A FAST DIFFUSION EQUATION WHICH GENERATES
A MONOTONE LOCAL SEMIFLOW II:
GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOR

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Abstract. Global existence and large-time asymptotic behavior of mild solutions to the
Cauchy problem for the fast diffusion equation \( \partial_t n = d \cdot \partial_x (n^{-1} \cdot \partial_x n) \), \((x, t) \in \mathbb{R} \times \mathbb{R}_+\), with
the boundary conditions \( \lim_{x \to -\infty} n^{-1} \cdot \partial_x n = c \) and \( \lim_{x \to \infty} n = b \) are investigated. Here,
b, c, d \in (0, \infty)\) are given constants. It is proved that, when viewed as an abstract evolution equation in a suitable Sobolev space \( Y \), this problem has a unique mild solution which exists
globally in time, is \( C^\infty \) in \( \mathbb{R} \times (0, \infty) \) and satisfies the boundary conditions for every \( t \in \mathbb{R}_+\)
whenever \( n(x, 0) \in Y \) is given. These solutions form a semigroup of monotone contractions
in \( \overline{Y} = \text{closure of} \ Y \) in the translation of \( L^1(\mathbb{R}) \) by the Heaviside step function. Each solution
approaches a traveling wave in the \( L^1(\mathbb{R}) \)-metric as \( t \to \infty \).

1. Introduction. This paper is the second one from a series of two papers
studying a fast diffusion equation. The purpose of this paper is to study global (in time)
existence and asymptotic behavior of a solution \( n(x, t) \) to the Cauchy problem
for the following fast diffusion equation on the real line:

\[
\begin{align*}
&\partial_t n = d \cdot \partial_x (n^{-1} \cdot \partial_x n) \quad \text{for} \quad -\infty < x < \infty, \quad t > 0; \quad (1.1) \\
n(x, 0) = n_0(x) \quad \text{for} \quad -\infty < x < \infty; \quad (1.2) \\
&\lim_{x \to -\infty} n^{-1}(x, t) \cdot \partial_x n(x, t) = c \quad \text{for} \quad t > 0; \quad (1.3) \\
&\lim_{x \to \infty} n(x, t) = b \quad \text{for} \quad t > 0. \quad (1.4)
\end{align*}
\]

Here, \( n : \mathbb{R} \times \mathbb{R}_+ \to (0, \infty) \) is the unknown function whose initial value at \( t = 0 \) is
a given function \( n_0 : \mathbb{R} \to (0, \infty) \), and \( b, c, d \in (0, \infty) \) are given constants.

Equation (1.1) arises in a number of nonlinear diffusion problems in mathematical physics and population dynamics, cf. Takáč [12] for references. The boundary condition (1.3) means constant flux at \( x = -\infty \), while (1.4) means constant density
at \( x = \infty \). Equation (1.1) on the bounded interval \((0, 1)\) with Dirichlet boundary
conditions \( n(0, t) = n(1, t) = b > 0 \ (t > 0) \) and the asymptotic behavior of \( n(x, t) \)

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as $t \to \infty$ was studied in Berryman and Holland [1]. However, the asymptotic behavior in their case is completely different from ours. In this paper, we investigate global (in time) existence and asymptotic behavior (as $x \to \pm \infty$ or $t \to \infty$) of a solution $n(x,t)$ of our problem (1.1-4). In [12, Theorem 4.1] we constructed a monotone local semiflow $\Phi$, $\Phi_t n_0 = n(\cdot,t)$ for $0 \leq t < \tau(n_0) \leq \infty$ associated with (1.1-4) in a Sobolev space $X$ over $\mathbb{R}$ where $\partial_x \ln(n_0/N_0)$ belongs to a weighted $L^2$-space over $\mathbb{R}$ for some traveling wave $N(x,t) = N_0(x-\gamma t)$ with the velocity $\gamma = cd/b$. We will show that $\Phi$ is global (i.e., $\tau(n_0) = \infty$) and $L^1(\mathbb{R})$-contractive (Theorem 3.1') in a subspace $Y$ of $X$ where $\partial_x \ln(n_0/N_0)$ belongs to a weighted $L^\infty$-space over $\mathbb{R}$. Also, $n(\cdot,t) - N(\cdot,t) \to 0$ (as $t \to \infty$) in the $L^1(\mathbb{R})$-norm for a uniquely determined (by $n_0 \in Y$) traveling wave $N$ (Theorem 3.2').

This paper is organized as follows: in Section 2, we formulate an equivalent problem on $(0,1)$ instead of $\mathbb{R}$. We use this problem to state our main results in Section 3. The proofs of these results are in Sections 4 and 5. In Section 4, we show that the portion of $\Phi$ generating classical solutions of our problem is global and contractive. This allows us to construct a semigroup of nonlinear contractions in a subset of $L^1_+(\mathbb{R})$ for the original problem (1.1-4), where $L^1_+(\mathbb{R}) = \{ \phi : \phi - h \in L^1(\mathbb{R}) \}$ with $h = \text{characteristic function of } \mathbb{R}^+$. In Section 5, we make use of a strict contraction argument to show that every semiorbit converges to an equilibrium. Finally, in Section 6, we compare the behavior of $\Phi$ to other parabolic dynamical systems.

As a general reference for quasilinear parabolic equations, we use the monograph by Ladyzhenskaya, Solonnikov and Uraltseva [9] for maximum principles Protter and Weinberger [11] and for monotone local semiflows Hirsch [5].

2. Transformation into a bounded interval. By a dilation of coordinates we may always assume that $b = c = d = 1$ in (1.1-4), cf. [12]. Let $\xi_0 \in \mathbb{R}$ and $t \in \mathbb{R}^+$ be fixed. We define a function (and a new variable replacing $x$)

$$z \equiv z(x) = (1 + e^{\xi_0 - x + t})^{-1/2} \quad \text{for } x \in \mathbb{R}. \quad (2.1)$$

Then $z$ is a homeomorphism of $\mathbb{R}$ onto $(0,1)$ with $\lim_{x \to -\infty} z(x) = 0$, $\lim_{x \to \infty} z(x) = 1$ and derivatives

$$z' = \frac{1}{2} z(1 - z^2) \quad \text{and} \quad z'' = \frac{1}{4} z(1 - z^2)(1 - 3z^2). \quad (2.2)$$

The traveling waves for (1.1, 3, 4) form a one-parametric family (in $\xi \in \mathbb{R}$, cf. [12])

$$N_{0,\xi}(x - t) = (1 + e^{\xi - x + t})^{-1} \quad \text{for } x \in \mathbb{R}, \quad t \in \mathbb{R}^+. \quad (2.3)$$

We write $M = N_{0,\xi_0}$ and $M_\xi = N_{0,\xi}$ for brevity. Note that

$$M(x - t) = z^2; \quad (2.4)$$
$$M'(x - t) = z^2(1 - z^2); \quad (2.5)$$
$$M''(x - t) = z^2(1 - z^2)(1 - 2z^2). \quad (2.6)$$
Next, we write the unknown \( n \) in (1.1-4) as

\[
n(x - t, t) = z^2 e^{-u(z, t)}
\]  

(2.7)

where \( u : (0,1) \times \mathbb{R}_+ \rightarrow \mathbb{R} \), thus obtaining a new equivalent problem for the unknown \( u \equiv u(z, t) \):

\[
\partial_t u = \frac{1}{4} (1 - z^2)^2 e^u \cdot \partial_z^2 u + \frac{1}{4} (1 - z^2)[(1 - 3z)e^u + 2z] \cdot \partial_z u + (1 - z^2)(e^u - 1)
\]  

(2.8)

for \( (z, t) \in Q \);

\[
u(z, 0) = u_0(z) \quad \text{for} \quad z \in \Omega;
\]

\[
limit_{z \rightarrow 0^+} z \cdot \partial_z u(z, t) = 0 \quad \text{for} \quad t > 0;
\]

\[
limit_{z \rightarrow 1^-} u(z, t) = 0 \quad \text{for} \quad t > 0.
\]

(2.9)

(2.10)

(2.11)

Here, \( \Omega = (0,1), Q = (0,1) \times (0,\infty) \) and \( u_0(z) = 2 \ln z - \ln n_0(x - t) \). This reformulation of our problem makes use of (2.1-6). The traveling waves are transformed into equilibria. The equilibria of (2.8, 10, 11) are now given by

\[
U_\omega(z) = -\ln(M_\xi(x - t)/M(x - t)) = \ln[\omega + (1 - \omega)z^2] \quad \text{for} \quad z \in \Omega,
\]

(2.12)

where \( \omega = e^{\xi - \xi_0} \in (0, \infty) \) is the new parameter. Note that

\[
U_\omega'(z) = \frac{2(1 - \omega)z}{\omega + (1 - \omega)z^2};
\]

(2.13)

\[
U_\omega''(0) = 0, \quad U_\omega''(0) = 2(\omega^{-1} - 1);
\]

(2.14)

\[
U_\omega'(1) = 2(1 - \omega);
\]

(2.15)

\[
\partial_\omega U_\omega(z) = \frac{1 - z^2}{\omega(1 - z^2) + z^2} > 0 \quad \text{for} \quad z \in \Omega;
\]

(2.16)

\[
\partial_\omega U_\omega'(z) = \frac{-2z}{[\omega(1 - z^2) + z^2]^2} < 0 \quad \text{for} \quad z \in \Omega.
\]

(2.17)

In the rest of this paper, we study the initial-boundary value problem (IBVP) (2.8-11). We leave the obvious reformulation of our main results for the problem (1.1-4) to the reader and to Section 6 (Discussion).

3. Main results. In order to state our results for the problem (2.8-11) we introduce the following notation: we set \( \overline{\Omega} = [0,1] \) and denote by \( W \) the Sobolev space of all absolutely continuous functions \( f : \overline{\Omega} \rightarrow \mathbb{R} \) satisfying \( f(1) = 0 \) and

\[
\|f\|_W = \left( \int_0^1 f'(z)^2 \, dz \right)^{1/2} < \infty.
\]
Let $Q_T = (0, 1) \times (0, T)$ for $T \in (0, \infty)$. We denote by $\mathcal{W}_T$ the Sobolev space of all strongly measurable functions $u : (0, T) \rightarrow W$ satisfying

$$
\|u\|_{\mathcal{W}_T} = \left\{ \int_0^T \|u(\cdot, t)\|_W^2 + \int_0^1 |\partial_z^2 u(z, t)|^2(1-z)^2dz \\
+ \int_0^1 |\partial_t u(z, t)|^2(1-z)^{-2}dz dt \right\}^{1/2} < \infty
$$

together with $\partial_z u(0, t) = 0$ and $\lim_{z \rightarrow 1-} (1-z) \cdot \partial_z u(z, t) = 0$, for almost every $t \in (0, T)$, in the sense of traces in $L^2(0, T)$. Here, $\partial_z^2 u$ and $\partial_t u$ are distributional derivatives in $Q_T$. In particular, every $u \in \mathcal{W}_T$ satisfies the boundary conditions (2.10, 11) for almost every $t \in (0, T)$, cf. [12, equation (4.1-3)].

**Definition 3.0.** A function $u : Q_T \rightarrow \mathbb{R}$ is called a mild solution of the initial boundary value problem (2.8-11) in $Q_T$ if:

(i) $u \in C^{2,1}(Q_T)$; i.e., $\partial_z^2 u$ and $\partial_t u$ exist and are continuous in $Q_T$;

(ii) $u$ satisfies (2.8) in $Q_T$;

(iii) $u \in \mathcal{W}_T$;

(iv) $u : (0, T) \rightarrow W$ is continuous with $\lim_{t \rightarrow 0^+} u(\cdot, t) = u_0 \in W$, a limit in $W$.

A mapping $\Phi : D(\Phi) \subset W \times \mathbb{R}_+ \rightarrow W$ with domain $D(\Phi)$ is called a local semiflow in $W$ if it satisfies:

(i) $D(\phi)$ is open in $W \times \mathbb{R}_+$;

(ii) $\Phi$ is continuous;

(iii) $W \times \{0\} \subset D(\Phi)$ and $\Phi(\cdot, 0) = id_W$, the identity mapping on $W$; and

(iv) if $(f, t) \in D(\Phi)$ and $0 \leq s \leq t$, then $(f, s) \in D(\Phi)$, $(\Phi(f, s), t-s) \in D(\Phi)$ and $\Phi(\Phi(f, s), t-s) = \Phi(f, t)$.

We write $\Phi_t \equiv \Phi(\cdot, t), t \in \mathbb{R}_+$. A mapping $\phi : D(\phi) \subset W \rightarrow W$ is called monotone if $f, g \in D(\phi)$ and $f \leq g$ in $\Omega$ imply $\phi(f) \leq \phi(g)$ in $\Omega$. Finally, a local semiflow $\Phi$ in $W$ is called monotone if every $\Phi_t$ $(t \in \mathbb{R}_+)$ is monotone.

The following result was proved in [12, Thm. 4.1]:

There exists a unique local semiflow $\Phi$ in $W$ which has the following property:

(a) given $f \in W$, there exists $\tau = \tau(f) \in (0, \infty]$ such that $u = \Phi(f, \cdot)$ is a unique mild solution of (2.8-11) in every $Q_T$, $T \in (0, \tau)$, and either $\tau = \infty$ or else

$$
\lim_{t \rightarrow \tau^-} \|u(\cdot, t)\|_W = \infty. \quad (3.1)
$$

Moreover, $\Phi$ is monotone and every $u = \Phi(f, \cdot), f \in W$, is $C^\infty$ in $Q_T$.

The number $\tau = \tau(f)$ from (a) is called the escape time for $f \in W$. Given $0 < \omega_1 < \omega_2 < \infty$, we set $V_{\omega_1, \omega_2} = \{ f \in V : U_{\omega_2}' \leq f' \leq U_{\omega_1}' \}$, almost everywhere in $\Omega$. Recall that $U_\omega$ is defined by (2.12) and satisfies (2.17). Define $V = \bigcup \{ V_{\omega_1, \omega_2} : 0 < \omega_1 < \omega_2 < \infty \}$. Notice that $f'(0) = f(1) = 0$ for each $f \in V$. It follows from (2.15) that $V$ is not a linear space. More precisely, (2.13, 17) yield

$$
\frac{-2z}{1-z^2} < f'(z) < \frac{2}{z} \quad \text{for} \quad z \in \Omega, \quad f \in V. \quad (3.2)
$$
We denote by \( E \) the set of all (equivalence classes of) Lebesgue measurable functions \( f : \Omega \rightarrow (-\infty, \infty] \) with
\[
\int_0^1 |e^{-f(z)} - 1| \frac{z}{1-z^2}dz < \infty.
\] (3.3)

Given \( f, g \in E \), we set
\[
\rho(f, g) = 2 \int_0^1 |e^{-f(z)} - e^{-g(z)}| \frac{z}{1-z^2}dz.
\]

Then \((E, \rho)\) is a complete metric space. Obviously, \( V \subset W \subset E \) as sets, cf. \([12, \text{equation (4.3)}]\). Denote \( \overline{V} = \text{closure of } V \text{ in } E \). Given \( f \in E \), it can be shown that \( f \in \overline{V} \text{ if and only if } f \) is locally absolutely continuous in \( \Omega \) and
\[
-\frac{2z}{1-z^2} \leq f'(z) \leq \frac{2}{z} \quad \text{a.e. in } \Omega.
\] (3.4)

To formulate a conservation law we introduce
\[
\mathcal{M}(f) = 2 \int_0^1 (e^{-f(z)} - 1) \frac{z}{1-z^2}dz \quad \text{for } f \in E.
\] (3.5)

Finally, a local semiflow \( \overline{\Psi} : D(\overline{\Psi}) \subset \overline{V} \times \mathbb{R}_+ \rightarrow \overline{V} \) is called a semigroup of contractions in \( \overline{V} \) if \( D(\overline{\Psi}) = \overline{V} \times \mathbb{R}_+ \) and \( \rho(\overline{\Psi}_t(f), \overline{\Psi}_t(g)) \leq \rho(f, g) \) for all \( f, g \in \overline{V} \) and \( t \in \mathbb{R}_+ \).

Our two main results are as follows:

**Theorem 3.1.** The local semiflow \( \Phi \) from \([12, \text{Theorem 4.1}]\) has the following additional properties:

(a') \( V \times \mathbb{R}_+ \subset D(\Phi) \) and \( \Phi_t(V_{\omega_1, \omega_2}) \subset V_{\omega_1, \omega_2} \) for all \( t \in \mathbb{R}_+ \) and \( 0 < \omega_1 < \omega_2 < \infty \);

(b') if \( u_0 \in W \) and \( u_t = \Phi(u_0, t) \), then
\[
\mathcal{M}(u_t) = \mathcal{M}(u_0) \quad \text{for } 0 \leq t < \tau(u_0);
\]

(c') the restriction \( \Psi = \Phi|_{V \times \mathbb{R}_+} \) of \( \Phi \) to \( V \times \mathbb{R}_+ \) has a unique extension to a semigroup \( \Psi \) of contractions in \( \overline{V} \);

(d') \( \overline{\Psi} \) is monotone.

**Theorem 3.2.** Let \( \overline{\Psi} \) be as above, \( u_0 \in \overline{V} \) and \( \omega = \exp(-\mathcal{M}(u_0)) \). Then,
\[
\rho(u_t, U_\omega) \rightarrow 0 \quad \text{as } t \rightarrow \infty.
\] (3.6)

Furthermore, given any \( f, g \in V \) and \( t \in (0, \infty) \), the equality in
\[
\rho(\Psi_t(f), \Psi_t(g)) \leq \rho(f, g)
\] (3.7)
occurs if and only if either \( f \leq g \) in \( \Omega \) or else \( f \geq g \) in \( \Omega \).

These two theorems have a simple interpretation in terms of the original Cauchy problem (1.1-4) where we assume \( b = c = d = 1 \) without loss of generality. Namely,
let $L^1_+(\mathbb{R}) = \{ \phi : \phi - h \in L^1(\mathbb{R}) \}$ have the $L^1(\mathbb{R})$-metric translated by $h = \text{characteristic function of } \mathbb{R}_+$. Let $J : E \rightarrow L^1_+(\mathbb{R})$ be defined by $J(f)(x) = z(x)^2e^{-f(z(x))}$ for $f \in E$ and $x \in \mathbb{R}$, cf. (2.1, 7). Then $J$ is an isometry of $E$ onto a closed subset $J(E)$ of $L^1_+(\mathbb{R})$, by (2.2):

$$\|J(f) - J(g)\|_{L^1(\mathbb{R})} = \int_{-\infty}^{\infty} |J(f) - J(g)|dx = \rho(f,g) \quad \text{for} \quad f, g \in E. \quad \text{(3.8)}$$

Moreover, for every $f \in E$ we have

$$\int_{-\infty}^{\infty} [J(f) - J(0)]dx = M(f). \quad \text{(3.9)}$$

Given $t \in \mathbb{R}_+$ and $n_0 \in J(\mathbb{V})$, we define

$$S_t(n_0)(x-t) = J(\mathbb{V}_t(J^{-1}(n_0)))(x) \quad \text{for} \quad x \in \mathbb{R}, \quad \text{(3.10)}$$

and notice that $n_t = S(t_0, t) \equiv S_t(n_0) \in J(\mathbb{V})$. We set $Y = J(\mathbb{V})$ and observe that the closure $\mathbb{V}$ of $Y$ in $L^1_+(\mathbb{R})$ satisfies $\mathbb{V} = J(\mathbb{V})$. Finally, we endow $X = J(W)$ with the metric

$$\sigma(\phi, \psi) = \|J^{-1}(\phi) - J^{-1}(\psi)\|_W \quad \text{for} \quad \phi, \psi \in X. \quad \text{(3.11)}$$

Then $(X, \sigma)$ is a complete metric space and

$$\sigma(\phi, \psi) = \left[ \int_{-\infty}^{\infty} \left| \frac{d}{dx}\ln\left(\frac{\phi}{\psi}\right)\right|^2 w_{\xi_0}(x)dx \right]^{1/2} \quad \text{(3.12)}$$

where $w_{\xi_0}(x) = 2(1 + e^{\xi_0 - x})^{1/2}(1 + e^{x - \xi_0})$ with $\xi_0 \in \mathbb{R}$ fixed by (2.1, 2, 4, 7). The mapping $R : D(R) \subset X \times \mathbb{R}_+ \rightarrow X$ is defined in the same way as $S$ in (3.10) with $\Phi$ in place of $\mathbb{V}$, and

$$(\phi, t) \in D(R) \iff (J^{-1}(\phi), t) \in D(\Phi).$$

The last important property of $J$ is that $-J$ is an order isomorphism of $E$ into $L^1_+(\mathbb{R})$: given any $f, g \in E$, we have

$$f \leq g \quad \text{a.e. in } \Omega \iff J(f) \geq J(g) \quad \text{a.e. in } \mathbb{R}. \quad \text{(3.13)}$$

The following result is a direct consequence of [12, Thm. 4.1]:

$R$ is a monotone local semiflow in $X$. Moreover, given $n_0 \in X$, the function $n : \mathbb{R} \times [0, \tau) \rightarrow (0, \infty)$ defined by $n = R(n_0, \cdot)$, where $\tau = \tau(n_0) > 0$ is the escape time for $n_0$, has the following properties:

(i') $n$ is $C^\infty$ in $\mathbb{R} \times (0, \tau)$ and satisfies (1.1) in $\mathbb{R} \times (0, \tau)$;

(ii') for every $T \in (0, \tau)$, we have

$$e^{-\frac{3}{2}x}\int_0^T |\partial_x \ln n(x,t) - 1|^2 dt \rightarrow 0 \quad \text{as} \quad x \rightarrow -\infty, \quad \text{(3.13)}$$

and

$$e^{\frac{3}{2}x}(1 - n(x,t)) \rightarrow 0 \quad \text{as} \quad x \rightarrow \infty, \quad \text{uniformly in} \quad t \in [0, T]. \quad \text{(3.14)}$$

The convergence results (3.13, 14) follow directly from [12, equations (4.1, 3), resp.]. Theorems 3.1 and 3.2 can be restated as follows:
Theorem 3.1'. $S$ is a semigroup of monotone contractions in $\bar{Y} \subset L^1_+(\mathbb{R})$. Moreover, we have $S_t(n_0) = R_t(n_0) \in Y$ for all $(n_0, t) \in Y \times \mathbb{R}_+$, and

$$\int_{-\infty}^{\infty} [S_t(n_0) - n_0]dx = 0 \quad \text{for} \quad (n_0, t) \in \bar{Y} \times \mathbb{R}_+. \quad (3.15)$$

Theorem 3.2'. Let $n_0 \in \bar{Y}$ and $\xi = \xi_0 - \int_{-\infty}^{\infty} (n_0 - N_0, \xi_0)dx$, where $N_0, \xi_0$ is defined by (2.3) for $\xi_0 \in \mathbb{R}$. Then,

$$\|S_t(n_0) - N_0, \xi(\cdot - t)\|_{L^1(\mathbb{R})} \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty. \quad (3.16)$$

In order to stress an important advantage of our approximation of $\bar{Y}$ by $Y$ in comparison with the semigroup approach used in [6, 7, 8, 10], we observe that a function $\phi \in L^1_+(\mathbb{R})$ belongs to $\bar{Y}$ if and only if $\phi = J(f)$ for some $f \in E$ satisfying (3.4). Combining this observation with (2.1, 2, 3, 7), we conclude that $\bar{Y}$ is the set of all locally absolutely continuous functions $\phi : \mathbb{R} \longrightarrow (0, \infty)$ satisfying

$$\phi|_{(-\infty, 0)} \in L^1(-\infty, 0), \quad 1 - \phi|_{\mathbb{R}_+} \in L^1(\mathbb{R}_+) \quad (3.17)$$

and

$$0 \leq \frac{d}{dx} \ln \phi \leq 1 \quad \text{a.e. in} \ \mathbb{R}. \quad (3.18)$$

In particular, if $n_t = S_t(n_0)$, $t \in \mathbb{R}_+$, for a given $n_0 \in \bar{Y} \setminus Y$, it is not clear in what sense equation (1.1) and the boundary condition (1.3) are satisfied. Notice that (1.4) holds by (3.17, 18). The only case we know about in which (1.1, 3) are satisfied is the approximation of $n_t$ by elements from $Y \subset X$. A different approximation method was used in McKean [10] to prove the central limit theorem for Carleman's equation in $L^1(\mathbb{R}) = \{\phi \in L^1(\mathbb{R}) : \phi \geq 0 \text{ almost everywhere in } \mathbb{R}\}$. Our monotonicity result for $S$ can be derived also from the semigroup approach by a result of Crandall and Tartar [2] who showed that Carleman's equation enjoys analogous monotonicity properties.

The proofs of Theorems 3.1 and 3.2 are given in Sections 4 and 5, respectively.

4. Global existence and a conservation law. We prove each part of Theorem 3.1 separately.

\textbf{Proof of (a')}: We fix $0 < \omega_1 < \omega_2 < \infty$ and observe that $V_{\omega_1, \omega_2} = V_{\omega_1, \infty} \cap V_{0, \omega_2}$, where $V_{\omega_1, \infty} = \{f \in W : f' \leq U'_{\omega_1} \text{ a.e. in } \Omega\}$ and $V_{0, \omega_2} = \{f \in W : f' \geq U'_{\omega_2} \text{ a.e. in } \Omega\}$. Combining this fact with (3.1), we conclude that it suffices to prove (a') for each $V_{\omega_1, \infty}$ and $V_{0, \omega_2}$ in place of $V_{\omega_1, \omega_2}$ separately and for every $t \in (0, T)$, where $T = T(\omega_1, \omega_2) \in (0, \infty)$ is a constant depending only on $\omega_1$ and $\omega_2$. Furthermore, choosing $\xi_0 \in \mathbb{R}$ in (2.1) such that either $\omega_1 = e^{\xi_0}$ or $\omega_2 = e^{\xi_0}$, we can force either $\omega_1 = 1$ or $\omega_2 = 1$, respectively, as it follows from the following simplification.

\textbf{Remark}. Consider a mild solution $u = \Phi(u_0, \cdot)$ of the IBVP (2.8-11) in $Q_T$, for some $u_0 \in W$ and $T \in (0, \tau(u_0))$, and an equilibrium $U_{\omega}$ of $\Phi$, for some $\omega \in (0, \infty)$. We fix $\xi_0 \in \mathbb{R}$ and find $\xi \in \mathbb{R}$ with $e^{\xi - \xi_0} = \omega$ as in Section 2. We define a function

$$z^* \equiv z^*(z) = z[\omega + (1 - \omega)z^2]^{-1/2} \quad \text{for} \quad z \in [0, 1]. \quad (4.1)$$
Then \( z^* \) is a homeomorphism of \([0, 1]\) onto itself with the derivative
\[
\frac{dz^*}{dz} = \omega [\omega + (1 - \omega)z^2]^{-3/2} > 0 \quad \text{for} \quad z \in [0, 1],
\]
and satisfies also \( z^* = (1 + e^{\xi - x + t})^{-1/2} \) in accordance with (2.1). Finally, we define another function
\[
u^*(z^*, t) = u(z, t) - U_w(z) \quad \text{for} \quad (z^*, t) \in \overline{\Omega} \times [0, T).
\]
It is clear that \( u^* \) is a mild solution of the IBVP (2.8-11) in \( Q_T \) with \( z^* \) in place of \( z \) and \( u_0^*(z^*) = u_0(z) - U_w(z), \ z^* \in \overline{\Omega} \), in place of \( u_0(z) \). Conclusion: Whenever studying the difference \( u - U_w \), we may assume \( U_w = 0 \) in \( \Omega \), i.e., \( \omega = 1 \).

Since the proofs of (a') for both \( V_{1,\infty} \) and \( V_{0,1} \) are completely analogous, we will prove (a') for \( V_{1,\infty} \) only. Thus, it suffices to show the following result.

**Proposition 4.1.** Given \( \mu \in (0, \infty) \), there exists a constant \( T \in (0, \infty) \) such that \( \tau(u_0) > T \) for every \( u_0 \in B^W_{\mu} = \{ f \in W : ||f||_W \leq \mu \} \), and if also \( u_0^* \leq 0 \) almost everywhere in \( \Omega \), then \( u'_t \leq 0 \) almost everywhere in \( \Omega \) for every \( t \in (0, T) \) where \( u_t = \Phi(u_0, t) \).

**Proof:** Fix \( \mu \in (0, \infty) \). By [12, Prop. 6.7] there exists a constant \( T \in (0, \infty) \) depending only on \( \mu \) such that, given \( u_0 \in B^W_{\mu} \), the IBVP (2.8-11) in \( Q_T \) has a unique mild solution \( u = \Phi(u_0, \cdot) \). Assume now that \( u_0^* \leq 0 \) almost everywhere in \( \Omega \). Since each \( \Phi_t|_{B^W_{\mu}} : B^W_{\mu} \rightarrow W, \ t \in (0, T) \), is continuous, it suffices to show \( u'_t \leq 0 \) for all \( u_0 \in D \) where \( D \) is a dense subset of \( \{ f \in B^W_{\mu} : f' \leq 0 \ \text{a.e. in} \ \Omega \} \subset W \). We choose \( D = \{ f \in B^W_{\mu} \cap C^3(\overline{\Omega}) : f' \leq 0 \ \text{and} \ f'(0) = f(1) = 0 \} \). So we fix \( u_0 \in D \). Again, by [12, Prop. 6.7, equation (6.9)], we have
\[
u = \Phi(u) \quad \text{weakly in} \ \mathcal{L}^2(Q_T) \ 	ext{as} \ \alpha \rightarrow 0^+, \ \text{where every} \ \nu(\alpha), \ \alpha \in (0, \frac{1}{2}) \ \text{is the unique classical solution of the IBVP}
\]
\[
\partial_t u = \frac{1}{4} \phi_1 e^u \cdot \partial_z^2 u + \frac{1}{4} (1 - z^2) [(\phi_0 - 3z) e^u + 2z \cdot \partial_z u + (1 - z^2)(e^u - 1) \quad (4.5)
\]
for \((z, t) \in Q_T; \)
\[
u(z, 0) = u_0(z) \quad \text{for} \quad z \in \Omega; \quad (4.6)
\]
\[
\partial_z u(0, t) = 0 \quad \text{for} \quad t \in (0, T); \quad (4.7)
\]
\[
u(1, t) = 0 \quad \text{for} \quad t \in (0, T) \quad (4.8)
\]
in the space \( C^{2+\beta, 1+\beta/2}(\overline{\Omega}_T) \) of all functions \( u : \overline{\Omega}_T \rightarrow \mathbb{R} \) such that \( u, \partial_z u, \partial_z^2 u \) and \( \partial_t u \) are continuous in \( \overline{\Omega}_T \) and \((\beta, \beta/2)\)-Hölder continuous in \((z, t)\) uniformly in \( \overline{\Omega}_T \), for any fixed \( \beta \in (0, 1) \). Here, \((\phi_0, \phi_1) = (\phi_0(\alpha), \phi_1(\alpha)) \) is a pair of \( C^\infty \)-functions \( \phi_0, \phi_1 : \mathbb{R} \rightarrow \mathbb{R} \) which are even (about 0) and satisfy:
\[
(\phi_0) \ \phi_0(z) = 1/|z| \ \text{for} \ |z| \geq \alpha, \ \text{and} \ 0 \leq \phi_0(z) \leq 1/|z| \ \text{for} \ |z| < \alpha;
\]
\[
(\phi_1) \ \phi_1(z) = (1 - z^2)^2 \ \text{for} \ |z| \leq 1 - \alpha, \ \phi_1(1) > 0, \ \text{and} \ (1 - z^2)^2 \leq \phi_1(z) \leq 1 \ \text{for} \ z \in \mathbb{R}.
\]
The existence and uniqueness of such \( \nu(\alpha) \), for each \( \alpha \in (0, \frac{1}{2}) \), follow from [12, Thm. 6.4] combined with the global regularity result [9, Chap. IV, §5, Thm. 5.3]. We conclude from (4.4) that it suffices to prove the following.
Lemma 4.2. Let $\alpha \in (0, \frac{1}{2})$, $\beta \in (0,1)$ and $u_0 \in D$ be fixed. Then the IBVP (4.5-8) in $Q_T$ has a unique classical solution $u \in C^{2+\beta,1+\beta/2}(\overline{Q_T})$, i.e., all equations (4.5-8) hold pointwise everywhere, and $u \geq 0$ and $\partial_z u \leq 0$ in $\overline{Q_T}$.

Proof: First we prove $u \geq 0$ in $\overline{Q_T}$. We observe that $u_0 \in D$ entails $u_0 \geq 0$ in $\overline{\Omega}$. Hence, we may apply the parabolic maximum and boundary point principles [11, Chap. 3, Sec. 2 and 3] to (4.5) to obtain $u \geq 0$ in $\overline{Q_T}$ together with

$$\partial_z u(1,t) < 0 \quad \text{for} \quad t \in (0,T).$$

(4.9)

To prove $\partial_z u \leq 0$ in $\overline{Q_T}$, we differentiate (4.5) with respect to $z$ and set $v = \partial_z u$, thus arriving at

$$-\partial_t v + a(z,t) \cdot \partial_z^2 v + b(z,t) \cdot \partial_z v + c(z,t)v = 2z(e^u - 1) \geq 0$$

(4.10)

in $Q_T$, where $a = \frac{1}{2} \phi_1 e^u$, $b, c \in C(Q_T)$ with $a > 0$ in $\overline{Q_T}$, and $u \in C^\infty(Q_T)$ by the interior regularity theorem of Friedman [4, Chap. 3, Sec. 5, Thm. 11]. Since $v \in C(\overline{Q_T})$ satisfies $v \leq 0$ on $\overline{\Omega} \times \{0\}$ and $\{0,1\} \times [0,T]$, we may apply the maximum principle to (4.10) to get $v \leq 0$ in $\overline{Q_T}$ as desired. Thus, we have proved (a').

Proof of (b'): For all $\alpha \in (0, \frac{1}{2})$ and $f \in E$, we introduce

$$M^{(\alpha)}(f) = 2 \int_0^{1-\alpha} (e^{-f(z)} - 1) \frac{z}{1-z^2} dz.$$

Notice that $M^{(\alpha)}(f) \to M(f)$ as $\alpha \to 0+$ by the Lebesgue dominated convergence theorem. Next we fix $\alpha \in (0, \frac{1}{2})$ and $u_0 \in W$. Then, $u = \phi(u_0, \cdot)$ is $C^\infty$ in $Q_T$ where $\tau = \tau(u_0) > 0$, and so (2.8) implies

$$\frac{d}{dt} M^{(\alpha)}(u_t) = -2 \int_0^{1-\alpha} e^{-u} \cdot \partial_t u \frac{z}{1-z^2} dz$$

(4.11)

$$= - \int_0^{1-\alpha} \partial_z \left[ \frac{1}{2} z(1-z^2) \cdot \partial_z u + z^2(1-e^{-u}) \right] dz$$

$$= - \frac{1}{2} \left[ z(1-z^2) \cdot \partial_z u \right]_{z=\alpha}^{z=1-\alpha} - \left[ z^2(1-e^{-u}) \right]_{z=\alpha}^{z=1-\alpha}$$

for all $t \in (0,\tau)$. Integrating this equality over any subinterval $[0,t] \subset [0,\tau)$ and letting $\alpha \to 0+$, we arrive at $M(u_t) - M(u_0) = 0$, by the trace relations [12, equation (4.1-3)] combined with the Cauchy-Schwartz inequality since $u \in W_t$. This verifies (b').

Proof of (c'): First, we treat the case $u_0, v_0 \in V$ with $u_0 \leq v_0$ in $\overline{\Omega}$. Then also $u_t \leq v_t$ in $\overline{\Omega}$ since $\Phi_t$ is monotone for each $t \in \mathbb{R}_+$. Hence, (b') shows that

$$\rho(u_t, v_t) = M(u_t) - M(v_t) = M(u_0) - M(v_0) = \rho(u_0, v_0).$$

Now, let $u_0, v_0 \in V$ be arbitrary. Then, $\bar{u}_0 = \min\{u_0, v_0\}$ and $\bar{v}_0 = \max\{u_0, v_0\}$ belong to $V$ since their derivatives satisfy

$$\min\{u'_0, v'_0\} \leq \min\{\bar{u}'_0, \bar{v}'_0\} \leq \max\{u'_0, v'_0\} \quad \text{a.e. in} \quad \Omega,$$
by the elementary inequalities (for all \(a_1, a_2, b_1, b_2 \in \mathbb{R}\))

\[
\min\{a_1 - a_2, b_1 - b_2\} \leq \min_{\max} \{a_1, b_1\} - \min_{\max} \{a_2, b_2\} \leq \max\{a_1 - a_2, b_1 - b_2\}.
\]

Consequently, the monotonicity of \(\Phi_t (t \in \mathbb{R}_+)\) implies

\[
\Phi_t(\tilde{u}_0) \leq \min_{\max} \{u_t, v_t\} \leq \Phi_t(\tilde{v}_0) \quad \text{in } \overline{\Omega}.
\]

Making use of the identity \(|a - b| = \max\{a, b\} - \min\{a, b\}\), for \(a, b \in \mathbb{R}\), we arrive at

\[
\rho(u_t, v_t) \leq \rho(\Phi_t(\tilde{u}_0), \Phi_t(\tilde{v}_0)) = \rho(\tilde{u}_0, \tilde{v}_0) = \rho(u_0, v_0)
\]

for every \(t \in \mathbb{R}_+\), by \(\Phi_t \circ \tilde{u}_0 \leq \tilde{v}_0\) in \(\overline{\Omega}\). In particular, the restriction \(\Psi = \Phi |_{V \times \mathbb{R}_+}\) of \(\Phi\) to \(V \times \mathbb{R}_+\) forms a semigroup of contractions in \(V\) with respect to the metric \(\rho\) since the imbedding \(W \hookrightarrow E\) is continuous. This clearly implies \((c')\).

\[(d')\] follows directly from the monotonicity of \(\Phi\) combined with the density of the order relation \(O_v = O_E \cap (V \times V)\) in \(O_E = \{(f, g) \in E \times E : f \leq g \text{ almost everywhere in } \Omega\} \subset E \times E\) and the continuity of \(\overline{\Psi}_t : V \rightarrow V, t \in \mathbb{R}_+\).

The proof of Theorem 3.1 is complete.

5. Convergence to an equilibrium. We prove Theorem 3.2 in two steps.

Step 1. Let \(f, g \in V\). If either \(f \leq g\) in \(\Omega\) or \(f \geq g\) in \(\Omega\), then the equality in (3.7) holds, by the proof of Theorem 3.1 (c'). Assume now that the equality in (3.7) holds for some fixed \(t \in (0, \infty)\), but \(f(\zeta_1) > g(\zeta_1)\) and \(f(\zeta_2) < g(\zeta_2)\) for some \(\zeta_1, \zeta_2 \in \Omega\). We set \(u_0 = f\), \(v_0 = g\) and continue using the notation from the proof of Theorem 3.1 (c'). We denote \(\phi_s = \min\{u_s, v_s\}\) and \(\psi_s = \max\{u_s, v_s\}\) for \(s \in \mathbb{R}_+\). Then (4.13) entails

\[
\int_0^1 |e^{-u_s} - e^{-v_s}| \frac{z}{1 - z^2} dz = \int_0^1 (e^{-\Phi_s(\tilde{u}_0)} - e^{-\Phi_s(\tilde{v}_0)}) \frac{z}{1 - z^2} dz
\]

for each \(s \in [0, t]\). By (4.12), this is equivalent to

\[
\phi_s = \Phi_s(\tilde{u}_0) \quad \text{and} \quad \psi_s = \Phi_s(\tilde{v}_0) \quad \text{for} \quad 0 \leq s \leq t.
\]

On the other hand, consider the functions \(F(\cdot, s) = u_s - \Phi_s(\tilde{u}_0) \geq 0\) and \(G(\cdot, s) = \Phi_s(\tilde{v}_0) - u_s \geq 0\) in \(\overline{\Omega}\), \(0 \leq s \leq t\). Notice that \(F(\zeta_1, 0) > 0\) and \(G(\zeta_2, 0) > 0\) by our hypotheses. Furthermore, both \(F\) and \(G\) satisfy a parabolic PDE of the form (cf. [12, equation (6.4)])

\[
\partial_s w = a(z, t) \cdot \partial^2 w + b(z, t) \cdot \partial z w + c(z, t) w \quad \text{in } \Omega_t,
\]

where \(a, b, c, w \in C^\infty(Q_t)\) with \(a > 0\) in \(Q_t\) and \(0 \leq w \in C(\overline{Q}_t)\). By the strong maximum principle (cf. [11]) we have \(w > 0\) throughout \(Q_t\) unless \(w \equiv 0\) in \(\overline{Q}_t\). Hence, we have \(F > 0\) and \(G > 0\) throughout \(Q_t\), and therefore \(\phi_s < u_s\) and \(u_s < \psi_s\) in \(\Omega\) for \(0 < s < t\), by (5.1), a contradiction. We conclude that equality in (3.7) forces either \(f \leq g\) in \(\Omega\) or else \(f \geq g\) in \(\Omega\).
Step 2. To prove (3.6), we fix $u_0 \in V$ and set $\omega = \exp(-M(u_0))$. By Theorem 3.1 (b'), we have
\[ M(u_t) = M(U_\omega) \quad \text{for} \quad t \in \mathbb{R}_+. \tag{5.3} \]
If $u_t = U_\omega$ in $\Omega$ for some $t \in \mathbb{R}_+$, then (3.6) holds, since $U_\omega$ is an equilibrium for $\Phi$. Hence, by (5.3), we may assume that the set $\hat{U} = U \cup \{U_\omega\}$, where $U = \{u_t : t \in \mathbb{R}_+\}$ is unordered, i.e., if $f, g \in \hat{U}$ satisfy either $f \geq g$ in $\Omega$ or $f \leq g$ in $\Omega$ then $f = g$ in $\Omega$. Furthermore, Theorem 3.1 (a') shows that $U \subset V_{\omega_1, \omega_2}$ for some $0 < \omega_1 < \omega_2 < \infty$. It is an easy consequence of Arzelà-Ascoli's theorem that $V_{\omega_1, \omega_2}$ is a compact subset of $E$, because $V'_{\omega_1, \omega_2} = \{f' : f \in V_{\omega_1, \omega_2}\}$ is $w^*$-compact in $L^\infty(\Omega)$ by Alaoglu's theorem. We conclude that the function $\delta(t) = \rho(u_t, U_\omega)$ is strictly decreasing in $t \in \mathbb{R}_+$ by Step 1, and there exists $v_0 \in V_{\omega_1, \omega_2}$ such that $u_{t_n} \rightarrow v_0$ in $E$ for some sequence $t_n \rightarrow \infty$. Obviously,
\[ \lim_{t \rightarrow \infty} \delta(t) = \rho(v_0, U_\omega) \tag{5.4} \]
and
\[ M(v_0) = M(U_\omega). \tag{5.5} \]
If $v_0 = U_\omega$ in $\Omega$, then (3.6) holds. So let us assume $v_0(\zeta) \neq U_\omega(\zeta)$ for some $\zeta \in \Omega$. Then also the set $\hat{U}_0 = \hat{U} \cup \{v_0\}$ is unordered and $\epsilon(s) = \rho(\Phi_s(v_0), U_\omega)$, $s \in \mathbb{R}_+$, satisfies $\epsilon(s) < \epsilon(0) = \lim_{t \rightarrow \infty} \delta(t)$ for every $s > 0$. Now, fix any $s \in (0, \infty)$ and denote $\eta = [\epsilon(0) - \epsilon(s)]/2 > 0$. Then for every sufficiently large integer $n$, we have
\[ \rho(\Phi_s(u_{t_n}), \Phi_s(v_0)) \leq \rho(u_{t_n}, v_0) \leq \eta \]
by Theorem 3.1 (c'). Thus, the triangle inequality yields
\[ \delta(s + t_n) \leq \eta + \epsilon(s) = \epsilon(0) - \eta. \]
Finally, letting $n \rightarrow \infty$ we arrive at $\epsilon(0) \leq \epsilon(0) - \eta$ by (5.4), a contradiction. Hence, we must have $v_0 = U_\omega$ in $\Omega$, and (3.6) follows from (5.4).

Let now $u_0 \in V$ be arbitrary. To prove (3.6), we choose $\epsilon \in (0, \infty)$. Next, we find $v_0 \in V$, $\mu = \exp(-M(v_0))$ and $T \in \mathbb{R}_+$ such that
\[ \rho(u_0, v_0) \leq \epsilon/3 \quad \text{and} \quad \rho(v_t, U_\mu) \leq \epsilon/3 \quad \text{for} \quad t \geq T, \tag{5.6} \]
where $v_t = \Phi(v_0, t)$. Then also,
\[ \rho(u_t, v_t) \leq \epsilon/3 \quad \text{for} \quad t \in \mathbb{R}_+, \tag{5.7} \]
and
\[ \rho(U_\mu, U_\omega) = |M(U_\mu) - M(U_\omega)| = |M(v_0) - M(u_0)| \leq \rho(v_0, u_0) \leq \epsilon/3, \tag{5.8} \]
since $\Psi_t$ is contractive. Combining (5.6-8) with the triangle inequality, we obtain $\rho(u_t, U_\omega) \leq \epsilon$ for $t \geq T$. Since $\epsilon \in (0, \infty)$ was arbitrary, we have proved (3.6) and Theorem 3.2.
6. Discussion. As we have already mentioned in [12, Sec. 1], a global existence result for the IBVP (1.1-4) can be proved by constructing a semigroup of nonlinear contractions in a subset of $L^1_*(\mathbb{R})$, cf. Kurtz [8] for an analogous result for the IBVP (1.1, 2) with the no-flux boundary conditions. However, it is not clear in what sense the semigroup solution satisfies equations (1.1, 3, 4). Consequently, the strict contraction argument (cf. (3.7)) used in the proof of Theorem 3.2 (equation (3.6)) might be difficult to verify without the strong maximum principle which might not be available for the semigroup solutions. Also, our approach using typical PDE methods yields a semigroup $S$ of contractions in $\mathcal{Y}$, but its domain $\mathcal{Y}$ is an undesirably small subset of $L^1_*(\mathbb{R})$, cf. (3.18). Observe that (3.18) is equivalent to: 

\[ \phi(x) \text{ is nondecreasing and } \phi(x)e^{-x} \text{ is nonincreasing in } x \in \mathbb{R}. \]

Condition (3.18) is a severe restriction on the flux $\nu_0$. We know nothing about the large-time asymptotic behavior of the semigroup solution if the initial data do not satisfy (3.18).

Another interesting problem is to estimate the rate of convergence in (3.6). This might be possible by studying the corresponding linearization of equation (2.8) about the equilibrium $U_0$. However, we were not able to show that the solutions of the IBVP (2.8-11) and its linearization approach each other sufficiently fast as $t \to \infty$. Finally, the structure of convergence and the set of equilibria are similar to those for the discrete-time semigroups studied in [13, 14]. In particular, every trajectory $\{u_t = \Phi(u_0, t) : t \in [0, \tau(u_0))\}$, $u_0 \in W$, is contained in a $\Phi$-invariant smooth hypersurface $\mathcal{H}_0 = \{f \in W : M(f) = M(u_0)\}$ in $W$. Thus, to estimate the rate of convergence in (3.6), it suffices to study the restriction of $\Phi$ to $\mathcal{H}_0$. This restriction is strictly contractive (cf. (3.7)) in the metric $\rho$ since $\mathcal{H}_0$ is unordered.

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