

**GLOBAL EXISTENCE AND UNIQUENESS
FOR THE KUMMER TRANSFORMATION PROBLEM
SUBJECT TO NON-CAUCHY DATA**

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Abstract. Global existence and uniqueness for a non-Cauchy problem associated with a certain third-order nonlinear differential equation are established. Such a problem is related to the qualitative theory of transformations between pairs of linear second-order differential equations on the real domain. The Cauchy as well as other classical conditions were earlier considered by O. Borůvka. A connection between the Kummer transformation problem and the Liouville-Green (WKB) approximation has been found in the oscillatory case. The non-Cauchy data, in fact, consist of a given value of the solution at some point and of a prescribed asymptotic behavior near an oscillation point. Applications to special functions, central dispersions, and numerical analysis are given.

1. Introduction. It has been known since the pioneering work by E.E. Kummer in 1834 (published later in [6], cf. also [5, p. 109]) that the third-order nonlinear ordinary differential equation

$$Q(X)X'^2 = q(t) - \frac{1}{2}\{X, t\}, \quad (1.1)$$

where

$$\{X, t\} \equiv \frac{X'''}{X'} - \frac{3}{2} \frac{X''^2}{X'^2} \quad (1.2)$$

denotes the so-called “Schwarzian derivative”, plays a central role in the theory of transformations between any two linear homogeneous ordinary differential equations like

$$\frac{d^2y}{dt^2} + q(t)y = 0, \quad t \in j \equiv (a, b) \quad (1.3')$$

$$\frac{d^2Y}{dT^2} + Q(T)Y = 0, \quad T \in J \equiv (A, B), \quad (1.3'')$$

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$q \in C^0(j), Q \in C^0(J), -\infty \leq a < b \leq +\infty, -\infty \leq A < B \leq +\infty.$

Indeed, if one transforms equation (1.3'') into (1.3') by setting

$$y(t) = w(t)Y(X(t)), \tag{1.4}$$

where $w \in C^2(i), X \in C^3(i), wX' \neq 0$ for all $t \in i, i \subseteq j$ and $X(i) \subseteq J$, then the function $X(t)$ necessarily satisfies the differential equation (1.2) on the interval i . In the pair (w, X) that characterizes the transformation, $X(t)$ is called the *kernel* and $w(t)$ the *multiplier*. The key is, however, the function $X(t)$, as once $X(t)$ is given, $w(t)$ is determined uniquely up to a multiplicative constant $k \neq 0$ by

$$w(t) = k/\sqrt{|X'(t)|}, \tag{1.5}$$

cf. [5, pp. 109-111]. Equation (1.1) will be called *the Kummer differential equation*.

The special case of (1.1) obtained when $Q \equiv 1$, i.e.,

$$\alpha'^2 = q(t) - \frac{1}{2}\{\alpha, t\}, \tag{1.6}$$

is of special interest. Note that the kernel $T = \alpha(t)$ of any transformation between (1.3') and

$$\frac{d^2Y}{dT^2} + Y = 0 \tag{1.6'}$$

is a solution of (1.6).

Here and below, by a solution to (1.6) we mean any function $\alpha(t)$ three times differentiable and with $\alpha' \neq 0$ on j , that satisfies (1.6). It follows from (1.6) itself that $\alpha \in C^3(j)$, as $q \in C^0(j)$ and $\alpha' \neq 0$.

Let us introduce at this point the concept of "phase."

Definition 1.1. A function $\alpha(t)$ is called "a phase" of the differential equation (1.3'), if

$$\tan \alpha(t) = \frac{u(t)}{v(t)}, \tag{1.7}$$

$(u(t), v(t))$ being any given basis of (1.3'), and $\alpha \in C^3(j)$ (cf. [5, §§5.1, 5.5]).

Note that every given basis (u, v) defines the phase function α up to an additive constant multiple of π . Moreover, every phase has the property that

$$|\alpha(t_{k+1}) - \alpha(t_k)| = \pi \tag{1.8}$$

for every pair of consecutive zeros t_k, t_{k+1} (if any) of every solution to (1.3'), cf. [5, §5.4].

A characterization of the phase functions is given by the following lemma, that can essentially be found in [5, §§5.5, 5.7].

Lemma 1.2. A function $\alpha(t)$ is a phase of the differential equation (1.3') if and only if it is a solution (in the sense stated above) to equation (1.6).

Proof: If $\alpha(t)$ is a phase, then by differentiating in (1.7) we get

$$\alpha' = -\frac{W}{u^2 + v^2}, \tag{1.9}$$

where $W \equiv W[u, v]$ is the (constant) Wronskian of (u, v) . Equation (1.6) can then be obtained by further differentiation and using (1.3'). Note that $\alpha(t)$ turns out to be strictly decreasing or increasing according to the sign of W .

Conversely, every function α , with $\alpha \in C^3(j)$, $\alpha' \neq 0$ on j , is a phase of a differential equation like (1.3') with coefficient

$$\tilde{q}(t) = \alpha'^2 + \frac{1}{2}\{\alpha, t\}, \tag{1.10}$$

relatively to the basis

$$u = |\alpha'|^{-1/2} \sin \alpha, \quad v = |\alpha'|^{-1/2} \cos \alpha \tag{1.11}$$

(cf. [5, §5.7]). Comparing equation (1.10) with (1.6), it follows that $\tilde{q}(t) \equiv q(t)$ and therefore every solution to (1.6) is a phase of (1.3'). ■

In [9, 10] the asymptotic behavior of solutions to (1.6) has been studied under suitable hypotheses. It seems that a solution to (1.6) is uniquely determined prescribing a certain asymptotic behavior of α' as $t \rightarrow b^-$ plus the value of α at an inner point, while equation (1.6) is of the *third order*. The purpose of this paper is to establish global existence and uniqueness of solutions to equation (1.6) under such hypotheses.

Existence and uniqueness for the classical Cauchy problem for equation (1.6), as well as for the more general case (1.1), have been proven by Borůvka [2-5].

In the next section, the Liouville-Green approximation for (1.3') will be introduced and related to the phase theory. This will allow us to prove the main Theorem of the paper, that is existence and uniqueness for the phase equation (1.6). In Section 3, this result will be extended to the general Kummer equation (1.1). It should be clear that, as in Borůvka's theory, our results have a qualitative, global character and are concerned with the real domain. In Section 4, finally, examples and applications to certain special functions, to the so-called central dispersions [5, Ch. 13], and to numerical analysis will be given.

2. Main result. Several qualitative features of the solutions to equation (1.3') can be observed by the so-called Liouville-Green asymptotic approximation (cf. [8]). The following theorem is of primary importance.

Theorem 2.1 (Olver, [8, Theorem 2.2, p. 196]). *Suppose that $q(t) = f(t) + g(t)$ in (1.3'), with $f \in C^2(j)$, $f(t) > 0$ in j , $g \in C^0(j)$, and*

$$\mathcal{V}_{\xi,t}(\mathcal{F}) \equiv \left| \int_t^\xi |\mathcal{F}'(s)| ds \right| < \infty, \tag{2.1}$$

where

$$\mathcal{F}(t) \equiv \int^t (f^{-1/4} D^2 f^{-1/4} + g f^{-1/2}) ds \tag{2.2}$$

and ξ is any given point in the closure of $j = (a, b)$. Then, equation (1.3') possesses the family of Liouville-Green bases

$$\begin{aligned} U_{LG}(t) &\equiv f^{-1/4}(t) \exp \left\{ i \int^t f^{1/2}(s) ds \right\} [1 + \epsilon_1(t)] \\ V_{LG}(t) &\equiv f^{-1/4}(t) \exp \left\{ -i \int^t f^{1/2}(s) ds \right\} [1 + \epsilon_2(t)] \end{aligned} \tag{2.3}$$

with

$$|\epsilon_k(t)|, \quad f^{-1/2}|\epsilon'_k(t)| \leq \exp\{\mathcal{V}_{\xi,t}(\mathcal{F})\} - 1, \quad k = 1, 2. \tag{2.4}$$

If $g(t)$ is real, then $U_{LG}(t)$ and $V_{LG}(t)$ are complex conjugate.

The family of bases in (2.3) obtains when the constant of integration is varied. Hereafter the symbol LG will be used as an abbreviation for “Liouville-Green,” as in (2.3). As we are interested in real-valued solutions to (1.3') and (1.6), q, f, g being real, we introduce the real family of LG bases:

$$\begin{aligned} u_{LG}(t) &\equiv \operatorname{Re} U_{LG}(t) = \frac{U_{LG}(t) + V_{LG}(t)}{2}, \\ v_{LG}(t) &\equiv \operatorname{Im} U_{LG}(t) = \frac{U_{LG}(t) - V_{LG}(t)}{2i}. \end{aligned} \tag{2.5}$$

Remark 2.2. It is easily checked that the real LG family of bases is an equivalence class with respect to the relation $R : “(u_1, v_1)R(u_2, v_2)$ if $(u_1, v_1)^T = \mathbf{C}(u_2, v_2)^T$ with \mathbf{C} orthogonal and $\det \mathbf{C} = +1.$ ”

Lemma 2.3. Denoting by W_{LG} the Wronskian of any pair $(u_{LG}(t), v_{LG}(t))$ in the family (2.5), then $W_{LG} = 1.$

Proof: In view of Remark 2.2, the Wronskian is an invariant of the real LG family. Using (2.3), it is easily shown by a simple calculation that

$$W[U_{LG}(t), V_{LG}(t)] = -2i(1 + \epsilon_1(t))(1 + \epsilon_2(t)) + o(1), \tag{2.6}$$

where (2.4) has been used to assess the $\epsilon'_k(t)$ terms as $o(1)$ as $t \rightarrow \xi.$ W being a constant, taking the limit as $t \rightarrow \xi$ in (2.6) and using (2.4), we get

$$W[U_{LG}(t), V_{LG}(t)] = -2i.$$

From (2.5) we obtain finally

$$W_{LG} = 1. \tag{2.7}$$

Definition 2.4. We define as “a Liouville-Green phase,” $\alpha_{LG}(t),$ every solution α to (1.7) with $u = u_{LG}, v = v_{LG}.$

In view of (2.7), it follows from (1.9), (2.3), (2.5) that

$$\alpha'_{LG} = -\frac{1}{u^2_{LG} + v^2_{LG}} = -\frac{1}{U_{LG}V_{LG}} = -\frac{f^{1/2}}{(1 + \epsilon_1)(1 + \epsilon_2)}. \tag{2.8}$$

Note that this quantity is independent of the special choice of the constant of integration labelling the particular basis and thus is another invariant of the LG family.

The following lemma might be reconstructed from the results of [5, §5.17], and therefore we shall give no proof of it.

Lemma 2.5. *Let α and β be two phases of (1.3') relatively to the bases (u_1, v_1) , (u_2, v_2) respectively (cf. (1.7)). Then $\alpha = \pm\beta + \lambda$, λ being a real constant, if and only if (u_1, v_1) and (u_2, v_2) are obtained one from the other through an orthogonal followed by a homothetic transformation. Such a transformation can be represented by*

$$\mathbf{A} = k\mathbf{D}(\lambda) \quad \text{or} \quad \mathbf{A} = k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{D}(\lambda), \tag{2.9}$$

according to the \pm sign above, where

$$\mathbf{D}(\lambda) \equiv \begin{pmatrix} \cos \lambda & \sin \lambda \\ -\sin \lambda & \cos \lambda \end{pmatrix}, \quad k^2 = \left| \frac{W_2}{W_1} \right|, \tag{2.10}$$

W_i denoting the Wronskian of the basis (u_i, v_i) , $i = 1, 2$.

As a consequence, we can prove the following:

Corollary 2.6 (characterization of the LG phases). *The function α is a Liouville-Green phase if and only if it can be written as*

$$\alpha = \widehat{\alpha}_{LG} + \lambda, \tag{2.11}$$

where $\widehat{\alpha}_{LG}$ denotes a given LG phase and λ is any fixed real constant.

Proof: Applying the transformation $\mathbf{D}(\lambda)$ in (2.10) to an LG basis associated with $\widehat{\alpha}_{LG}$, we get by Lemma 2.5 and Remark 2.2 again an LG basis that is associated with $\widehat{\alpha}_{LG} + \lambda$. The converse is trivial by (2.8).

Here is the main theorem of the paper.

Theorem 2.7. *Suppose that the coefficient $q(t)$ in (1.6) satisfies all assumptions of Theorem 2.1 and, in addition,*

$$\int^t f^{1/2}(s) ds \rightarrow \infty \quad \text{as } t \rightarrow \xi. \tag{2.12}$$

Then there is a solution α to (1.6) such that

$$\ell \equiv \lim_{t \rightarrow \xi} \alpha' f^{-1/2} \tag{2.13}$$

exists if and only if $\ell = \pm 1$ and

$$\alpha' = -\text{sgn}(\ell)\alpha'_{LG}. \tag{2.14}$$

Finally, α is uniquely determined whenever a value of it is prescribed at some arbitrary point inside (a, b) .

Condition (2.12) means that equation (1.3') is oscillatory and each solution has zeros in every neighborhood of ξ . This can be easily seen from (2.5). Hereafter such a point, ξ , will be referred to as an oscillation point. Therefore, ξ coincides necessarily with a or b and thus is a singular point for (1.3').

Remark 2.8. Note that the two conditions, (2.13) with $\ell = +1$ or $\ell = -1$ and the prescribed value, uniquely determine a solution to the third-order equation (1.6).

Remark 2.9. Theorem 2.7 also yields a non-existence result in that if the limit in (2.13) exists, then, necessarily, $\ell = \pm 1$.

Proof of Theorem 2.7: It is convenient to prove first the theorem in terms of α' . Setting $\alpha' \equiv \sigma$, for short, (1.6) transforms into the second-order differential equation

$$\sigma^2 = q(t) + [\sigma^2, t], \tag{2.15}$$

where the differential operator

$$[\phi, t] \equiv -\frac{1}{4} \frac{\phi''}{\phi} + \frac{5}{16} \frac{\phi'^2}{\phi^2} \tag{2.16}$$

is defined for all functions $\phi \in C^2$, with $\phi \neq 0$. The necessity is easily proved as one can check immediately from (2.8) that α'_{LG} has the asymptotic behavior (2.13) with $\ell = -1$; $\ell = 1$ obtains when (v_{LG}, u_{LG}) is taken instead of (u_{LG}, v_{LG}) .

The sufficiency can be established as follows. If α is any given solution to (1.6) and thus a phase by Lemma 1.2, associated with a basis, say (u, v) , then

$$u^2 + v^2 = (\widehat{u}_{LG}, \widehat{v}_{LG}) \mathbf{A}^T \mathbf{A} \begin{pmatrix} \widehat{u}_{LG} \\ \widehat{v}_{LG} \end{pmatrix} = B_{11} \widehat{u}_{LG}^2 + 2B_{12} \widehat{u}_{LG} \widehat{v}_{LG} + B_{22} \widehat{v}_{LG}^2, \tag{2.17}$$

where $(\widehat{u}_{LG}, \widehat{v}_{LG})$ is a given LG basis, \mathbf{A} is the transforming matrix that takes $(\widehat{u}_{LG}, \widehat{v}_{LG})$ into (u, v) , and we set $\mathbf{B} \equiv \mathbf{A}^T \mathbf{A}$. Using the representations

$$\begin{aligned} \widehat{u}_{LG} &= f^{-1/4} \{ \cos \delta(t) + \eta_1(t) \} \\ \widehat{v}_{LG} &= f^{-1/4} \{ \sin \delta(t) + \eta_2(t) \} \end{aligned} \tag{2.18}$$

(cf. (2.5)), where $\delta(t)$ is the chosen primitive of $f^{1/2}$ and $\eta_k(t) = o(1)$, $k = 1, 2$, as $t \rightarrow \xi$, we get from (2.17)

$$f^{1/2}(u^2 + v^2) = B_{11} \cos^2 \delta(t) + 2B_{12} \sin \delta(t) \cos \delta(t) + B_{22} \sin^2 \delta(t) + o(1). \tag{2.19}$$

As α satisfies (2.13) by assumption, (1.9), (2.17) and (2.19) show that ℓ cannot vanish. Therefore we have from (1.8) and (2.13)

$$f^{1/2}(u^2 + v^2) \sim -\frac{W}{\ell}, \tag{2.20}$$

W being the constant Wronskian of (u, v) . This quantity is finite and clearly non-negative. Moreover, we obtain from (2.19) and (2.20), in view of (2.12),

$$B_{11} = B_{22} = -\frac{W}{\ell}, \quad B_{12} = 0. \tag{2.21}$$

Now, we can see that $-W/\ell > 0$ as, otherwise, $\mathbf{B} = 0$ and thus \mathbf{A} would be singular. This also shows that ℓ cannot be infinite. Therefore,

$$\mathbf{B} = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} = L\mathbf{I}, \quad L \equiv -\frac{W}{\ell} > 0, \tag{2.22}$$

that is,

$$\mathbf{A} = \sqrt{L}\mathbf{C} \tag{2.23}$$

with $\mathbf{C}^T = \mathbf{C}^{-1}$. This entails, by Lemma 2.5, that

$$\alpha' = \text{sgn}(\det \mathbf{C})\alpha'_{LG} = -\text{sgn}(\ell)\alpha'_{LG}. \tag{2.24}$$

The right-hand side in (2.24) has been obtained observing that $\text{sgn}(\det \mathbf{C}) = \text{sgn}(\det \mathbf{A}) = \text{sgn} W$ as $W_{LG} = 1$, and $\text{sgn} W = -\text{sgn}(\ell)$ as $L = -W/\ell > 0$. From this it follows that $\alpha' f^{-1/2} = -\text{sgn}(\ell)\alpha'_{LG} f^{-1/2} \sim \text{sgn}(\ell)$, but $\alpha' f^{-1/2} \sim \ell$ by (2.13). Therefore $\ell = \pm 1$. The last statement is trivial by (2.24).

Remark 2.10. Suppose that the Liouville-Green approximation holds as $t \rightarrow \xi$, for a given ξ , $-\infty \leq a \leq \xi \leq b \leq +\infty$. Then the following statements are equivalent:

- (a) $\int^t f^{1/2}(s) ds \rightarrow \infty$ as $t \rightarrow \xi$;
- (b) ξ is an oscillation point;
- (c) every phase $\alpha(t) \rightarrow \infty$ as $t \rightarrow \xi$.

In fact, (b) \implies (c) as α is monotonic and unbounded (by (1.8)) in this case; (c) \implies (b) as the basis in (1.11) is oscillatory, and thus all solutions are oscillatory in the neighborhood of ξ . Moreover, (a) \iff (b) by (2.18). We observe that the validity of the LG approximation as $t \rightarrow \xi$ is unnecessary in proving that (b) \iff (c). Such an assumption cannot be relaxed, however, in showing that (a) \iff (b). In view of Remark 2.10, (b) or (c) can replace (2.12) in Theorem 2.7.

As in several applications it was found that $\alpha'^2_{LG} \sim q$ (cf. (1.6), [7, 9, 10]), one could ask whether such an assumption might replace condition $\alpha'^2_{LG} \sim f$ (which is equivalent to (2.13), as $\ell = \pm 1$). The answer is no as examples can be constructed where all hypotheses of Theorem 2.7 hold but q is not asymptotic to f . This emphasizes the importance of a convenient splitting, $q = f + g$, in the LG approximation. A simple case is given by $f = t^{-2}$, $g = t^{-2}/4 + g_2(t)$ on the interval $(0, 1)$, and $\xi = 0$. In this case,

$$\int^t f^{1/2}(s) ds = \log t + \text{const.} \rightarrow -\infty \quad \text{as } t \rightarrow 0^+,$$

$\mathcal{F}'(t) = tg_2(t)$, and $\mathcal{V}_{0,t}(\mathcal{F}) < \infty$ if and only if $tg_2(t) \in L^1(0, 1)$. Now, every function $g_2(t)$ continuous on $[0, 1]$ fulfills such a requirement, while $q \sim f$ is equivalent to the condition $g/f = t^2g_2(t) + 1/4 \rightarrow 0$ as $t \rightarrow 0$. This, however, implies that $tg_2(t) \sim -1/4t$, and thus the contradiction $tg_2(t) \notin L^1(0, 1)$.

3. Existence and uniqueness for the Kummer differential equation.

In Section 1, we have defined the kernel $X(t)$ of a transformation and pointed out that it satisfies the Kummer differential equation (1.1). Hereafter we shall call $X(t)$ merely “a transformation” (cf. also (1.5)). If, similarly to the case of the phase equation (1.6), we define as a solution to (1.1) any function $X(t)$ with $X \in C^3(i)$, $i \subseteq j$, $X'(t) \neq 0$ in i , with range $X(i) \subseteq J$ and satisfying (1.1) in i , then the following result can be proved (cf. [5, §11.2 and §23.2]):

Lemma 3.1. *X(t) is a solution to equation (1.1) if and only if it is a transformation between equations (1.3'') and (1.3').*

In this Section, using Theorem 2.7, we shall prove a result of existence and uniqueness of solutions to equation (1.1). In fact, generally speaking, every transformation between (1.3'') and (1.3') can be decomposed first as a transformation from (1.3'') into (1.6'), and then a transformation from (1.6') into (1.3'). More precisely, for each transformation $X(t)$, $X : i \subseteq j \rightarrow X(i) \equiv I \subseteq J$, and any given phase, $\alpha(t)$, relative to $q(t)$, a function $\Gamma \equiv \alpha \circ X^{-1} : X(i) \rightarrow \alpha(i)$ is uniquely defined (see the diagram).

$$\begin{array}{ccc}
 i & \xrightarrow{X} & X(i) \\
 \alpha \downarrow & \swarrow & \Gamma \\
 \alpha(i) & &
 \end{array}$$

Such a function turns out to be a transformation as all the relevant properties can be easily checked [5, Ch. 23]. Moreover, $\Gamma(T)$ is a phase relative to $Q(T)$ in $I \equiv X(i)$ and thus a restriction of a phase of (1.3'') in J . This follows from the existence and uniqueness theorem for the phase equation proved in [5, §7.1]. Note that there are infinitely many representations of $X(t)$ as a composition of phase functions such as $X = \Gamma^{-1} \circ \alpha$ (in fact, α above can be chosen arbitrarily).

At this point, recall that every transformation $X : i \rightarrow X(i)$ maps oscillation [non-oscillation] points into oscillation [non-oscillation] points; that is, it preserves the oscillatory character [5, §1.6] of the differential equations on i and $X(i)$. We shall use this property below.

The main result of this section is contained in the following theorem.

Theorem 3.2. *Suppose that equations (1.3'), (1.3'') are given, with $q(t)$, $Q(T)$ both satisfying all hypotheses of Theorem 2.1, with $q = f + g$, $Q = F + G$, and reference points ξ , Ξ respectively. Let ξ , Ξ be oscillation points for (1.3'), (1.3''). Then, for every $t_0 \in j$ and $T_0 \in J$, there is a unique solution to equation (1.1) in $(t_0 \wedge \xi, t_0 \vee \xi)$ satisfying the three conditions:*

- (a) $X(t_0) = T_0$
- (b) $X(\xi) = \Xi$
- (c) the limit

$$\ell \equiv \lim_{t \rightarrow \xi} X'(t)F^{1/2}(X(t))f^{-1/2}(t) \tag{3.1}$$

exists. Moreover, $\ell = +1$ if both ξ and Ξ lie on the right or the left of t_0 , T_0 , respectively, and $\ell = -1$ otherwise. The structure of the solution $X(t)$ is given by

$$X'(t) = \text{sgn}(\ell) \frac{\alpha'_{LG}(t)}{\Gamma'_{LG}(X(t))}, \tag{3.2}$$

where α_{LG} and Γ_{LG} denote any two LG phases for (1.3'), (1.3'').

We emphasize that there is no solution to (1.1) satisfying conditions (a), (b), (c) above but with $\ell \neq +1, -1$. If the limit in (3.1) does not exist, there are infinitely many solutions to (1.1) satisfying (a) and (b).

Proof: We first show the uniqueness, that is, there exists at most one solution to (1.1), subject to conditions (a), (b), (c) on $i_0 \equiv (t_0 \wedge \xi, t_0 \vee \xi)$. Let $X(t)$ be such a

solution. Then $X : i_0 \rightarrow I_0 \equiv X(i_0)$ and $I_0 = (T_0 \wedge \Xi, T_0 \vee \Xi)$ as $X(t)$ is monotonic. According to the diagram above, choose a fixed LG phase of equation (1.3'), say $\widehat{\alpha}_{LG}$. Then the relation

$$\Gamma = \widehat{\alpha}_{LG} \circ X^{-1} \tag{3.3}$$

uniquely determines a phase of equation (1.3''). Differentiating in (3.3), we get

$$\Gamma'(T) = \frac{\widehat{\alpha}_{LG}'[X^{-1}(T)]}{X'[X^{-1}(T)]} \equiv \frac{\widehat{\alpha}_{LG}'(t)}{X'(t)} \equiv \frac{\alpha'_{LG}(t)}{X'(t)}, \tag{3.4}$$

and using (3.1) in (3.4) we get

$$\Gamma'(X(t)) \sim \frac{1}{\ell} \frac{\alpha'_{LG}}{f^{1/2}(t)} F^{1/2}(X(t)) \sim -\frac{1}{\ell} F^{1/2}(X(t)) \equiv -\frac{1}{\ell} F^{1/2}(T) \tag{3.5}$$

as $T \rightarrow \Xi$. In (3.5), the asymptotic behavior $\alpha'_{LG} f^{-1/2}(t) \sim -1$ has been used (cf. (2.8)). From (3.5) it follows that the limit

$$\lim_{T \rightarrow \Xi} F^{-1/2}(T) \Gamma'(T) \tag{3.6}$$

does exist and it equals $-1/\ell$. Therefore, by Theorem 2.7 we get $\ell = +1$ or $\ell = -1$, and

$$\Gamma'(T) = \text{sgn}(\ell) \Gamma'_{LG}(T). \tag{3.7}$$

Going back to (3.4), we have

$$X'(t) = \frac{\alpha'_{LG}(t)}{\Gamma'_{LG}(X(t))} \text{sgn}(\ell). \tag{3.8}$$

From (3.7), finally, $\Gamma(T) = \text{sgn}(\ell) \Gamma_{LG} + \text{const.}$, where the constant is determined by (a); i.e., by $\Gamma^{-1}(\widehat{\alpha}_{LG}(t_0)) = T_0$. Note that, for a fixed $\widehat{\alpha}_{LG}$, the so-determined function Γ does not depend on X . This proves uniqueness.

As for the existence, suppose first that the oscillation points ξ, Ξ lie on the same side with respect to t_0, T_0 , respectively. Then we construct a transformation $X(t)$ that is increasing. Choosing a LG phase $\widehat{\alpha}_{LG}$ on i_0 , $\widehat{\alpha}_{LG}$ turns out to be decreasing. Choosing then the unique LG phase on $I_0 = (T_0 \wedge \Xi, T_0 \vee \Xi)$, say $\widehat{\Gamma}_{LG}$, such that

$$\widehat{\Gamma}_{LG}(T_0) = \widehat{\alpha}_{LG}(t_0) \tag{3.9}$$

(cf. Corollary 2.6), the function

$$X(t) = (\widehat{\Gamma}_{LG}^{-1} \circ \widehat{\alpha}_{LG})(t) \tag{3.10}$$

is the solution we wanted. In fact, it is a transformation, as can be easily checked (cf. [5, Ch. 23]). Moreover, from (3.9), (3.10) we get (a), and obviously (b) (as every transformation takes oscillation points into oscillation points). Condition (3.1), finally, holds with $\ell = +1$.

Similarly, if ξ and Ξ lie on opposite sides of t_0, T_0 , we can construct a decreasing function X as

$$X(t) = (\widehat{\Gamma}^{-1} \circ \widehat{\alpha}_{LG})(t), \tag{3.10'}$$

where

$$\widehat{\Gamma} = -\widehat{\Gamma}_{LG} + 2\widehat{\alpha}_{LG}(t_0) \tag{3.10''}$$

and (a), (b), (c) are satisfied with $\ell = -1$. ■

At this point one could ask whether the results of Theorem 3.2 hold in the case of more general transformations (i.e., transformations defined in $i \supset i_0$). The answer is contained in the following:

Corollary 3.3. *Let X be the unique solution to equation (1.1) under conditions (a), (b), (c) (with $\ell = \pm 1$) that has been constructed on i_0 in Theorem 3.2. Then there is a unique extension of $X(t)$ to the interval i , $i_0 \subset i \subseteq j$, if and only if there exists a Liouville-Green phase, $\widehat{\alpha}_{LG}$, such that*

$$\widehat{\Gamma}(J) \supseteq \widehat{\alpha}_{LG}(i), \tag{3.11}$$

$\widehat{\Gamma}$ being the phase related to $\widehat{\alpha}_{LG}$ by (3.9) or (3.10'').

Note that if condition (3.11) is satisfied by a pair $\widehat{\alpha}_{LG}, \widehat{\Gamma}$, then it is satisfied by all; that is, choosing a different LG phase α_{LG} , then (3.11) is verified with the corresponding one, Γ .

Proof: We first notice that uniqueness of extensions follows in any case from uniqueness for the Cauchy problem [4, p. 197; 5, §24.1].

Suppose such an extension, $\tilde{X} : i \rightarrow \tilde{X}(i) \equiv I \subseteq J$, exists. Then the function $\Gamma = \widehat{\alpha}_{LG} \circ \tilde{X}^{-1}$, defined on I , is a phase that extends (in a unique way) $\widehat{\Gamma}$, i.e., $\Gamma|_{I_0} = \widehat{\Gamma}$. It follows that $\widehat{\Gamma}(I) = \widehat{\alpha}(i)$, and thus $\widehat{\Gamma}(J) \supseteq \widehat{\alpha}_{LG}(i)$.

Conversely, if (3.11) holds, the composition in (3.10) exists for $t \in i$. ■

So far, we made no assumptions on the oscillatory or non-oscillatory nature of the other end-point of the intervals j and J (throughout, ξ and Ξ have been assumed to be oscillation points and hence end-points). There are four possible cases. We shall say, for short, that equation (q) (or (Q)) is one-side or both-side oscillatory.

When (Q) is both-side oscillatory, $\widehat{\Gamma}(J) = \mathbb{R}$, and therefore (3.11) holds for every $i \subseteq j$. If

(I) (q) is also both-side oscillatory,

then the transformation $X(t)$ of Theorem 3.3 maps the whole j into the whole J (that is, $X(t)$ is “a complete transformation” in the terminology of Borůvka, (cf. [3; 5, Ch. 26])). If

(II) (q) is one-side oscillatory,

then $X(t)$ maps j into a proper sub-interval of J .

When (Q) is one-side oscillatory, then (3.11) represents an actual condition. In this case, if

(III) (q) is both-side oscillatory,

then the maximal interval i in (3.11) is a proper subset of j . If

(IV) (q) is one-side oscillatory,

on the other hand, the transformation X may be not a complete transformation between (Q) and (q), despite the fact that equations (Q) and (q) have the same oscillatory character. This occurrence is illustrated in the following example.

Example 3.4. Consider $q = Q \equiv 1$, $j = J = (0, +\infty)$. Then $\widehat{\alpha}_{LG}(t) = -t$ is an LG phase for equation (q), and $\Gamma_{LG}(T) = -T + c$ is, for every constant c , an LG phase for equation (Q). Imposing $\widehat{\alpha}(t_0) = \widehat{\Gamma}(T_0)$, for $t_0, T_0 \in (0, +\infty)$, we get $c = T_0 - t_0$. Therefore, $\widehat{\Gamma}_{LG} = -T + T_0 - t_0$ and $\widehat{\alpha}_{LG}(j) = (-\infty, 0)$, $\widehat{\Gamma}_{LG}(J) = (-\infty, T_0 - t_0)$. Finally, condition (3.11) is satisfied for every $i \subseteq j$ and for all choices of $t_0, T_0 \in (0, +\infty)$ with $T_0 \geq t_0$. When $T_0 < t_0$, (3.11) yields the maximal interval i that turns out to be $(t_0 - T_0, +\infty) \subset j$.

Remark 3.5. Comparing Theorem 3.2 with Theorem 2.7, one might recognize that three conditions instead of two are needed to uniquely determine a solution $X(t)$

to equation (1.1), other than the case of equation (1.6). In certain cases, however, condition (b) in Theorem 3.2 is automatically satisfied so that, again, only two conditions suffice. In case (III) and (IV), for instance, Ξ is the only oscillation point and thus, necessarily, the image of ξ . In case (I) or (II), condition (b) can also be dropped, provided that the LG approximation holds at both end-points of J with the same splitting of Q . Note that the latter is exactly the case of the phase equation (1.6), as $Q \equiv F \equiv 1$ and $J = (-\infty, +\infty)$.

4. Examples and applications.

4.1. A Liouville-Green basis for the Bessel differential equation. The pair $(\sqrt{t}J_\nu(t), \sqrt{t}Y_\nu(t))$ forms a basis for equation (1.3') with $q = 1 - (\nu^2 - 1/4)t^{-2}$ on $(0, +\infty)$. If α denotes a phase associated with such a basis, we obtain

$$\alpha' = -\frac{W[\sqrt{t}J_\nu(t), \sqrt{t}Y_\nu(t)]}{t[J_\nu^2(t) + Y_\nu^2(t)]} = -\frac{2}{\pi} \frac{1}{t[J_\nu^2(t) + Y_\nu^2(t)]} \rightarrow -1$$

as $t \rightarrow +\infty$ (cf. [1, Ch. 9]). By Theorem 2.7, this shows that $\alpha' = \alpha'_{LG}$. By Lemma 2.5, on the other hand, this implies that $(\sqrt{t}J_\nu(t), \sqrt{t}Y_\nu(t))$ is related to any LG basis by an orthogonal transformation followed by a homothetic transformation. But $W[\sqrt{t}J_\nu(t), \sqrt{t}Y_\nu(t)] = 2/\pi$, so that $(\sqrt{\pi t/2}J_\nu(t), \sqrt{\pi t/2}Y_\nu(t))$ is an LG basis up to an orthogonal transformation with determinant +1, and therefore an LG basis itself (cf. Remark 2.2). Notice that, our results being global in character, here an LG basis is explicitly known on the whole interval $(0, +\infty)$, rather than just in a neighborhood of $+\infty$.

4.2. An LG basis for the Airy equation. Similarly, it can be shown that $(\sqrt{\pi}Bi(-t), \sqrt{\pi}Ai(-t))$ is an LG basis for the Airy differential equation, $y'' + ty = 0$, on $(-\infty, +\infty)$, cf. [1, pp. 446, 449]. We leave to the reader the easy task to check all details.

4.3 Central dispersions. A “central dispersion” is the kernel of the transformations of equations like (1.3') into themselves, and therefore the solution θ to equation (1.1) with $Q \equiv q$, i.e.,

$$\theta'^2 q(\theta) = q(t) - \frac{1}{2}\{\theta, t\} \tag{4.1}$$

(see [5, Ch. 13]). By Theorem 3.2 there is exactly one solution θ to (4.1) with $\theta(t_0) = t_0$ and $\theta^1 f^{1/2}(\theta) f^{-1/2}(t) \rightarrow \ell > 0$ as $t \rightarrow \xi$ (condition (b) in Theorem 3.2, i.e., $\theta(\xi) = \xi$, is automatically satisfied).

As the identity $\theta(t) = t$ is always (trivially) a central dispersion, the previous result shows that this is the required solution. In other words, the identity is the only increasing central dispersion with an inner fixed point and satisfying the asymptotic condition above.

4.4. An application to numerical analysis. It has been known after Borůvka's work that the concept of phase plays a central role in describing all solutions to equations like (1.3'). In particular, a basis can be constructed (cf. (1.11)) and zeros of solutions obtained from it (cf. (1.8)). In [4, p. 244], Borůvka

himself emphasized the importance of developing efficient numerical methods for approximating phase functions, his theory being only qualitative.

In [9, 10], we worked out an asymptotic-numerical method to approximate zeros of solutions to equations of the form (1.3'), in the oscillatory case on a half-line. We refer to [9, 10] for the precise hypotheses made on q . In particular, the validity of the LG approximation at $\xi = +\infty$ was required, as we wanted to approximate an LG phase.

Setting $\phi = \alpha'^2$ in (1.6), we got

$$\phi = q(t) + [\phi, t], \tag{4.2}$$

where $[\phi, t]$ is defined in (2.16). The reduction of (1.6) to (4.2) presents remarkable advantages both from the analytical and the computational standpoints. Then we considered the iterative scheme

$$\phi_0 = q(t), \quad \phi_{n+1} = q(t) + [\phi_n, t], \quad n = 0, 1, 2, \dots, \tag{4.3}$$

and proved estimates like

$$|\phi_n(t) - \phi_{LG}(t)| \leq H_n(t), \quad t \in J_n, \quad n = 0, 1, 2, \dots, \tag{4.4}$$

where $\phi_{LG} \equiv \alpha'^2_{LG}$, $J_n = (\rho_n, +\infty)$ is a decreasing sequence of half-lines (ρ_n increases linearly with n), and $H_n(t) \rightarrow 0$ as $t \rightarrow +\infty$ for each fixed n . Moreover,

$$\sup_{t \in J_n} H_n(t) \rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{4.5}$$

Note that the asymptotic result (4.4)–(4.5) implies a weaker type of asymptotic convergence, in that the set J_n also depends on n . Below, we shall refer to the case where

$$\sup_{t \in J_n} |\phi_n(t) - \phi_{LG}(t)| \rightarrow 0 \quad \text{as } n \rightarrow \infty \tag{4.6}$$

by writing, for short,

$$\phi_n \xrightarrow{J_n} \phi_{LG} \quad \text{as } n \rightarrow \infty. \tag{4.7}$$

One may ask whether such a convergence uniquely determines the limit-function ϕ_{LG} . The answer is no, as if (4.7) holds, then $\phi_n \xrightarrow{J_n} \psi$ if and only if

$$\psi - \phi_{LG} = o(1) \quad \text{as } t \rightarrow +\infty. \tag{4.8}$$

A relation like $\phi_n \xrightarrow{J_n} \psi$, however, uniquely determines a positive C^2 -solution to (4.2). In fact, being such a solution, ψ equals α'^2 for some phase α . Moreover, as in [9, 10] f^{-1} is bounded near $t = +\infty$, it follows from (4.8) that

$$\psi f^{-1} = \phi_{LG} f^{-1} + o(1) f^{-1} = \phi_{LG} f^{-1} + o(1) \quad \text{as } t \rightarrow +\infty. \tag{4.9}$$

In [9, 10], $f \sim at^m$ as $t \rightarrow +\infty$, with $a > 0$, $m \geq 0$. Uniqueness then follows from Theorem 2.7.

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REFERENCES

- [1] M. Abramowitz and I.A. Stegun, "Handbook of Mathematical Functions", Dover, New York, 1968.
- [2] O. Borůvka, *Sur la transformation des intégrales des équations différentielles linéaires ordinaires du second ordre*, Ann. Mat. Pura Appl., 41 (1956), 325–342.
- [3] O. Borůvka, *Sur les transformations différentielles linéaires complètes du second ordre*, Ann. Mat. Pura Appl., 49 (1960), 229–251.
- [4] O. Borůvka, *Théorie des transformations des équations différentielles linéaires du deuxième ordre*, Rend. Mat., 26, ser. V (1967), 187–246.
- [5] O. Borůvka, "Linear Differential Transformations of the Second Order," The English Universities Press, London, 1971.
- [6] E.E. Kummer, *De generali quadam aequatione tertii ordinis*, J. Reine Angew. Math., 100 (1887), 1–9.
- [7] F.W.J. Olver, *A new method for the evaluation of zeros of Bessel functions and of other solutions of second-order differential equations*, Proc. Cambridge Philos. Soc., 46 (1950), 570–580.
- [8] F.W.J. Olver, "Asymptotics and Special Functions," Academic Press, New York, 1974.
- [9] R. Spigler and M. Vianello, *Approximating zeros of solutions of second-order linear ODE's by "phase function" methods*, in "Asymptotic and Computational Analysis," Proc. of a Conference in honor of F.W.J. Olver on his 65th birthday, Winnipeg, June 1989, Marcel Dekker, New York, 1990, pp. 707–722.
- [10] R. Spigler and M. Vianello, *A numerical method for evaluating zeros of solutions of second-order linear differential equations*, Math. Comp., 55 (1990), 591–612.