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ON THE STABILITY OF THE ZERO SOLUTION OF A ONE-DIMENSIONAL MATHEMATICAL MODEL OF VISCOELASTICITY

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Abstract. The initial-boundary value problem for a mathematical model of a one-dimensional physical linear viscoelastic medium is considered. The Lyapunov's stability of the zero solution and, hence, nonlocal solvability for small initial data are established. First we prove a coercive solvability of some abstract differential equation of second order. Then by means of the abstract results obtained we reduce the original problem to an operator equation with a contractive operator.

1. Introduction. This work continues the investigations on the mathematical models of physical linear viscoelastic (multi-dimensional) and thermoelastic (one-dimensional) mediums carried out in [6–8]. The nonlinearities in the models above arose because of the difference between Lagrangian and Eulerian coordinates used for describing elasticity and viscosity respectively. In the works mentioned the local (in time) existence and uniqueness theorems were obtained.

Here we consider the initial-boundary value problem

$$u_{tt} - \mu_1 u_{xx} - \mu_2[(1+u_x)^{-1}u_{tx}]_x = f(t,x), \quad t \geq 0,\ 0 \leq x \leq 1,\ \ \mu_1, \mu_2 \geq 0; \quad (1)$$

$$u(0,x) = u_0(x),\ \ u_t(0,x) = u_1(x)\ \ (0 \leq x \leq 1), \quad u(t,0) = u(t,1) = 0\ \ (t \geq 0) \quad (2)$$

that describes in Lagrangian coordinates the motion of the one-dimensional viscoelastic medium (cf. [8]). We shall establish the Lyapunov's stability of the zero solution of problem (1)–(2) and, hence, the nonlocal solvability for "small" f, u_0 and u_1.

An important part in the proof of stability is an investigation of a coercive solvability and properties of some abstract differential equation of second-order in a Banach space. There is an extensive bibliography on second order abstract differential equations (cf. [3]). Peculiarity and goals in our case require a different approach to the solution of such equations.

2. Formulation of results. The solution of problem (1)–(2) is defined to be a function $u(t,x)$ having all (generalized) derivatives contained in the equation

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(1) belonging to $L_q(Q)$, $Q = [0,\infty) \times [0,1]$ for some $1 < q < +\infty$ and satisfying equation (1) and conditions (2). By W_q^k we denote the space of functions which have generalized derivatives up to order k (integer or fractional) summable in power q on $[0,1]$.

We denote the norm in W_q^k by $|\cdot|_k$, and in $L_q(Q)$ by $\|\cdot\|_0$. Let us consider first the linear problem

$$u_{tt} - \mu_1 u_{xx} - \mu_2 u_{txx} = f(t,x), \quad (t,x) \in Q; \tag{3}$$

$$u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x) \;\; (0 \le x \le 1), \quad u(t,0) = u(t,1) = 0, \quad t \ge 0. \tag{4}$$

Theorem 1. *Let* $1 < q < +\infty$, $f \in L_q(Q)$, $u_0 \in \overset{0}{W}{}_q^2$, $u_1 \in \overset{0}{W}{}_q^{2-2/q}$, $\mu_2^2 > \frac{4}{\pi^2}\mu_1$. *Then the problem* (3)–(4) *has a unique solution, and the estimate*

$$\begin{aligned} \|u_{tt}\|_0 + \|u_{txx}\|_0 + \|u_{xx}\|_0 + \|u\|_0 + \sup_{t\ge 0} |u|_2 + \sup_{t\ge 0} |u_t|_{2-2/q} \\ \le M\left[\|f\|_0 + |u_0|_2 + |u_1|_{2-2/q}\right] \end{aligned} \tag{5}$$

holds.

Remark 1. Let $3 < q < +\infty$. Then from (5) and embedding theorems for Sobolev spaces (cf. for example [1]) it follows the continuity with respect to t and x of the functions u, u_t, u_x, u_{tx}, the inequality

$$\begin{aligned} \sup_{(t,x)\in Q} \left[|u(t,x)| + |u_x(t,x)| + |u_t(t,x)| + |u_{tx}(t,x)|\right] \\ \le M\left[\|f\|_0 + |u_0|_2 + |u_1|_{2-2/q}\right], \end{aligned} \tag{6}$$

and the relation

$$\lim_{t\to+\infty}\left[\max_{0\le x\le 1} |u(t,x)| + \max_{0\le x\le 1} |u_x(t,x)| + |u(t,x)|_2 + |u_t(t,x)|_0\right] = 0. \tag{7}$$

For the proof of (7) the simple idea used is that from the summability on semiaxis with the power $q > 1$ of a function (of one variable) and its derivative, it follows that the function tends to zero at infinity.

Remark 2. The same constant M in various inequalities or in chains of inequalities has generally speaking different values.

Consider problem (1)–(2). Theorem 1 gives an opportunity to prove the main result.

Theorem 2. *Let the conditions of Theorem* 1 *be satisfied. Let further* $q > 3$. *Then for sufficiently small* $R > 0$ *there exists* $\delta > 0$ *such that for any* u_0, u_1 *and* f *with*

$$|u_0|_2 \le \delta, \quad |u_1|_{2-2/q} \le \delta, \quad \|f\|_0 \le \delta, \tag{8}$$

problem (1)–(2) *has a unique solution* $u(t,x)$ *and the estimate*

$$\|u_{tt}\|_0 + \|u_{txx}\|_0 + \|u_{xx}\|_0 + \sup_{t\ge 0} |u(t,x)|_2 + \sup_{t\ge 0} |u_t(t,x)|_{2-2/q} \le R \tag{9}$$

holds.

As well as in the case of Theorem 1 the estimate (9) implies the continuity of u, u_t, u_x, u_{tx}, the inequality

$$\sup_{(t,x)\in Q} [|u(t,x)| + |u_t(t,x)| + |u_x(t,x)| + |u_{tx}(t,x)|] \le RM \tag{10}$$

and the relation

$$\lim_{t\to+\infty} \left[\max_{0\le x\le 1} |u(t,x)| + \max_{0\le x\le 1} |u_x(t,x)| + |u(t,x)|_2 + |u_t(t,x)|_0 \right] = 0. \tag{11}$$

The proofs of Theorems 1 and 2 are in §§5 and 6, respectively. The investigation of the abstract differential equation is carried out in §§3 and 4.

3. Auxiliary results.

3.1. Consider in an arbitrary Banach space E the problem

$$u''(t) + \mu_2 A u'(t) + \mu_1 A u(t) = f(t), \quad t \ge 0, \tag{12}$$

$$u(0) = u_0, \quad u'(0) = u_1. \tag{13}$$

Here A is a strongly positive operator in E. We recall (cf. [2]) that the operator A is strongly positive if its domain of definition $\mathcal{D}(A)$ is dense in E and for any λ with $\operatorname{Re}\lambda \le \sigma_0$ ($\sigma_0 > 0$) the operators $\lambda I - A$ have bounded inverse operators $(\lambda I - A)^{-1}$ and the inequality

$$\|(\lambda I - A)^{-1}\| \le M(1 + |\lambda|)^{-1} \tag{14}$$

holds.

The solution of problem (12)–(13) is defined to be an E-valued function $u(t)$ ($t \ge 0$) such that $u'(t)$ is absolutly continuous, $u(t), u'(t) \in \mathcal{D}(A)$ almost everywhere, all terms in the left-hand side of (12) belong to $L_q = L_q(0,\infty;E)$, $1 < q < +\infty$ and equation (12) and conditions (13) are satisfied. The formal solution of (12)–(13) is given by the formula

$$\begin{aligned} u(t) = {} & \left[(A_1 - A_2)^{-1} \int_0^t T_1(t-s) f(s)\, ds + (A_2 - A_1)^{-1} \int_0^t T_2(t-s) f(s)\, ds\right] \\ & + \left[(A_1 - A_2)^{-1} T_1(t) + (A_2 - A_1)^{-1} T_2(t)\right] u_1 \\ & + \left[A_2 (A_2 - A_1)^{-1} T_1(t) + A_1 (A_1 - A_2)^{-1} T_2(t)\right] u_0. \end{aligned} \tag{15}$$

Here

$$A_1 = \frac{1}{2}\left[\mu_2 I + (\mu_2^2 I - 4\mu_1 A^{-1})^{1/2}\right] A, \tag{16}$$

$$A_2 = \frac{1}{2}\left[\mu_2 I - (\mu_2^2 I - 4\mu_1 A^{-1})^{1/2}\right] A, \tag{17}$$

$$T_1(t) = \exp(-tA_1), \quad T_2(t) = \exp(-tA_2) \quad (t \geq 0). \tag{18}$$

In §§3.2–3.5, we shall show that for

$$\sigma_0 > 4\mu_1\mu_2^{-2} \tag{19}$$

formula (15) makes sense and indeed represents the solution of problem (12)–(13).

3.2. By means of the formulas of Cauchy-Riemann (cf. [3]), we can define the fractional powers A^α of the strongly positive operator A, both positive and negative. From the estimate (14) it follows that there exists $\psi \in (0, \frac{\pi}{2})$ such that the spectrum $\sigma(A)$ of the operator A is situated in the set $K_0 = S[\psi] \cap \Pi_0$ where

$$S[\psi] = \{z : |\arg z| < \psi\}, \quad \Pi_0 = \{z : \operatorname{Re} z > \sigma_0\}.$$

Any strongly positive operator $-A$ generates the analytic semigroup $T(z) = \exp(-zA)$ in the sector

$$S_\psi = \{z : |\arg z| < \frac{\pi}{2} - \psi\}.$$

For the semigroup $T(z)$ the estimates

$$\|T(z)\| \leq M \exp(-\sigma_0 \cos(\arg z)|z|), \quad z \in S_\psi, \tag{20}$$

$$\|A^\alpha T(z)\| \leq M|z|^{-\alpha} \exp(-\sigma_0 \cos(\arg z)|z|), \quad z \in S_\psi, \ z \neq 0, \ \alpha > 0 \tag{21}$$

hold.

By means of $T(t)$ $(t \geq 0)$ it is possible (cf. [9]) to construct for A the Banach space $E_{1-1/q}$ of such elements of E which have a finite norm

$$]v[= \Big(\int_0^\infty \|AT(t)v\|^q dt\Big)^{1/q}. \tag{22}$$

Let K be a bounded closed set situated inside the sector S_ψ. Changing A to μA $(\mu \in K)$ it is possible to construct the space $E^\mu_{1-1/q}$. It is easy to show that the spaces $E^\mu_{1-1/q}$ and $E_{1-1/q}$ coincide and their norms $]\cdot[_\mu$ and $]\cdot[$ are equivalent uniformly with respect to $\mu \in K$.

3.3. Let A be strongly positive. Let the complex parameter μ belong to the set

$$K = \{\mu : 0 < \alpha_1 < \operatorname{Re}\mu < \alpha_2 < +\infty, \ |\arg\mu| < \frac{\pi}{2} - \psi\}. \tag{23}$$

Consider the following problem

$$u'(t) + \mu A u(t) = f(t) \quad (t \geq 0), \tag{24}$$

$$u(0) = u_0. \tag{25}$$

The solution $u(t)$ of problem (24)–(25) is defined to be an E-valued absolutely continuous function on the semiaxes $[0, \infty)$, such that $u'(t)$, $Au(t) \in L_q(0, \infty; E)$ and equation (24) and condition (25) are satisfied.

We call problem (24)–(25) coercive solvable (c.s.), if for any $f \in L_q(0,\infty;E)$ and $u_0 \in E_{1-1/q}$ the problem has a unique solution and the estimate

$$\|u'\|_{L_q} + \|Au\|_{L_q} \leq M\left[\|f\|_{L_q} +]u_0[\right] \tag{26}$$

holds. Here M does not depend on f and u_0.

If M does not depend also on $\mu \in K$ for any fixed $0 < \alpha_1 < \alpha_2 < +\infty$, we call problem (24)–(25) uniformly c.s. (u.c.s.).

The strong positiveness of A implies the strong positiveness of μA for $\mu \in K$ and the relation

$$T_\mu(t) = T(\mu t), \quad t \geq 0 \tag{27}$$

for the analytic semigroup $T_\mu(t)$ (corresponding to $-\mu A$).

From u.c.s. of problem (24)–(25) it follows that its solution is given by the formula

$$u(t) = T(\mu t)u_0 + \int_0^t T(\mu(t-s))f(s)\,ds, \tag{28}$$

and for the operator

$$R_\mu(f) = \int_0^t T(\mu(t-s))f(s)\,ds \tag{29}$$

the inequality

$$\|AR_\mu(f)\|_{L_q} \leq M\|f\|_{L_q} \tag{30}$$

with constant M independent of $\mu \in K$ holds. Moreover (cf. [10]), the function $u(t)$ is continuous as a function with the values in $E_{1-1/q}$ and

$$\sup_{t\geq 0}]u(t)[\leq M\left(\|f\|_{L_q} +]u_0[\right) \tag{31}$$

holds. In particular

$$\sup_{t\geq 0}]R_\mu(f)[\leq M\|f\|_{L_q}, \tag{32}$$

$$\sup_{t\geq 0}]T(\mu t)u_0[\leq M]u_0[. \tag{33}$$

We emphasize finally that the constant M above depends certainly on α_1 and α_2.

3.4. Let us give an example of u.c.s. Let $E = L_q(0,1)$, $1 < q < +\infty$. Consider in $L_q(0,1)$ the operator $Au \equiv -u''(x)$ with a domain of definition $\mathcal{D}(A) = \overset{0}{W}{}_q^2(0,1)$. This operator is strongly positive and $\sigma(A) = \{(k\pi)^2,\ k = 1,2,\dots\}$. The inequality (14) is valid for all λ situated on the left of $K_0 = S[\psi] \cap \Pi_0$, in which σ_0 is an arbitrary number smaller than π^2 and $\psi \in (0, \frac{\pi}{2})$ may be chosen arbitrarily small. The space $E_{1-1/q}$ here coincides with $\overset{0}{W}{}_q^{2-2/q}$. For details see [3]–[5]. Thus, the problem (25)–(26) in this case is u.c.s.

3.5. Let us study some properties of operators (16)–(18).

Lemma 1. *There exist bounded linear operators*

$$Q_1 = \frac{1}{2}\left[\mu_2 I + (\mu_2^2 I - 4\mu_1 A^{-1})^{1/2}\right], \tag{34}$$

$$Q_2 = 2\mu_1\left[\mu_2 I + (\mu_2^2 I - 4\mu_1 A^{-1})^{1/2}\right]^{-1}, \tag{35}$$

$$Q_3 = (\mu_2 I - 4\mu A^{-1})^{1/2}, \tag{36}$$

$$Q_4 = Q_1 Q_3, \quad Q_5 = Q_2 Q_3. \tag{37}$$

Q_i and A commute on $\mathcal{D}(A)$, and the relations

$$\begin{aligned} &A_1 = Q_1 A, \quad A_2 = Q_2, \quad (A_1 - A_2)^{-1} = A^{-1} Q_3, \\ &A_1(A_1 - A_2)^{-1} = Q_4, \quad A_2(A_1 - A_2)^{-1} = A^{-1} Q_5 \end{aligned} \tag{38}$$

hold.

Lemma 2. *The operator $-A_1$ is the generator of an analytic semigroup $T_1(t)$, and the estimates*

$$\|T_1(t)\| \le M\exp(-\sigma_1 t), \quad t \ge 0 \tag{39}$$

$$\|AT_1(t)\| \le M\exp(-\sigma_1 t)t^{-1}, \quad t > 0 \tag{40}$$

with $\sigma_1 = \frac{1}{2}(\cos\frac{\psi}{2})\sigma_0(\mu_2 + (\mu_2^2 - 4\mu_1\sigma_0^{-1})^{1/2}$ hold.

Lemma 3. *Let problem (24)–(25) be u.c.s., $f \in L_q$. Then the function*

$$\phi(t) = \int_0^t T_1(t-s)f(s)\,ds \quad (\equiv R_1(f)) \tag{41}$$

is absolutly continuous, $\phi(t) \in \mathcal{D}(A)$ almost everwhere, and the estimates

$$\|AR_1(f)\|_{L_q} \le M\|f\|_{L_q}, \tag{42}$$

$$\sup_{t\ge 0}]R_1(f)[\le M\|f\|_{L_q} \tag{43}$$

hold. Here M does not depend on f.

Lemma 4. *The family of the bounded operators $T_2(t)$ satisfies the inequality*

$$\|T_2(t)\| \le M\exp(-\sigma_2 t), \quad t \ge 0 \tag{44}$$

with $\sigma_2 = \mu_1\mu_2^{-1}$.

Lemma 5. *The function*

$$\phi(t) = \int_0^t T_2(t-s)f(s)\,ds \quad (\equiv R_2(f)) \tag{45}$$

is absolutly continuous for any $f \in L_q$, and the estimate

$$\|R_2(f)\|_{L_q} \le M\|f\|_{L_q} \tag{46}$$

holds.

Lemma 6. *For any $u \in E_{1-1/q}$ there exists the inclusion $AT_1(t)u \in L_q$ and the estimate*

$$\|AT_1(t)u\|_{L_q} \leq M]u[\tag{47}$$

holds.

Now we shall prove Lemmas 1–6.

Proof of Lemma 1: We prove first that the operator Q_1 (and therefore A_1) makes sense.

The estimate (14) and condition $\sigma_0 > 4\mu_1\mu_2^{-2}$ imply the strong positiveness of the bounded operator $Q_0 = \mu_2^2 I - 4\mu_1 A^{-1}$. Therefore there exists (and is bounded) the positive square root Q_3 of Q_0. But then the operator $Q_1 = \frac{1}{2}[\mu_2 I + Q_3]$ is defined and bounded. The first formula (38) is proved.

As it is easy to see the operator Q_1 is boundedly invertible. Supposing $I = Q_1 Q_1^{-1}$ and making use of easy transformations of the right-hand side of (16) we obtain the second formula (38) where $Q_2 = \mu_1 Q_1^{-1}$.

Further the proof is based on elementary operations. The commutation of the operators Q_i and A is obvious. ∎

Proof of Lemma 2: Considering Q_1 as a function of the operator A and making use of the spectral mapping theorem (cf. [11]) it is easy to prove that the spectrum of the operator Q_1 is situated in $K = S(\psi) \cap \Pi$, where

$$S(\psi) = \{z : |\arg z| < \frac{\psi}{2}\},$$

$$\Pi_1 = \{z : 0 < \mu_2 < \operatorname{Re} z < \frac{1}{2}(\mu_2 + (\mu_2^2 - 4\mu_1\sigma_0^{-1})^{1/2})\}.$$

Let the closed smooth contour Γ situated in K surround the spectrum of Q_1. It is easy to see that the operator-function

$$T_1(t) = \frac{1}{2\pi i}\int_\Gamma T(\mu t)(\mu I - Q_1)^{-1} d\mu, \quad t \geq 0 \tag{48}$$

is an analytic semigroup with the generator $-A_1$. The estimates (39)–(40) for $T_1(t)$ follow from (20)–(21). ∎

Proof of Lemma 3: First we prove estimates (42)–(43). Represent $z(t) = AR_1(f)$ in the form

$$z(t) = \frac{1}{2\pi i}\int_\Gamma (\mu I - Q_1)^{-1} A \int_0^t T(\mu(t-s)) f(s)\, ds\, d\mu.$$

From this it follows that

$$\|z\|_{L_q} \leq \frac{1}{2\pi}\int_\Gamma \|(\mu I - Q_1)^{-1}\|\, \|A\textstyle\int_0^t T(\mu(t-s)) f(s)\, ds\|_{L_q} |d\mu|. \tag{49}$$

Because of u.c.s. of (24)–(25)

$$\left\|A\int_0^t T(\mu(t-s)) f(s)\, ds\right\|_{L_q} \leq M\|f\|_{L_q},$$

from (49) and boundedness of Γ the estimate (42) follows. Estimate (43) is established in a similar way by means of (32). The absolute continuity of $\phi(t)$ is established now in the obvious way. ∎

Proof of Lemma 4: The proof follows from the boundedness of A_2 and the location of the spectrum of A_2 in the strip

$$\Pi_2 = \{z : \mu_1\mu_2^{-1} < \operatorname{Re} z < 2\mu_1(\mu_2 + (\mu_2^2 - 4\mu_1\sigma_0^{-1})^{1/2})^{-1}\}.$$

The proof of Lemma 5 is obvious.

Proof of Lemma 6: From (48) it follows that

$$AT_1(t)u = \frac{1}{2\pi i}\int_\Gamma (\mu I - Q_1)^{-1}[AT(\mu t)u]\,d\mu.$$

Therefore

$$\|AT_1(t)u\| \le \frac{1}{2\pi}\int_\Gamma \|(\mu I - Q_1)^{-1}\|\,\|AT(\mu t)u\|\,|d\mu|$$

and

$$\|AT_1(t)u\|_{L_q} \le \frac{1}{2\pi}\int_\Gamma \|(\mu I - Q_1)^{-1}\|\,\|AT(\mu t)u\|_{L_q}|d\mu|. \tag{50}$$

Since the norms of $E^{\mu}_{1-1/q}$ and $E_{1-1/q}$ are equivalent,

$$\|AT(\mu t)u\|_{L_q} =]u[_\mu \le M]u[.$$

Hence, from (50) inequality (47) follows. ∎

4. In this section we establish the main auxiliary result.

Theorem A. *Let problem* (24)–(25) *be u.c.s. Let* $\sigma_0 > 4\mu_1\mu_2^{-2}$. *Then for any* $u_0 \in \mathcal{D}(A)$, $u_1 \in E_{1-1/q}$, $f \in L_q$ *the problem* (12)–(13) *has a unique solution* $u(t)$, *and formula* (15) *and the estimate*

$$\begin{aligned}&\|u''(t)\|_{L_q} + \|Au'(t)\|_{L_q} + \|Au(t)\|_{L_q} + \sup_{t\ge 0}\|Au(t)\| + \sup_{t\ge 0}]u'(t)[\\ &\quad\le M\left(\|f\|_{L_q} + \|Au_0\| +]u_1[\right)\end{aligned} \tag{51}$$

hold.

Proof: From Lemmas 3 and 5 it follows that the formula (15) defines the solution of the problem (12)–(13). We establish the estimate (51).

Consider first the case $u_0 = 0$, $u_1 = 0$. Then

$$u(t) = Q_3A^{-1}\int_0^t T_1(t-s)f(s)\,ds - Q_3A^{-1}\int_0^t T_2(t-s)f(s)\,ds. \tag{52}$$

Differentiating $u(t)$ by t and using Lemmas 3 and 5 we get

$$u'(t) = Q_4\int_0^t T_1(t-s)f(s)\,ds + Q_5A^{-1}\int_0^t T_2(t-s)f(s)\,ds, \tag{53}$$

$$u''(t) = Q_1Q_4A\int_0^t T_1(t-s)f(s)\,ds - Q_5Q_2A^{-1}\int_0^t T_2(t-s)f(s)\,ds + f(t). \quad (54)$$

From these formulas and Lemmas 3 and 5 it follows that $u'(t)$, $u''(t) \in L_q$ and

$$\|u''(t)\|_{L_q} + \|u'(t)\|_{L_q} + \|u(t)\|_{L_q} + \sup_{t\geq 0}]u'(t)[\leq M\|f\|_{L_q} \quad (55)$$

holds. Further from formulas

$$Au(t) = Q_3\int_0^t T_1(t-s)f(s)\,ds - Q_3\int_0^t T_2(t-s)f(s)\,ds, \quad (56)$$

$$Au'(t) = Q_4A\int_0^t T_1(t-s)f(s)\,ds + Q_5\int_0^t T_2(t-s)f(s)\,ds, \quad (57)$$

and Lemmas 3 and 5 we obtain the inclusion $Au(t)$, $Au'(t) \in L_q$, and the estimate

$$\|Au'(t)\|_{L_q} + \|Au(t)\|_{L_q} + \sup_{t\geq 0}\|Au(t)\| \leq M\|f\|_{L_q}. \quad (58)$$

Thus

$$\|u''(t)\|_{L_q} + \|Au'(t)\|_{L_q} + \|Au(t)\|_{L_q} + \sup_{t\geq 0}]u'(t)[+ \sup_{t\geq 0}\|Au(t)\| \leq M\|f\|_{L_q}. \quad (59)$$

Consider now the case $f = 0$, $u_0 = 0$. Then

$$u(t) = Q_3A^{-1}T_1(t)u_1 - Q_3A^{-1}T_2(t)u_1. \quad (60)$$

The differentiation by t in the relation above and the use of Lemma 1 yield

$$\begin{aligned} u'(t) &= Q_4T_1(t)u_1 + Q_5A^{-1}T_2(t)u_1, \\ u''(t) &= Q_4Q_1AT_1(t)u_1 + Q_2Q_5A^{-1}T_2(t)u_1, \\ Au'(t) &= Q_4AT_1(t)u_1 + Q_5T_2(t)u_1. \end{aligned} \quad (61)$$

From this, the inclusion $u_1 \in E_{1-1/q}$ and Lemma 6 it follows that

$$\|u''(t)\|_{L_q} + \|Au'(t)\|_{L_q} + \|Au(t)\|_{L_q} + \sup_{t\geq 0}]u'(t)[+ \sup_{t\geq 0}\|Au(t)\| \leq M]u_1[. \quad (62)$$

Let us consider the case $f = 0$, $u_1 = 0$. Now

$$u(t) = Q_5A^{-1}T_1(t)u_0 - Q_4T_2(t)u_0. \quad (63)$$

From this, it follows that

$$\begin{aligned} u'(t) &= -Q_1Q_5T_1(t)u_0 - Q_4Q_2T_2(t)u_0, \\ u''(t) &= Q_1^2Q_5AT_1(t)u_0 + Q_4Q_2^2T_2(t)u_0. \end{aligned}$$

It is easy to see that if $u_0 \in \mathcal{D}(A)$ (and hence in $E_{1-1/q}$), then $u''(t) \in L_q$ and the estimates

$$\|u''(t)\|_{L_q} + \|u'(t)\|_{L_q} \le M\|Au_0\|, \tag{64}$$

$$\sup_{t\ge 0}]u'(t)[\le M\|Au_0\| \tag{65}$$

hold. The application of A to the formulas obtained for $u(t)$ and $u'(t)$ yields the inclusions $Au(t)$, $Au'(t) \in L_q$ and the estimate

$$\|Au(t)\|_{L_q} + \|Au'(t)\|_{L_q} + \sup_{t\ge 0}\|Au(t)\| \le M\|Au_0\|. \tag{66}$$

From this and (64)–(65) we have

$$\|u''(t)\|_{L_q} + \|Au'(t)\|_{L_q} + \|Au(t)\|_{L_q} + \sup_{t\ge 0}\|Au(t)\| + \sup_{t\ge 0}]u'(t)[\le M\|Au_0\|. \tag{67}$$

Inequalities (59), (62) and (67) prove (51). ∎

5. Proof of Theorem 1. Let $E = L_q(0,1)$ and A be the operator introduced in §3.4. Consider problem (3)–(4) as the abstract problem (24)–(25). The application of Theorem A yields the solvability of (3)–(4) and the estimate (5). Theorem 1 is proved.

Let L^{-1} be the linear operator assigning to every $f \in L_q$ the solution u of problem (3)–(4) with $u_0 = 0$ and $u_1 = 0$. From Theorem 1 and inequalities (5) and (6) we have

Lemma 7. *For any $w \in L_q$ the following inequalities hold.*

$$\sup_{t\ge 0}\Big|\frac{\partial^2}{\partial x^2}L^{-1}(w)\Big|_0 \le M\|w\|_0, \tag{68}$$

$$\Big\|\frac{\partial^3}{\partial t\partial x^2}L^{-1}(w)\Big\|_0 \le M\|w\|_0, \tag{69}$$

$$\sup_{(t,x)\in Q}\Big|\frac{\partial}{\partial x}L^{-1}(w)\Big| \le M\|w\|_0, \tag{70}$$

$$\sup_{(t,x)\in Q}\Big|\frac{\partial^2}{\partial t\partial x}L^{-1}(w)\Big| \le M\|w\|_0. \tag{71}$$

Here M does not depend on w.

6. Proof of Theorem 2. Let $p(t,x)$ be the solution of the linear problem (3)–(4) with $f=0$. Let us represent the solution u of problem (1)–(2) in the form

$$u = z + p.$$

Then $z(t,x)$ is the solution of the problem

$$\begin{aligned} z_{tt} - \mu_2 z_{txx} - \mu_1 z_{xx} &= f - \mu_2(z_{xx}+p_{xx})(1+z_x+p_x)^{-2}(z_{tx}+p_{tx}) \\ &- \mu_2(p_x+z_x)(1+z_x+p_x)^{-1}p_{txx} - \mu_2(p_x+z_x)(1+z_x+p_x)^{-1}z_{txx}, \end{aligned} \tag{72}$$

$$z(0,x) = z_t(0,x) = 0, \quad z(t,0) = z(t,1) = 0. \tag{73}$$

Denote the right-hand side of (72) by w. A use of the linear operator L^{-1} gives the relation

$$\begin{aligned} w = f &- \mu_2\Big(\frac{\partial^2}{\partial x^2}L^{-1}(w) + p_{xx}\Big)\Big(1 + \frac{\partial}{\partial x}L^{-1}(w) + p_x\Big)^{-2}\Big(\frac{\partial^2}{\partial t \partial x}L^{-1}(w) + p_{tx}\Big) \\ &- \mu_2\Big(p_x + \frac{\partial}{\partial x}L^{-1}(w)\Big)\Big(1 + \frac{\partial}{\partial x}L^{-1}(w) + p_x\Big)^{-1}p_{txx} - \mu_2\Big(p_x + \frac{\partial}{\partial x}L^{-1}(w)\Big) \\ &\times \Big(1 + \frac{\partial}{\partial x}L^{-1}(w) + p_x\Big)^{-1}\frac{\partial^3}{\partial t \partial x^2}L^{-1}(w). \end{aligned} \tag{74}$$

Let us represent the right-hand side $\tilde{K}(w)$ of the relation (74) in the form

$$\tilde{K}(w) = f - \mu_2 K_1(w) - \mu_2 K_2(w) - \mu_2 K_3(w)$$

and rewrite (74) as an operator equation

$$w = \tilde{K}(w). \tag{75}$$

We shall prove that the operator $\tilde{K}$ maps the set

$$S_R = \{w : \|w\|_0 \le R\}$$

into itself for sufficiently small R, if f, u_0 and u_1 (which define $p(t,x)$) are also small. Afterwards we shall prove that $\tilde{K}$ is contractive on S_R.

We prove first that the operator $\tilde{K}$ is defined on S_R. From Theorem 1 it follows that for $p(t,x)$ the estimate

$$\sup_{t\ge 0}|p(t,x)|_0 + \|p_{txx}\|_0 + \sup_{t,x}|p_{tx}(t,x)| + \sup_{t,x}|p_x(t,x)| \le M(|u_0|_2 + |u_1|_{2-2/q}) \tag{76}$$

holds. Let

$$|u_0|_2 \le \delta, \quad |u_1|_{2-2/q} \le \delta. \tag{77}$$

Then from (76) it follows that

$$\sup_{t,x}|p_{tx}(t,x)| \le M\delta, \quad \sup_{t,x}|p_x(t,x)| \le M\delta, \quad \sup_{t,x}|p_{txx}(t,x)| \le M\delta. \tag{78}$$

Let R and δ be so small that for u_0 and u_1 which satisfy (77), and for any w which satisfies

$$\|w\|_0 \le R, \tag{79}$$

the estimate

$$\sup_{t,x}\Big|\frac{\partial}{\partial x}L^{-1}(w) + p_x\Big| \le \frac{1}{2} \tag{80}$$

holds. Then the estimate

$$\sup_{t,x}\Big|1 + \frac{\partial}{\partial x}L^{-1}(w) + p_x\Big|^{-1} \le 2 \tag{81}$$

is obviously valid.

Let us show that the operator $\tilde{K}$ maps S_R into itself. The estimate of $K_1(w)$ in the L_q-norm and the application of Lemma 7 yield the inequality

$$\|K_1(w)\|_0 \le \Big\|\frac{\partial^2}{\partial x^2}L^{-1}(w)+p_{xx}\Big\|_0 \sup_{t,x}\Big|1+\frac{\partial}{\partial x}L^{-1}(w)+p_x\Big|^{-2}\sup_{t,x}\Big|\frac{\partial^2}{\partial t\partial x}L^{-1}(w)+p_{tx}\Big|.$$

By (69), (78) and (81) we obtain

$$\|K_1(w)\|_0 \le 4[M_1\|w\|_0 + M_2\delta][M_3\|w\| + M_4\delta] \le M_5(R+\delta)^2.$$

Choosing $\delta < R$ we have

$$\|K_1(w)\|_0 \le Rq(R),$$

where $q(R) = M_5R$. The similar estimates for K_2 and K_3 are obtained in the same manner. Assuming

$$\|f\|_0 \le q(R)R,$$

we get

$$\|K_1(w)\|_0 \le 4q(R)R.$$

Since $q(R) \to 0$ as $R \to 0$, we obtain that for sufficiently small R and δ the operator $\tilde{K}$ maps S_R into itself.

We prove now that the operator $\tilde{K}$ is contractive on S_R. It suffices to prove this for each K_i. We demonstrate this for example for K_3. For w_1, $w_2 \in S_R$ we have

$$\begin{aligned}
K_3(w_1) - K_3(w_2) &= \frac{\partial}{\partial x}L^{-1}(w_1-w_2)\big(1+\frac{\partial}{\partial x}L^{-1}(w_1)+p_x\big)^{-1}\frac{\partial^3}{\partial t\partial x^2}L^{-1}(w_1)\\
&+ \big(p_x + \frac{\partial}{\partial x}L^{-1}(w_2)\big)\frac{\partial}{\partial x}L^{-1}(w_2-w_1)\big(1+\frac{\partial}{\partial x}L^{-1}(w_1)+p_x\big)^{-1}\\
&\times\big(1+\frac{\partial}{\partial x}L^{-1}(w_2)+p_x\big)^{-1}\frac{\partial^3}{\partial t\partial x^2}L^{-1}(w_1)\\
&+ \big(p_x + \frac{\partial}{\partial x}L^{-1}(w_2)\big)\big(1+\frac{\partial}{\partial x}L^{-1}(w_2)+p_x\big)^{-1}\frac{\partial^3}{\partial t\partial x^2}L^{-1}(w_1-w_2)\\
&= \sum_{i=1}^{3} S_i.
\end{aligned}$$

Estimating S_1, we get

$$\begin{aligned}
\|S_1\|_0 &\le \sup_{t,x}\Big|\frac{\partial}{\partial x}L^{-1}(w_1-w_2)\Big| \sup_{t,x}\Big|1+\frac{\partial}{\partial x}L^{-1}(w_1)+p_x\Big|^{-1}\Big\|\frac{\partial^3}{\partial t\partial x^2}L^{-1}(w_1)\Big\|_0\\
&\le M\|w_1-w_2\|_0\,\|w_1\|_0.
\end{aligned}$$

Choose R and δ so small that

$$\|S_1\|_0 \le \frac{q}{3\mu_2}\|w_1-w_2\|_0$$

for some $q \in (0,1)$. The estimates for S_2 and S_3 are obtained in the similar way. Thus for sufficiently small R and δ the operator $\tilde{K}$ turns out to be a contractive operator on S_R.

From the principle of contractive mappings it follows that if R is sufficiently small there exists a small δ such that for data smaller than δ the operator equation (75) has a unique solution on S_R.

The solvability of equation (75) implies the solvability of problem (72)–(73) and, hence, solvability of problem (1)–(2) and estimate (9). Proof is complete.

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