aligned} \tag{4.47}$$

and the proof is complete. ∎

Our sought-after result is obtained as a Corollary.

Corollary 4.3. *We write $\mathcal{A}_F$ instead of $\mathcal{A}_F(b)$ throughout.*

(i) for $u = [u_0, u_1] \in \mathcal{D}(\mathcal{A}_F)$ and with reference to (4.32) and (4.41) we have the unique expansion in E:

$$\mathcal{A}_F u = \sum_n \mu_n^{+,-}(b)(u, \Phi_n^{+,-*})_E \Phi_n^{+,-}(b). \tag{4.48}$$

(ii) The operator $\mathcal{A}_F = \mathcal{A}_F(b)$ generates a s.c. group on E given by the unique expansion on E:

$$\begin{aligned} \begin{vmatrix} u(t,u) \\ u_t(t,u) \end{vmatrix} &= e^{\mathcal{A}_F t} u = \sum_n e^{\mu_n^{+,-}(b)t} \left(u, \Phi_n^{+,-*}(b)\right)_E \Phi_n^{+,-}(b) \\ &= e^{\alpha_k t} \sum_n e^{\pm i(\pi/2)nt} \left(\begin{vmatrix} u_0 \\ u_1 \end{vmatrix}, \Phi_n^{+,-*}(b) \right)_E \Phi_n^{+,-}(b), \quad k = 1,2 \end{aligned} \tag{4.49}$$

which provides the unique solution of problem (4.1) with $w_1 = g_1 = 0$, $w_2 g_2 = b$ and $u = [u_0, u_1]$. Moreover

$$e^{\alpha_k t} \frac{1}{C_{2\alpha}} \left\| \begin{vmatrix} u_0 \\ u_1 \end{vmatrix} \right\|_E^2 \le \|e^{\mathcal{A}_F t} u\|_E^2 \le e^{\alpha_k t} \frac{1}{C_{1a}} \left\| \begin{vmatrix} u_0 \\ u_1 \end{vmatrix} \right\|_E^2, \quad t \ge 0, \ k = 1,2 \tag{4.50}$$

with $C_{1\alpha}$ and $C_{2\alpha} > 0$ as in (4.42).

(iii) For $b = w_2 g_2 < 0$, $b \neq -1$, then $\alpha_k < 0$ (by (4.21) and (4.25)) and the semigroup $\exp[\mathcal{A}_F t]$ is uniformly stable on E. Moreover its (exponential) rate of decay α_k may be arbitrarily preassigned via (4.20)–(4.21) for $k = 1$, and via (4.24)–(4.25) for $k = 2$, by selecting a suitable b.

Next, since A in (4.12) is skew-adjoint on E, if we take $[u_0, u_1] \in \mathcal{D}(\mathcal{A}_F)$ and let

$$z(t) = \begin{vmatrix} u(t) \\ u_t(t) \end{vmatrix} = e^{\mathcal{A}_F t} \begin{vmatrix} u_0 \\ u_1 \end{vmatrix} \tag{4.51}$$

we obtain as usual by (4.15) and (4.8)

$$\begin{aligned} 1/2 \frac{d}{dt} \left\| e^{\mathcal{A}_F t} \begin{vmatrix} u_0 \\ u_1 \end{vmatrix} \right\|_E^2 &= \text{Re } \left(\mathcal{A}_F z(t), z(t)\right)_E \\ &= \text{Re } \left\{ \left(\mathcal{A} N g (N^* \mathcal{A} u_t, w)_\Gamma, u_t\right)_\Gamma \right\} \\ &= \text{Re } \{ (N^* \mathcal{A} u_t, w)_\Gamma (g, N^* \mathcal{A} u_t)_\Gamma \} \end{aligned} \tag{4.52}$$

in the $L_2(\Gamma)$-inner product. Thus, specializing to the one-dimensional problem (4.1) where $w_1 = g_1 = 0$, $w_2 g_2 = b \neq \pm 1$ as before, we obtain from (4.52)

$$1/2 \frac{d}{dt} \left\| e^{\mathcal{A}_F t} \begin{vmatrix} u_0 \\ u_1 \end{vmatrix} \right\|_E^2 = -b |u_t(t,1)|^2. \tag{4.53}$$

Integrating (4.53) in time, we obtain a trace regularity result as a consequence of the interior regularity of Corollary 4.3.

Corollary 4.4. *(i) We have for* $[u_0, u_1] \in \mathcal{D}(\mathcal{A}_F)$

$$1/2\left\{\left\|e^{\mathcal{A}_F T}\begin{vmatrix} u_0 \\ u_1 \end{vmatrix}\right\|_E^2 - \left\|\begin{vmatrix} u_0 \\ u_1 \end{vmatrix}\right\|_E^2\right\} = -b\int_0^T |u_t(t,1)|^2\, dt. \tag{4.54}$$

(ii) Thus, extending by continuity to all $[u_0, u_1] \in E$, *we have for all* $0 < T < \infty$:

$$|b|\int_0^T |u_t(t,1)|^2\, dt \leq \ const_T \|[u_0, u_1]\|_E^2. \tag{4.55}$$

5. Theorem 2.1 is sharp. A class of conservative problems which can be uniformly (exponentially) stabilized by virtue of A-compact perturbations. Throughout this section we shall assume that:

(H.1): A (the elastic operator) is a self-adjoint operator on a Hilbert space X, strictly positive, with dense domain $\mathcal{D}(A)$ and compact resolvent, $R(\cdot, A)$ the case of interest in physical applications.

(H.2): B (the dissipation operator) is, for the time being, a positive, self-adjoint operator on X likewise with dense domain $\mathcal{D}(B)$ in X.

The object of our interest is the abstract differential equation

$$\ddot{x} + B\dot{x} + Ax = 0 \quad \text{on } X \tag{5.1}$$

$$\frac{d}{dt}\begin{vmatrix} x \\ \dot{x} \end{vmatrix} = \mathcal{A}_b \begin{vmatrix} x \\ \dot{x} \end{vmatrix} \qquad \text{on } E = \mathcal{D}(A^{1/2}) \times X \tag{5.2}$$

$$\mathcal{A}_B = \begin{vmatrix} 0 & I \\ -A & -B \end{vmatrix} \quad \text{with domain } \mathcal{D}(\mathcal{A}_B) \text{ containing } \mathcal{D}(A) \times \mathcal{D}(B) \tag{5.3}$$

where the inner product on E is defined by

$$([x_1, x_2], [y_1, y_2])_E = (A^{1/2}x_1, A^{1/2}y_1)_X + (x_2, y_2)_X. \tag{5.4}$$

The operator $\mathcal{A}_B$ is densely defined and dissipative on E, hence closable on E, and we shall use the same symbol $\mathcal{A}_B$ to denote its closure. The Lumer-Phillips theorem then readily shows that $\mathcal{A}_B$ generates a strongly continuous semigroup of contractions on E, denoted by $e^{\mathcal{A}_B t}$, e.g. [4], [5]. Stimulated by two conjectures raised in [3], we have already studied in [4], [5], [6] problem (5.1) or (5.2), under the additional hypothesis that B is "comparable to A^α", on the range $0 < \alpha \leq 1$.

Our hypothesis that B is comparable to A^α is as follows.

(H.3): There exist two constants $0 < \rho_1, \rho_2 < \infty$ and a constant $0 < \alpha \leq 1$ such that

$$\rho_1 A^\alpha \leq B \leq \rho_2 A^\alpha, \text{ i.e.} \tag{5.5a}$$

$$\rho_1(A^\alpha x, x)_X \leq (Bx, x)_X \leq \rho_2(A^\alpha x, x)_X, \quad x \in \mathcal{D}(A^{\alpha/2}). \tag{5.5b}$$

For the purposes of the present section, we restrict henceforth our attention to the case $0 < \alpha < 1/2$, unless otherwise noted.

(i) With α in this range $0 < \alpha < 1/2$ in (5.5), we have recently shown in [6] that: the strongly continuous semigroup of contractions $\exp(\mathcal{A}_B t)$ generated by the operator $\mathcal{A}_B$ in (5.3) (once closed) is also differentiable on $E = \mathcal{D}(A^{1/2}) \times X$ for all $t > 0$ (but generally not analytic). A fortiori, the spectrum determined growth assumption is satisfied for $\mathcal{A}_B$ [34] and there are constants M and $\delta = -\sup \operatorname{Re} \sigma(\mathcal{A}_B) > 0$, $\sigma(\mathcal{A}_B)$ being the spectrum of $\mathcal{A}_B$, such that

$$\|e^{\mathcal{A}_B t}\|_{\mathcal{L}(E)} \leq M e^{-\delta t}, \quad t \geq 0. \tag{5.6}$$

(Instead, in the range $1/2 \leq \alpha \leq 1$, the semigroup $\exp(\mathcal{A}_B t)$ is analytic on E, [4], [5].)

Assumption (5.5), and hence conclusion (5.6), holds a fortiori true if

$$\rho_1 A^\alpha \leq B \quad \text{on } \mathcal{D}(A^{\alpha/2}) \tag{5.7}$$

$$B^2 \leq \rho_2^2 A^{2\alpha} \quad \text{on } \mathcal{D}(A^\alpha) \tag{5.8}$$

(by Lowner Theorem, [14], [15], whereby (5.8) implies (5.5b)), in which case we also have by (5.8) that $BA^{-\alpha}$ is a bounded operator on X

$$BA^{-\alpha} \in \mathcal{L}(X). \tag{5.9}$$

(ii) On the other hand, one can readily show that with the notation

$$\mathcal{A}_0 = \begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix}, \quad \mathcal{P} = \begin{vmatrix} 0 & 0 \\ 0 & -B \end{vmatrix} \tag{5.10}$$

and B satisfying (5.7), (5.8), the perturbation $\mathcal{P}$ is $\mathcal{A}_0$-bounded on E for $0 < \alpha \leq 1/2$; and indeed, is $\mathcal{A}_0$-compact on E [15, p. 194] for $0 < \alpha < 1/2$. In fact, to check $\mathcal{A}_0$-compactness, let $x_n = [x_{n1}, x_{n2}] \in \mathcal{D}(\mathcal{A}_0) = \mathcal{D}(A) \times \mathcal{D}(A^{1/2})$, $n = 1, 2, \dots$ and let

$$\|x_n\|_E^2 = \|A^{1/2} x_{n1}\|_X^2 + \|x_{n2}\|_X^2 \leq C \tag{5.11}$$

$$\|\mathcal{A}_0 x_n\|_E^2 = \|A^{1/2} x_{n2}\|_X^2 + \|A x_{n1}\|_X^2 \leq C \tag{5.12}$$

for a constant C. Consider

$$\mathcal{P} x_n = B x_{n2} = B A^{-1/2} A^{1/2} x_{n2}. \tag{5.13}$$

For B satisfying (5.7), (5.8) with $0 < \alpha < 1/2$, we have that the operator $BA^{-1/2} = BA^{-\alpha} A^{\alpha - 1/2}$ is a bounded, compact operator on X, by (5.9) and compactness of A^{-1} (in (H.1)). This, along with (5.12) and (5.13), then guarantees that we can select a convergent subsequence from $\{;\mathcal{P} x_n\}$. Thus $\mathcal{P}$ is $\mathcal{A}_0$-compact on E, as desired.

(iii) Thus, in view of (5.6) the second order problem $\ddot{x} + Ax = u$, which is conservative on E for $u \equiv 0$, is uniformly stabilized on E by the feedback operator $u = -B\dot{x}$, B as in (5.7), (5.8) for $0 < \alpha < 1/2$, which yields a relatively compact perturbation $\mathcal{P}$ on E as in (5.10). Our assertion is proved.

REFERENCES

[1] G. Chen, *Energy decay estimates and exact controllability of the wave equation in a bounded domain,* J. Math. Pures et Appl. 58 (1979), 249-274.

[2] G. Chen, *A note on the boundary stabilization of the wave equation*, SIAM J. Control 19 (1981), 106-113.

[3] G. Chen and D.L. Russell, *A mathematical model for linear elastic systems with structural damping,* Quarterly of Applied Mathematics (1982), 433-454.

[4] S. Chen and R. Triggiani, *Proof of two conjectures of G. Chen and D. Russell on structural damping for elastic systems: The case* $\alpha = 1/2$, in "Springer-Verlag Lecture Notes in Mathematics", December 1988, Proceedings of a Seminar in Approximation and Optimization, held at the University of Habana, Cuba, January 10-12, 1987.

[5] S. Chen and R. Triggiani, *Proof of extensions of two conjectures on structural damping for elastic systems: The case* $1/2 \leq \alpha \leq 1$, Pacific J. Mathematics 136 (1989), 15-55.

[6] S. Chen and R. Triggiani, *Differentiable semigroups arising from elastic systems with gentle dissipation: The case* $0 < \alpha < 1/2$, Proc. Amer. Math. Soc., to appear.

[7] N. Dunford, *A survey of the theory of spectral operators*, Bulletin Amer. Math. Soc. 64 (1958), 217-274.

[8] N. Dunford and B.J. Pettis, *Linear operations on summable function*, Trans. Amer. Math. Soc. 47 (1940), 323-392.

[9] N. Dunford and J. Schwartz, "Linear Operators Vol. I" (1958), Vol. II (1963), Vol. III (1971); Interscience Publ., John Wiley.

[10] J.S. Gibson, *A note on stabilization of infinite dimensional linear oscillators by compact linear feedback*, SIAM J. Control and Optimiz. 18 (1980), 311-316.

[11] W. Grobner and N. Hofreiter, "Integral tafel," Erster Teil, Unbestimmte Integrale.

[12] I.C. Gohberg and M.G. Krein, *Introduction to linear nonself-adjoint operators,* in "Amer. Math. Soc. Translation of Mathem. Monographs 18," 1969, Providence, R.I.

[13] E. Hille and R.S. Phillips, *Functional analysis and semigroups,* in "Amer. Math. Soc. Colloq. Publ. 31", 1957, Providence, R.I.

[14] S.G. Krein, *Linear differential equations in Banach space,* in "Translations American Mathematical Society 29," American Mathematical Society, Providence, R.I., 1971.

[15] T. Kato, "Perturbation Theory of Linear Operators," Springer-Verlag, 1966.

[16] I. Lasiecka and R. Triggiani, *Finite rank, relatively bounded perturbations of semigroups generators, Part I: Well-posedness and boundary feedback hyperbolic dynamics,* Annali Scuola Normale Superiore, Pisa, Classe di Scienze, Serie IV (1985).

[17] I. Lasiecka and R. Triggiani, *Finite rank, relatively bounded perturbations of* C*-semigroups, Part II: Spectrum allocation and Riesz basis in parabolic and hyperbolic feedback systems,* Annali Matematica Pura e Applicata IV CXLIII (1986), 47-100.

[18] I. Lasiecka and R. Triggiani, *Dirichlet boundary stabilization of the wave equation with damping feedback of finite range,* J. Mathem. Anal. and Applic. 97 (1983), 112-130.

[19] I. Lasiecka and R. Triggiani, *Nondissipative boundary stabilization of the wave equation via boundary observation,* J. Math. Pures et Appliquees 63 (1984), 59-80.

[20] I. Lasiecka and R. Triggiani, *Regularity of hyperbolic equations under* $L_2(0,T;L_2(\Gamma))$ *boundary terms,* Applied Math. and Optimiz. 10 (1983), 275-286.

[21] I. Lasiecka and R. Triggiani, *A cosine operator approach to modeling* $L_2(0,T;L_2(\Gamma))$ *boundary input hyperbolic equations,* Applied Math. and Optimiz. 7 (1981), 35-83.

[22] J. Lagnese, *Decay of solutions of wave equations in a bounded region with bounday dissipation,* J. Diff. Eq. 50 (1983), 163-182.

[23] J.L. Lions, "Controle des Systems Distribues Singuliers," Gauthier-Villars, 1983.

[24] I. Lasiecka, J.L. Lions and R. Triggiani, *Nonhomogeneous boundary value problems for second order hyperbolic operators,* J. Mathem. Pures et Appliquees 69 (1986), 149-192.

[25] V.P. Mikhailov, "Partial Differential Equations," MIR Publishers, Moscow (English translation), 1978.

[26] J. Necas, "Les Methodes Directes en Theorie des Equations Elliptiques," Masson et Cie, Paris, 1964.

[27] A. Pazy, "Semigroups of Operators and Applications to Partial Differential Equations," Springer-Verlag, New York, 1983.

[28] D.L. Russell, *Decay rates for weakly damped systems in Hilbert spaces obtained via control-theoretic methods,* J. Diff. Eq. 19 (1975), 344-370.

[29] D.L. Russell, *Control theory of hyperbolic equations related to certain questions in harmonic analysis and spectral theory,* J. of Mathem. Analysis and Applications 40 (1972), 336-368.

[30] R. Triggiani, *A^{ϵ}-bounded, finite rank perturbations of s.c. group generators; counterexamples to generation and to another condition for well-posedness,* in "Lecture Notes in Mathematics 1076, Infinite Dimensional Systems, Proceedings of a Conference held at Retzholf, Austria," 1983, 227-253. Springer-Verlag, 1984, 227-253.

[31] R. Triggiani, *Improving stability properties of hyperbolic damped equations by boundary feedback,* in "Springer-Verlag Lecture Notes LNCIS N. 75," 1985, 400-409. Proceedings of Second International Conference on Control Theory for Distributed Parameter Systems, held at Vorau, Austria, July 1984, Eds.: Kappel, Kunish, Schappacher.

[32] R. Triggiani, *Wave equation on a bounded domain with boundary dissipation: an operator approach,* in "Operator Methods for Optimal Control Problems," Marcel Dekker Inc., 1987, 283-311. "Lecture Notes in Pure and Applied Mathem.," J. Math. Anal. and Appl., 137 (1989), 438-461.

[33] R. Triggiani, *A cosine operator approach to modeling $L_2(0,T;L_2(\Gamma))$-boundary input problems for hyperbolic systems,* in "Springer-Verlag Lecture Notes LNCIS 16," 1978. Proceedings 8th IFIP Conference, University of Wurzburg, W. Germany, 1977.

[34] R. Triggiani, *On the stabilizability problem in Banach space,* Addendum JMAA 52 (1975), 492-493.

[35] R. Triggiani, *Lack of uniform stabilization of noncontractive semigroups under compact perturbations,* Proceedings Amer. Math. Soc., 105 (1989), p. 375.

[36] J. Zabczyk, *A note on C_0-semigroups,* Bull. Acad. Polon. Sc. (Math. Astr. Phys.) 23 (1975), 895-898.

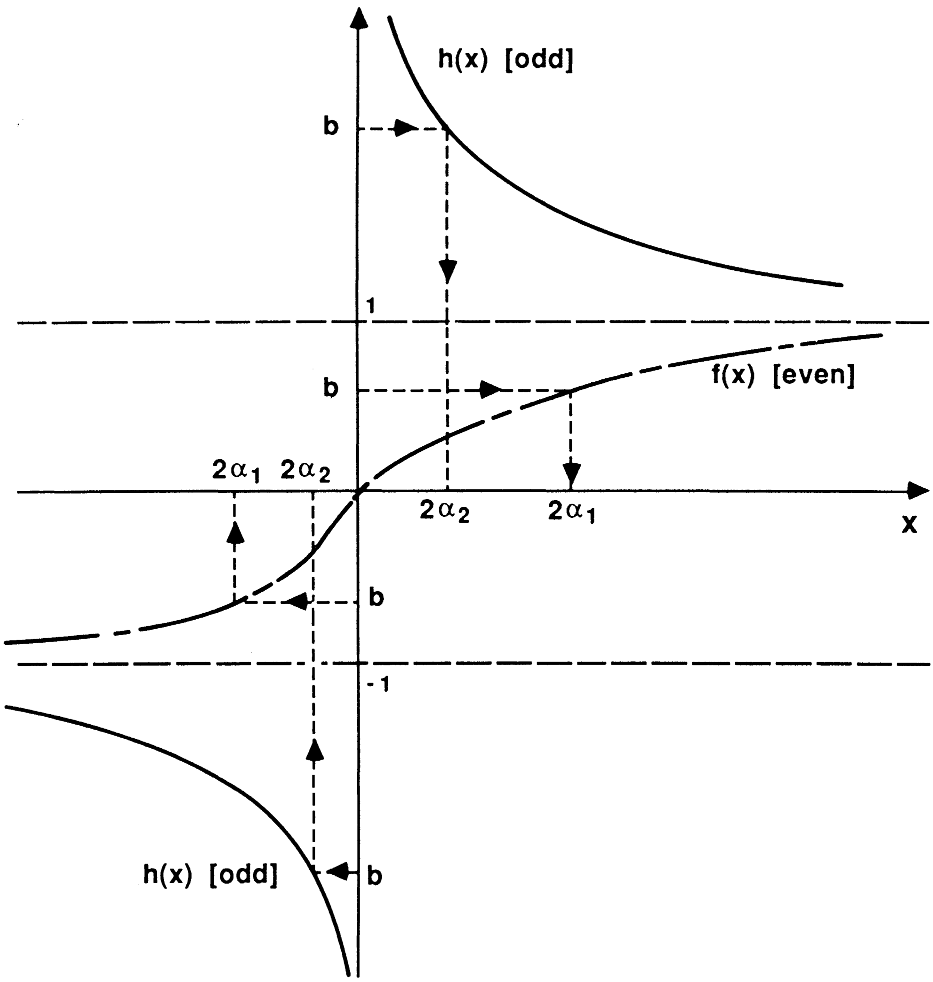

Figure 1: Determination of constants α_i given $b = w_2 g_2$, with $|b| \neq 1$

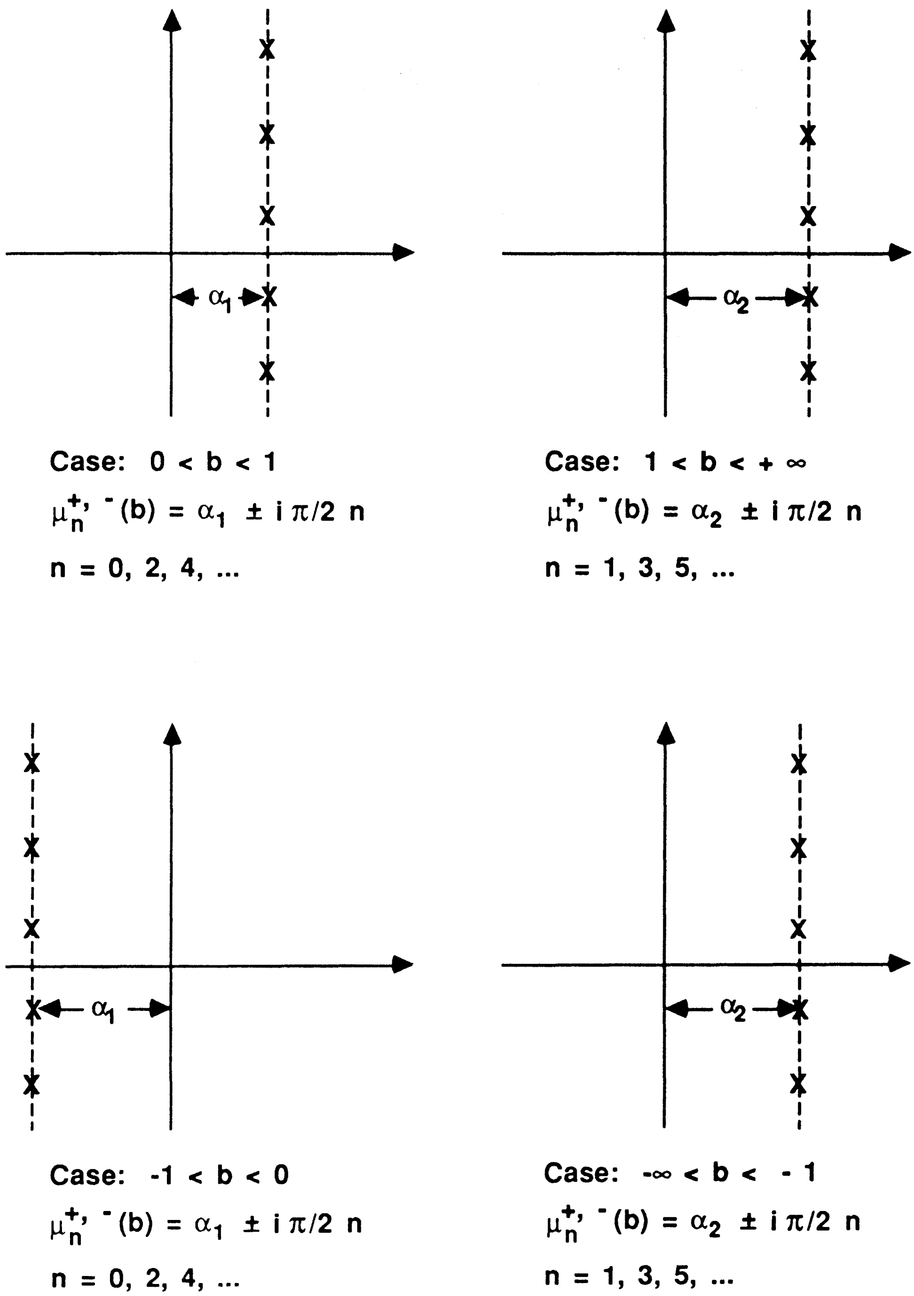

Figure 2: Eigenvalues corresponding to $b = w_2 g_2$