ASYMPTOTIC STABILITY FOR NONLINEAR DEGENERATE PARABOLIC EQUATIONS WITH NEUMANN BOUNDARY CONDITIONS

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(Submitted by: Peter Hess)

Abstract. In this paper we study a quasilinear degenerate parabolic equation with Neumann boundary condition of the form
\[ u_t - \Delta \beta(u) \geq f(t,x) \] in \( \mathbb{R}^N \), \( \partial_n \beta(u) \geq h(t,x) \) on \( \mathbb{R}^N \times \partial \Omega \), where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \), \( \beta \) is a given maximal monotone graph in \( \mathbb{R} \times \mathbb{R} \) and \( f, g \) are given functions. We shall show the existence of a periodic solution in time and its stability as \( t \to +\infty \).

1. Introduction. In this paper we study the following quasilinear parabolic equation:
\[ u_t - \Delta \tilde{\beta} = f, \quad \tilde{\beta} \in \beta(u), \quad \text{in } \mathbb{R}^N \times \Omega, \]
\[ \partial_n \tilde{\beta} = h \quad \text{on } \mathbb{R}^N \times \partial \Omega, \]
where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) \((N \geq 1)\) with sufficiently smooth boundary \( \Gamma = \partial \Omega \); \( \beta \) is a given maximal monotone graph in \( \mathbb{R} \times \mathbb{R} \); \( f \) and \( h \) are given functions on \( \mathbb{R}^N \times \Omega \) and on \( \mathbb{R}^N \times \partial \Omega \), respectively; the function \( u = u(t,x) \), with \( \tilde{\beta} = \tilde{\beta}(t,x) \in \beta(u) \), is the unknown; \( \partial_n \) denotes the outward normal derivative on \( \Gamma \).

Equation (1.1) represents mathematical models of some physical problems, and there are three interesting cases (a), (b) and (c) of \( \beta \) mentioned below:

(a) \( \beta \) is Lipschitz continuous on \( \mathbb{R} \) with linear growth at \( \pm \infty \);
(b) \( \beta^{-1} \) is Lipschitz continuous on \( \mathbb{R} \);
(c) the domain \( D(\beta) \) of \( \beta \) is bounded and not a singleton in \( \mathbb{R} \); i.e.,
\[ D(\beta) = [r_*, r^*] \] for some \(-\infty < r_* < r^* < +\infty \).

For instance, in the case (a) equation (1.1) includes the enthalpy formulation of Stefan problem (cf. [4, 9, 13, 19]), and in the case (b) or (c) it has been considered as a mathematical modeling of filtration in porous media and of flow in Hele-Shaw cells (cf. [1-3, 5-8, 12, 16, 17]).
Recently, when $\beta$ satisfies both (b) and (c), Hulshof [11] obtained some results on the existence, uniqueness, regularity and asymptotic equilibrium stability for the problem

$$\begin{cases}
    u_t - \Delta \tilde{\beta} = f, & \tilde{\beta} \in \beta(u), \quad \text{in } (t_0, +\infty) \times \Omega, \\
    \partial_n \tilde{\beta} = h & \text{on } (t_0, +\infty) \times \Gamma, \\
    u(t_0, \cdot) = u_0 & \text{in } \Omega.
\end{cases}$$

(1.3)

Also the case (a) was treated by Haraux-Kenmochi [10]. In the author’s previous paper [14] the case (c) was discussed and it was proved that problem (1.3) has a unique solution $u$ in a very weak sense; in fact, a solution is constructed and its uniqueness is shown in the dual space $H^1(\Omega)^* \cap H^1(\Omega)$, and the space $H^1(\Omega)^*$ seems to be the largest one in which the problem is well-posed. Our case includes $\beta$ having the multivalued $\beta^{-1}$, so that one can not expect much regularity for solutions. However, as far as the existence and uniqueness of solutions are concerned, some results in [14] are more general than in [11].

In this paper we shall still restrict $\beta$ to the case (c) and mainly discuss the periodic behavior of solutions to (1.1), (1.2). In fact, with the help of some results in [14] we shall construct a periodic solution, and investigate the structure of periodic solutions as well as their stability. We refer to [6, 18] for related results.

**Notations.** In general, for a Banach space $V$ we denote by $| \cdot |_V$ the norm in $V$ and by $V^*$ the dual space with the usual dual norm $V$. Also, we use the symbol “$\rightarrow$” or “lim” to indicate convergence in the strong topologies of Banach spaces, unless otherwise stated.

In this paper we use the following notations:

- $|E|$ : the volume of measurable set $E$ in $\mathbb{R}^N$;
- $H = L^2(\Omega)$, $(u,v) = \int_\Omega u(x)v(x) \, dx$ for $u,v \in H$;
- $X = H^1(\Omega)$, $a(u,v) = \int_\Omega \nabla u(x) \cdot \nabla v(x) \, dx$ for $u,v \in X$;
- $\langle \cdot, \cdot \rangle$ : the duality pairing between $X^*$ and $X$;
- $H_0 = \{z \in H; \int_\Omega z \, dx = 0\}$ : the Hilbert space with inner product $\langle \cdot, \cdot \rangle_0$ and norm $| \cdot |_{H_0}$ given by

$$\langle u, v \rangle_0 = (u,v), \quad |u|_{H_0} = |u|_H \quad \text{for } u,v \in H_0;$$

- $X_0 = X \cap H_0$ : the Banach space with norm $|u|_{X_0} = |\nabla u|_H$ for $u \in X_0$;
- $\langle \cdot, \cdot \rangle_0$ : the duality pairing between $X_0^*$ and $X_0$;
- $F$ : the duality mapping from $X_0$ onto $X_0^*$.

Sometimes, $H$ and $H_0$ are identified with their dual spaces by the inner products. In such cases we have

$$X \subset H \subset X^*, \quad X_0 \subset H_0 \subset X_0^*$$

with compact and dense imbeddings. With the projection $P$ from $H$ onto $H_0$, i.e.,

$$Pz = z - \frac{1}{|\Omega|} \int_\Omega z \, dx, \quad z \in H,$$

we see that $X_0 = P(X)$ and $P$ is the projection from $X$ onto $X_0$, and

$$\langle Pw, z \rangle_0 = (w,z) \quad \text{for } w \in H, z \in H_0;$$

(1.4)
\( \langle Pw, z \rangle_0 = (Pw, z)_0 = \langle w, z \rangle = \langle w, z \rangle \) for \( w \in H, \ z \in X_0 \);

\( a(w, z) = a(Pw, Pz) = \langle FPw, Pz \rangle_0 \) for \( w, z \in X \).

Moreover, \( X_0^* \) becomes a Hilbert space with inner product \( \langle \cdot, \cdot \rangle_* \) given by

\[ \langle w, z \rangle_* = \langle w, F^{-1}z \rangle_0 = \langle z, F^{-1}w \rangle_0 \] for \( w, z \in X_0^* \).

For simplicity we write \( | \cdot |_* \) for the dual norm of \( | \cdot |_{X_0^*} \); i.e., \( |z|_* = (z, z)_{X_0^*}^{1/2} \) for any \( z \in X_0^* \).

### 2. Main results.

Throughout the remainder of this paper, let

\[ f \in L^2_{loc}(\mathbb{R}; H), \quad h \in L^2_{loc}(\mathbb{R}; L^2(\Gamma)) \]

and assume that \( \beta \) is a maximal monotone graph in \( \mathbb{R} \times \mathbb{R} \) and

\[ \overline{D(\beta)} = [r_*, r^*] \] for some \( -\infty < r_* < r^* < +\infty \),

where \( D(\beta) \) is the domain of \( \beta \); i.e., \( D(\beta) = \{ r \in \mathbb{R} ; \beta(r) \neq \emptyset \} \).

We now give a notion of solution to the problem

\[ \begin{aligned}
P & : \begin{cases}
u_t - \Delta \bar{\beta} = f, & \bar{\beta} \in \beta(u) \text{ in } Q, \\
\partial_n \bar{\beta} = h & \text{ on } \Sigma,
\end{cases}
\end{aligned} \]

where \( Q = (t_0, t_1) \times \Omega \) and \( \Sigma = (t_0, t_1) \times \Gamma \) for \( -\infty < t_0 < t_1 < +\infty \).

**Definition 2.1.** Let \( J \) be a compact interval of the form \( [t_0, t_1] \) in \( \mathbb{R} \). Then a function \( u : J \to H \) is called a (weak) solution of \( P = P(\beta; f, h) \) on \( J \), if the following properties are fulfilled:

(w1) \( u \) is weakly continuous from \( J \) into \( H \);

(w2) there is a function \( \bar{\beta} \) on \( Q \) such that

\[ \bar{\beta} \in \beta(u) \text{ a.e. on } Q, \]

\[ \bar{\beta} \in L^2(t_0', t_1; X) \text{ for any } t_0' \text{ with } t_0 < t_0' < t_1, \]

and

\[ -\int_Q u\xi_t \, dx \, dt + \int_J a(\bar{\beta}, \xi) \, dt = \int_Q f\xi \, dx \, dt + \int_\Sigma h\xi \, d\sigma \, dt \]

for any \( \xi \in D(t_0, t_1; X) \), where \( Q = (t_0, t_1) \times \Omega, \Sigma = (t_0, t_1) \times \Gamma, D(t_0, t_1; X) \) is the space of all \( X \)-valued smooth functions on \( J \) with compact support in \( (t_0, t_1) \) and \( d\sigma \) denotes the usual surface element on \( \Gamma \).
As was seen in [14; Remark 2.1], in Definition 2.1 it follows from (w1) and (w2) that
\[ u \in W^{1,2}(t_0', t_1'; X^*) \quad \text{for every } t_0 < t_0' < t_1; \]
and (2.5) is equivalent to
\[ \langle u'(t), z \rangle + a(\tilde{\beta}(t), z) = \langle f^*(t), z \rangle \quad \text{for any } z \in X, \text{ a.e. } t \in J, \] (2.5')
where \( f^* \in L^2(J; X^*) \) is the function given by
\[ \langle f^*(t), z \rangle = \int_{\Omega} f(t, x)z(x) \, dx + \int_{\Gamma} h(t, \sigma)z(\sigma) \, d\sigma \quad \text{for any } z \in X. \] (2.6)

Therefore (w1) and (w2) can be rewritten as the following:
(w1)' \( u \in C(J; X^*) \cap L^\infty(J; H), u \in W^{1,2}(t_0', t_1'; X^*) \) for every \( t_0 < t_0' < t_1; \)
(w2)' there is a function \( \bar{\beta} \) in \( Q \) for which (2.3), (2.4) and (2.5)' hold.

Furthermore, if we employ the set \( \beta_J[u] \) defined by
\[ \beta_J[u] = \{ \bar{\beta} : (2.3), (2.4) \text{ and } (2.5)' \text{ hold} \} \]
for each \( u : J \to H \) with \( u(t, x) \in D(\beta) \) almost everywhere on \( Q \), then (w2)' is equivalent to
(w2)'' \( \beta_J[u] \neq \emptyset. \)

**Definition 2.2.** For any interval \( J' \) in \( \mathbb{R} \), a function \( u : J' \to H \) is called a (weak) solution of \( P = P(\beta; f, h) \) on \( J' \), if the restriction of \( u \) to any compact subinterval \( J \) of \( J' \) is a solution of \( P \) on \( J \) in the sense of Definition 2.1.

For a general interval \( J' \) in \( \mathbb{R} \) and a function \( u : J' \to H \) with \( u(t, x) \in D(\beta) \) almost everywhere on \( J' \times \Omega \), we define
\[ \beta_{J'}[u] = \{ \tilde{\beta} : \tilde{\beta} \text{ is a function on } J' \times \Omega, \text{ and } \tilde{\beta} \in \beta_J[u] \}
\text{for each compact subinterval } J \text{ of } J'. \]

It is easy to see that \( u \) is a solution of \( P \) on \( J' \) if and only if \( u \) is weakly continuous from \( J' \) into \( H \) and \( \beta_{J'}[u] \neq \emptyset. \)

**Definition 2.3.** Let \( u \) be a solution of \( P = P(\beta; f, h) \) on \( \mathbb{R} \), and \( T \) be a positive number. Then we say that \( u \) is a \( T \)-periodic solution of \( P \) on \( \mathbb{R} \), if \( u(t + T) = u(t) \) for any \( t \in \mathbb{R} \).

The main results of this paper are stated as follows.

**Theorem 2.1** (Existence of periodic solutions). Let \( T \) be a positive number, and assume that \( f \) and \( h \) are \( T \)-periodic on \( \mathbb{R} \); i.e.,
\[ f(t + T) = f(t), \quad h(t + T) = h(t) \quad \text{for a.e. } t \in \mathbb{R}, \] (2.7)
and
\[ \int_0^T \int_{\Omega} f \, dx \, dt + \int_0^T \int_{\Gamma} h \, d\sigma \, dt = 0. \] (2.8)
Now, let $c_0$ be any number such that

$$r_* < \frac{1}{|\Omega|}\left\{c_0 + \int_0^t \int_\Omega f\,dx\,d\tau + \int_0^t \int_{\Gamma} h\,d\sigma\,d\tau\right\} < r^* \text{ for any } t \in [0, T]. \quad (2.9)$$

Then there exists at least one $T$-periodic solution $\omega$ of $P = P(\beta; f, h)$ on $\mathbb{R}$ such that

$$\int_\Omega \omega(0, x)\,dx = c_0. \quad (2.10)$$

The next theorem is concerned with the relationship between two periodic solutions.

**Theorem 2.2.** Let $T$ be as in Theorem 2.1 and assume (2.7) and (2.8) hold. Let $\omega_1, \omega_2$ be two $T$-periodic solutions of $P = P(\beta; f, h)$ on $\mathbb{R}$ such that

$$\int_\Omega \omega_1(0, x)\,dx = \int_\Omega \omega_2(0, x)\,dx. \quad (2.11)$$

Then there is a function $\alpha_0$ in $H_0$ such that

$$\omega_1(t) - \omega_2(t) = \alpha_0 \text{ for any } t \in \mathbb{R}, \quad (2.12)$$

and moreover

$$\beta_R[\omega_1] = \beta_R[\omega_2]. \quad (2.13)$$

As is seen from the next example, in general, a $T$-periodic solution is not unique even in the class of solutions $\omega$ with $\int_\Omega \omega(0, x)\,dx = c_0$ for a fixed $c_0$.

**Example 2.1.** Consider the steady-state problem in one-dimensional space. Let $\beta$ be a multivalued function on $\mathbb{R}$ defined by

$$\beta(r) = \begin{cases} 
(-\infty, -1] & \text{for } r = -2, \\
-1 & \text{for } -2 < r < 0, \\
[-1, -1] & \text{for } r = 0, \\
1 & \text{for } 0 < r < 2, \\
[1, +\infty) & \text{for } r = 2, \\
\emptyset & \text{otherwise.}
\end{cases}$$

We then see that $\beta$ is maximal monotone in $\mathbb{R} \times \mathbb{R}$ and $D(\beta) = [-2, 2]$. Now, let $z_0$ be any smooth function on $\Omega = (-2, 2)$ such that

$$\begin{cases} 
-1 \leq z_0 < 0 \text{ on } (-2, -1), & z_0 = 0 \text{ on } [-1, 1], \\
0 < z_0 \leq 1 \text{ on } (1, 2) \text{ and } \int_\Omega z_0\,dx = 0,
\end{cases} \quad (2.14)$$

and let $\tilde{\beta}$ be any smooth function on $\Omega$ such that

$$\tilde{\beta} = -1 \text{ on } (-2, -1), \quad -1 \leq \tilde{\beta} \leq 1 \text{ on } [-1, 1], \quad \tilde{\beta} = 1 \text{ on } (1, 2). \quad (2.15)$$
Then it is easy to check that $\tilde{\beta}$ is the classical solution of

$$-\tilde{\beta}_{xx} = f \quad \text{in } \Omega, \quad \partial_n \tilde{\beta} = 0 \quad \text{on } x = -2, 2,$$

where $f = -\tilde{\beta}_{xx}$. Therefore, the function $u(t, x) = z_0(x)$ gives a $T$-periodic solution of $P(\beta; f, 0)$ on $\mathbb{R}$ for any positive number $T$, since $\tilde{\beta} \in \beta(z_0)$ almost everywhere on $\Omega$. For a fixed $\tilde{\beta}$ satisfying (2.5) there are an infinite number of functions $z_0$ having property (2.4) with $\tilde{\beta} \in \beta(z_0)$ almost everywhere on $(-2, 2)$. In fact, if $z_0$ satisfies these properties, so does $cz_0$ for any positive constant $c$ with $0 < c < 1$.

This shows that $P$ has an infinite number of $T$-periodic solutions $\omega$ of the form $\omega = cz_0$, $0 < c < 1$, on $\mathbb{R}$ with $\int_{-2}^{2} \omega(0, x) \, dx = 0$. Such a type of non-uniqueness of periodic solutions is caused by the bad behavior of $\beta$.

As to the asymptotic stability of solutions we prove:

**Theorem 2.3.** Let $T$ and $c_0$ be as in Theorem 2.1 and suppose (2.7)-(2.9) hold. Let $u$ be any solution of $P = P(\beta; f, h)$ on $J' = (t_0, +\infty)$, $-\infty < t_0 < +\infty$, such that

$$\int_{\Omega} u(n_0 T, x) \, dx = c_0 \quad \text{for some } n_0 \in \mathbb{N} \text{ with } n_0 T \geq t_0. \quad (2.16)$$

Then there exists a $T$-periodic solution $\omega$ of $P$ on $\mathbb{R}$ such that

$$\int_{\Omega} \omega(0, x) \, dx = c_0 \quad (2.17)$$

and

$$u(t) - \omega(t) \to 0 \quad \text{weakly in } H \quad \text{as } t \to +\infty. \quad (2.18)$$

Finally we prove the periodicity of global solutions.

**Theorem 2.4.** Let $T$ and $c_0$ be as in Theorem 2.1, and suppose (2.7)-(2.9) hold as well. Then any solution $u$ of $P = P(\beta; f, h)$ on $\mathbb{R}$ with $\int_{\Omega} u(0, x) \, dx = c_0$ is $T$-periodic on $\mathbb{R}$.

The proofs of these theorems will be given in Section 5.

### 3. Cauchy problems.

In this section we consider the Cauchy problem for $P(\beta; f, h)$.

**Definition 3.1.** Let $J'$ be any interval of the form $[t_0, t_1]$ or $(0, +\infty)$, $-\infty < t_0 < t_1$, and let $u_0 \in H$. Then a function $u : J' \to H$ is called a (weak) solution of $CP = CP(\beta; f, h; u_0)$ on $J'$, if $u$ is a solution of $P(\beta; f, h)$ on $J'$ and $u(t_0) = u_0$.

In order to formulate an existence-uniqueness result for $CP$, we introduce a proper, non-negative, l.s.c. and convex function $\beta^*$ on $\mathbb{R}$ whose subdifferential is $\beta$; i.e., $\partial \beta^* = \beta$ in $\mathbb{R}$. Clearly, by (2.2),

$$\beta^*(r) \geq C_0 |r|^2 - C'_0 \quad \text{for any } r \in \mathbb{R}, \quad (3.1)$$

where $C_0, C'_0$ are some positive constants. Further, define the non-negative, proper, l.s.c. and convex function $G$ on $H$ by the formula

$$G(z) = \int_{\Omega} \beta^*(z(x)) \, dx, \quad z \in H; \quad (3.2)$$

it is evident by (3.1) that $D(G) = \{z \in H : \beta^*(z) \in L^1(\Omega)\}$. 

Theorem 3.1 (cf. [14; Theorems 3.1, 3.2]). Let \( J = [t_0, t_1] \), \(-\infty < t_0 < t_1 < +\infty\), and let \( u_0 \in H \) such that
\[
r_* \leq u_0 \leq r^* \quad \text{a.e. on } \Omega,
\]
\[
r_* < a(t) := \frac{1}{|\Omega|} \left\{ \int u_0 x dx + \int_0^t \int_0^t f dx d\tau + \int_0^t g d\sigma d\tau \right\} < r^*
\]
for any \( t \in J \). Then \( CP = CP(\beta; f, h; u_0) \) has a solution \( u \) on \( J \) such that
\[
G(u) \in L^1(J), \quad (t - t_0)G(u) \in L^\infty(J), \quad (t - t_0)^{1/2} u' \in L^2(J; X^*) \quad (3.5)
\]
and \( G(u) \) is absolutely continuous on \([t'_0, t_1]\) for every \( t'_0 \) with \( t_0 < t'_0 < t_1 \). Moreover, for any \( \tilde{\beta} \in \beta_J[u] \), we have
\[
(t - t_0)^{1/2} \tilde{\beta} \in L^2(J; X) \quad (3.6)
\]
and
\[
\frac{d}{dt} G(u(t)) + |\nabla \tilde{\beta}(t)|^2_H = (f^*(t), \tilde{\beta}(t)) \quad \text{for a.e. } t \in J, \quad (3.7)
\]
where \( f^* \) is the function given by (2.6). In particular, if \( G(u_0) < +\infty \), then \( G(u) \) is absolutely continuous on \( J \),
\[
u' \in L^2(J; X^*) \quad (3.8)
\]
and
\[
\beta_J[u] \subset L^2(J; X). \quad (3.9)
\]

Next, let \( u_1, u_2 \) be two solutions of \( P(\beta; f, h) \) on an interval \( J' \) in \( \mathbb{R} \) such that
\[
\int_\Omega u_1(t, x) dx = \int_\Omega u_2(t, x) dx \quad \text{for some } t_0 \in J'. \quad (3.10)
\]
Then, taking \( z = 1 \) in the variational identities (2.5)' corresponding to two solutions \( u_1 \) and \( u_2 \), and integrating them over \([t_0, t]\) (or \([t, t_0]\)) for \( t \in J' \), we have
\[
\int_\Omega u_1(t, x) dx = \int_\Omega u_2(t, x) dx; \quad \text{i.e. } u_1(t) - u_2(t) \in H_0 \quad \text{for any } t \in J'. \quad (3.11)
\]
By this observation, it is quite natural to estimate the difference of two solutions in the space \( H_0 \) or \( X_0^* \). The following proposition mentions the contraction property of solutions in the space \( X_0^* \).

Proposition 3.1 (cf. [14; Proposition 2.1]). Let \( J' \) be any interval in \( \mathbb{R} \), and let \( u_1, u_2 \) be solutions of \( P = P(\beta; f, h) \) on \( J' \) for which (3.10) holds. Then
\[
\frac{1}{2} |u_1(t) - u_2(t)|^2 + \int_s^t |\tilde{\beta}_1 - \tilde{\beta}_2, u_1 - u_2| d\tau = \frac{1}{2} |u_1(s) - u_2(s)|^2 \quad (3.12)
\]
for any \( s, t \in J' \) and \( \tilde{\beta}_i \in \beta_{J'}[u_i], i = 1, 2 \). In particular, \( |u_1(t) - u_2(t)|_* \) is non-increasing in \( t \in J' \).

This proposition asserts that the Cauchy problem \( CP \) has at most one solution on any compact interval \( J = [t_0, t_1] \), and the solution depends continuously upon initial value \( u_0 \) with respect to the topology of \( X_0^* \).

4. Lemmas. In this section we prepare some lemmas which are useful in proving the main results.
Lemma 4.1. Let \( J = [t_0, t_1] \) be any compact interval, and let \( u \) be in \( W^{1,2}(J; X^*) \cap L^\infty(J; H) \) such that \( G(u(t)) \) is absolutely continuous on \( J \). Then

\[
\frac{d}{dt} G(u(t)) = \langle u'(t), z \rangle \quad \text{for any } z \in \partial G(u(t)) \cap X, \text{ a.e. } t \in J,
\]

where \( \partial G \) stands for the subdifferential of \( G \) in \( H \).

**Proof:** For any number \( \delta \) and any \( z \in \partial G(u(t)) \cap X \) we have by definition

\[
(u(t + \delta) - u(t), z) \leq G((u(t + \delta)) - G(u(t)).
\]

Dividing both sides by \( \delta > 0 \) (resp. \( \delta < 0 \)) and letting \( \delta \to 0 \), we obtain

\[
\langle u'(t), z \rangle \leq (\text{resp. } \geq) \frac{d}{dt} G(u(t)).
\]

Thus (4.1) holds.

Lemma 4.2. Let \( J = [t_0, t_1] \) be any compact interval in \( \mathbb{R} \), and let \( u_1, u_2 \) be two solutions of \( P(\beta; f, h) \) on \( J \) such that

\[
u_1(t) - u_2(t) \in H_0 \quad \text{for any } t \in J
\]

and

\[
|u_1(t) - u_2(t)|_* = \text{const.} \quad \text{for any } t \in J.
\]

Then we have

(i) \( \partial G(u_1(t)) \cap X = \partial G(u_2(t)) \cap X \) for a.e. \( t \in J \);

(ii) \( \nabla \tilde{\beta}_1 = \nabla \tilde{\beta}_2 \) a.e. on \( J \times \Omega \) for any \( \tilde{\beta}_i \in \beta_J[u_i], i = 1, 2 \);

(iii) \( u_1' = u_2' \) a.e. on \( J \) and there is \( \alpha_0 \in H_0 \) such that

\[
u_1(t) - u_2(t) = \alpha_0 \quad \text{for any } t \in J;
\]

(iv) \( \beta_J[u_1] = \beta_J[u_2] \);

(v) in particular, if \( \beta \) is strictly monotone on \( D(\beta) \), then \( u_1 = u_2 \).

**Proof:** Let \( \tilde{\beta}_i \) be any function in \( \beta_J[u_i], i = 1, 2 \). By Proposition 3.1 with (4.2),

\[
\int_J (\tilde{\beta}_1 - \tilde{\beta}_2, u_1 - u_2) d\tau = 0.
\]

Since \( \beta \) is monotone, we see that

\[
(\tilde{\beta}_1 - \tilde{\beta}_2)(u_1 - u_2) = 0 \quad \text{a.e. on } J \times \Omega.
\]

(4.3)

Now, for any \( z \in H \) and almost every \( t \in J \) we have, by (4.3),

\[
(\tilde{\beta}_1(t), z - u_2(t)) = (\tilde{\beta}_1(t), z - u_1(t)) + (\tilde{\beta}_1(t), u_1(t) - u_2(t))
\]

\[
= (\tilde{\beta}_1(t), z - u_1(t)) + (\tilde{\beta}_2(t), u_1(t) - u_2(t))
\]

\[
\leq G(z) - G(u_1(t)) + G(u_1(t)) - G(u_2(t))
\]

\[
= G(z) - G(u_2(t)).
\]
This implies that $\tilde{\beta}_1(t) \in \partial G(u_2(t))$ for almost every $t \in J$. Similarly, $\tilde{\beta}_2(t) \in \partial G(u_1(t))$ for almost every $t \in J$. Thus we have (i). Next, applying (4.1) in Lemma 4.1, and using (2.5)' for $u_1$, we see that for almost every $t \in J$ and $i = 1, 2$,

$$\frac{d}{dt} G(u_1(t)) = \langle u'_1(t), \tilde{\beta}_i(t) \rangle = -a(\tilde{\beta}_1(t), \tilde{\beta}_i(t)) + \langle f^*(t), \tilde{\beta}_i(t) \rangle.$$

Hence,

$$a(\tilde{\beta}_1(t), \tilde{\beta}_1(t) - \tilde{\beta}_2(t)) = \langle f^*(t), \tilde{\beta}_1(t) - \tilde{\beta}_2(t) \rangle \quad \text{for a.e. } t \in J.$$

Similarly,

$$a(\tilde{\beta}_2(t), \tilde{\beta}_2(t) - \tilde{\beta}_1(t)) = \langle f^*(t), \tilde{\beta}_2(t) - \tilde{\beta}_1(t) \rangle \quad \text{for a.e. } t \in J.$$

Adding these two equalities, we obtain

$$a(\tilde{\beta}_1(t) - \tilde{\beta}_2(t), \tilde{\beta}_1(t) - \tilde{\beta}_2(t)) = 0 \quad \text{for a.e. } t \in J,$$

which implies (ii) as well as (iii). The assertion (iv) results from (ii). Finally, if $\beta$ is strictly monotone on $D(\tilde{\beta})$, then it follows from (4.3) that $u_1 = u_2$ almost everywhere on $J \times \Omega$. Thus we get (v).

**Lemma 4.3.** Let $J'$ be any interval of the form $[t_0, t_1)$, $-\infty < t_0 < t_1 \leq +\infty$. Let $u_n$ and $u$ be the solutions of $CP(\beta; f, h; u_{0,n})$ and $CP(\beta; f, h; u_0)$ on $J'$, respectively. If $r_* \leq u_{0,n} \leq r^*$, $r_* \leq u_0 \leq r^*$ almost everywhere on $\Omega$, $u_{0,n} - u_0 \in H_0$ and $u_{0,n} - u_0 \rightharpoonup 0$ in $X_0^*$ (or equivalently, $u_{0,n} - u_0 \rightharpoonup 0$ weakly in $H_0$), then

$$u_n(t) \rightharpoonup u(t) \text{ weakly in } H \text{ and uniformly in } t \in J',$$

and hence $u_n \rightharpoonup u$ in $C(J'; X^*)$.

**Proof:** Since $u_{0,n} - u_0 \in H_0$, we see that $u_n(t) - u(t) \in H_0$ for any $t \in J'$. By Proposition 3.1, $u_n(t) - u(t) \rightharpoonup 0$ in $X_0^*$ uniformly in $t \in J'$. Combining this with the boundedness of $\{u_n - u\}$ in $L^\infty(J'; H_0)$, we obtain that $u_n(t) - u(t) \rightharpoonup 0$ weakly in $H_0$ and uniformly in $t \in J'$. This shows (4.4).

5. Proofs of main theorems.

**Proof of Theorem 2.1:** Consider the set

$$K(c_0) = \{ z \in H : r_* \leq z \leq r^* \text{ a.e. on } \Omega, \int_\Omega z \, dx = c_0 \}.$$

Then, clearly, $K(c_0)$ is a non-empty, bounded, closed and convex subset of $H$ and it is compact in $X^*$. By virtue of Theorem 3.1, for each $z \in K(c_0)$, there is a unique solution $u(z; t)$ of $CP(\beta; f, h; z)$ on $[0, T]$. Now, define a mapping $S : K(c_0) \rightarrow X^*$ by putting

$$Sz = u(z; T) \quad \text{for } z \in K(c_0).$$

Then we note from (2.8) that $S$ maps $K(c_0)$ into itself. Moreover, by Lemma 4.3, $S$ is continuous in $K(c_0)$ with respect to the topology of $X^*$. Therefore, it is possible
to apply the fixed point theorem to $S$ in $K(c_0)$, and we can find $z_0$ in $K(c_0)$ such that $Sz_0 = z_0$. The solution $u(t) = u(z_0; t)$ satisfies that $u(0) = u(T)$, so that the $T$-periodic extension of $u$ onto the whole line $\mathbb{R}$ is a desired $T$-periodic solution of $P$ on $\mathbb{R}$.

**Proof of Theorem 2.2:** This theorem is a direct consequence of Lemma 4.2. In fact, since $|\omega_1(t) - \omega_2(t)|_*$ is non-increasing in $t \in \mathbb{R}$ (cf. Proposition 3.1), it follows from the $T$-periodicity of $|\omega_1(t) - \omega_2(t)|_*$ that

$$|\omega_1(t) - \omega_2(t)|_* = \text{const.} \quad \text{for any } t \in \mathbb{R}.$$

Therefore, the assertions of the theorem are due to Lemma 4.2.

**Remark 5.1.** In particular, if $\beta$ is strictly convex on $D(\beta)$, then the $T$-periodic solution to $P$ is unique. This is also due to (v) of Lemma 4.2.

**Remark 5.2.** In (2.12) of Theorem 2.2, suppose that $\alpha_0 \neq 0$ in $H_0$; i.e., $\omega_1 \neq \omega_2$. Then, $\beta_R[\omega_i]$ is a singleton for each $i = 1, 2$. In fact, let $\tilde{\beta}_1$ and $\tilde{\beta}_2$ be any two functions in $\beta_R[\omega_1]$ ($= \beta_R[\omega_2]$). Then, by (ii) and (iii) of Lemma 4.2 there are functions $\alpha_0 \in H_0$ and $\alpha_1 \in L^2(0,T)$ such that

$$\omega_1(t,x) - \omega_2(t,x) = \alpha_0(x), \quad \tilde{\beta}_1(t,x) - \tilde{\beta}_2(t,x) = \alpha_1(t) \quad \text{a.e. on } (0,T) \times \Omega.$$

Therefore, noting (4.3), we have

$$\alpha_0(x)\alpha_1(t) = 0 \quad \text{for a.e. } (t,x) \in (0,T) \times \Omega.$$

Since $\alpha_0 \neq 0$ on a subset of $\Omega$ with positive measure, $\alpha_1(t) = 0$ for almost every $t \in (0,T)$, so that $\tilde{\beta}_1 = \tilde{\beta}_2$ almost everywhere on $(0,T) \times \Omega$. This shows that $\beta_R[\omega_1]$ ($= \beta_R[\omega_2]$) is a singleton.

**Proof of Theorem 2.3:** Let $u$ be as in the statement of Theorem 2.3, and let $\omega_1$ be a $T$-periodic solution of $P$ on $\mathbb{R}$ such that $\int_\Omega \omega_1(0,x) \, dx = c_0$; the existence of such a $T$-periodic solution $\omega_1$ is due to Theorem 2.1. Now put

$$d_\infty = \lim_{t \to +\infty} |u(t) - \omega_1(t)|_*, \quad u_n(t) = u(t + nT) \quad \text{for } t \geq 0.$$

Note that $u_n$ is a solution of $P$ on $[0, +\infty)$ for large $n$. Since $r_* \leq u_n(0, \cdot) \leq r^*$ almost everywhere on $\Omega$, $\{u_n(0)\}$ is relatively weakly compact in $H$, and we can find a sequence $\{n_k\}$ in $N$ so that $u_{n_k}(0) \to u_0$ weakly in $H$ (hence $u_{n_k}(0) \to u_0$ in $X^*$) (as $k \to +\infty$). Also, we note from (2.16) that $u_{n_k}(0) - u_0 \in H_0 \subseteq X_0^*$, so that $u_{n_k}(0) - u_0 \to 0$ in $X_0^*$. Therefore, using Lemma 4.3, we see that

$$u_{n_k}(t) \to \omega(t) \quad \text{in } X^* \text{ and uniformly in } t \in [0, +\infty),$$

where $\omega$ is a solution of $CP(\beta; f, h; u_0)$ on $[0, +\infty)$. We observe now that

$$d_\infty = \lim_{k \to +\infty} \{|u(t + n_kT) - \omega_1(t + n_kT)|_* = |\omega(t) - \omega_1(t)|_* \quad \text{for any } t \geq 0,$$

so that Lemma 4.2 implies that

$$\omega(t) - \omega_1(t) = \alpha_0 \quad \text{for any } t \geq 0 \text{ and some } \alpha_0 \in H_0.$$
This shows that \( \omega \) is \( T \)-periodic on \([0, +\infty)\). We denote the \( T \)-periodic extension of \( \omega \) on the whole line \( \mathbb{R} \) by the same symbol \( \omega \), and note that
\[
\lim_{s \to +\infty} |u(s) - \omega(s)|_* = \lim_{k \to +\infty} |u(t + n_k T) - \omega(t)|_* = 0.
\]
Thus, \( u(t) - \omega(t) \to 0 \) in \( X_0^\ast \) as \( t \to +\infty \). Taking into account the fact that \( u(t) - \omega(t) \in H_0 \) for any \( t \in \mathbb{R} \) and \( \{u(t) - \omega(t) : t \in \mathbb{R}\} \) is bounded in \( H_0 \), we obtain (2.18).

**Proof of Theorem 2.4:** Let \( \omega \) be a \( T \)-periodic solution of \( P \) on \( \mathbb{R} \) with
\[
\int_\Omega \omega(0, x) \, dx = c_0.
\]
Since \( r_\ast \leq u \leq r^\ast \) and \( r_\ast \leq \omega \leq r^\ast \) almost everywhere on \( \mathbb{R} \times \Omega \), it follows from Proposition 3.1 that
\[
d_{-\infty} = \lim_{t \to -\infty} |u(t) - \omega(t)|_*
\]
exists. Now, put
\[
u_n(t) = u(t - nT) \quad \text{for } t \in \mathbb{R} \text{ and } n \in \mathbb{N}.
\]
Clearly, \( u_n \) is again a solution of \( P \) on \( \mathbb{R} \) with \( \int_\Omega u_n(0, x) \, dx = c_0 \) and \( r_\ast \leq u_n \leq r^\ast \) almost everywhere on \( \mathbb{R} \times \Omega \). Also, we note from Theorem 3.1 that
\[
u_n \in W^{1,2}(J; X^\ast) \quad \text{for every compact } J \subset \mathbb{R}.
\]
Hence, with the help of Lemma 4.3, it is possible to find a sequence \( \{n_k\} \) in \( \mathbb{N} \) so that
\[
u_{n_k} \to \omega_1 \quad \text{in } C(J; X^\ast) \text{ for every compact } J \text{ with } 0 \in J,
\]
as \( k \to +\infty \), where \( \omega_1 \) is a solution of \( P \) on \( \mathbb{R} \) with \( \int_\Omega \omega_1(0, x) \, dx = c_0 \). In this case we see that
\[
d_{-\infty} = \lim_{k \to +\infty} |u(t - n_k T) - \omega(t - n_k T)|_* = |\omega_1(t) - \omega(t)|_\ast \quad \text{for any } t \in \mathbb{R}.
\]
Therefore, on account of Lemma 4.2, there is \( \alpha_0 \in H_0 \) such that \( \omega_1(t) = \omega(t) + \alpha_0 \) for any \( t \in \mathbb{R} \), so that \( \omega_1 \) is \( T \)-periodic on \( \mathbb{R} \). Besides,
\[
\lim_{s \to -\infty} |u(s) - \omega(s)|_* = \lim_{k \to +\infty} |u(t - n_k T) - \omega_1(t - n_k T)|_* = |\omega_1(t) - \omega_1(t)|_* = 0.
\]
This implies that \( u \equiv \omega_1 \), because \( 0 = \lim_{s \to -\infty} |u(s) - \omega_1(s)|_* \geq |u(t) - \omega_1(t)|_* \) for any \( t \in \mathbb{R} \).

**6. Order property for periodic solutions.** Under the same situation as in the statement of Theorem 2.1, denote by \( P(c_0) \) the set of all \( T \)-periodic solutions \( \omega \) satisfying (2.10), of \( P(\beta; f, h) \) on \( \mathbb{R} \). Then, Theorem 2.1 says that \( P(c_0) \neq \emptyset \), and Theorem 2.2 gives structural properties of \( P(c_0) \). However, we have further some questions on the relationship between \( P(c_{01}) \) and \( P(c_{02}) \) for different constants \( c_{01} \) and \( c_{02} \). Concerning this point we prove:
Proposition 6.1. Let $T$ be a positive number, and $f$ and $h$ be as in Theorem 2.1; conditions (2.7) and (2.8) are satisfied as well. Now, let $c_{0i}$, $i = 1, 2$, be two numbers such that

$$r_* < \frac{1}{|\Omega|}\left\{c_{0i} + \int_0^t \int_\Omega f\, dx\, d\tau + \int_0^t \int_\Gamma h\, d\sigma\, d\tau \right\} < r^*$$

for any $t \in [0, T]$ and $i = 1, 2$. If $c_{01} < c_{02}$, then there are $\omega_i \in \mathcal{P}(c_{0i})$, $i = 1, 2$, such that $\omega_1 \leq \omega_2$ almost everywhere on $\mathbb{R} \times \Omega$.

Corollary 1. In addition to the assumptions of Proposition 6.1, suppose that $\beta$ is strictly monotone on $D(\beta)$. Then $\mathcal{P}(c_{0i})$ is a singleton; i.e., $\mathcal{P}(c_{0i}) = \{\omega_i\}$, for $i = 1, 2$, and $\omega_1 \leq \omega_2$ almost everywhere on $\mathbb{R} \times \Omega$.

Corollary 2. In addition to the assumptions of Proposition 6.1, suppose that $\beta$ is singlevalued. Then, for any $\omega_i \in \mathcal{P}(c_{0i})$, $i = 1, 2$, we have $\beta(\omega_1) \leq \beta(\omega_2)$ almost everywhere on $\mathbb{R} \times \Omega$.

The proposition is proved by making use of the following lemma on the order-preserving property of solutions.

Lemma 6.1. Assume all the conditions of Proposition 6.1 are satisfied, and let $u_{0i}$, $i = 1, 2$, be functions in $H$ such that

$$r_* \leq u_{01} \leq u_{02} \leq r^* \text{ a.e. on } \Omega, \quad \int_\Omega u_{01}\, dx = c_{0i}, \quad i = 1, 2.$$ 

Then the solutions $u_i$ of $CP(\beta; f, h; u_{0i})$ on $\mathbb{R}^+_+ = [0, +\infty)$, $i = 1, 2$, satisfy that $u_1 \leq u_2$ almost everywhere on $\mathbb{R}^+_+ \times \Omega$.

**Proof:** Let $\{f_\lambda\} \subset C^1(\mathbb{R}^+_+; H)$ and $\{h_\lambda\} \subset C^1(\mathbb{R}^+_+; H^{1/2}(\Gamma))$ such that $f_\lambda \to f$ in $L^2_{loc}(\mathbb{R}^+_+; H)$ and $h_\lambda \to h$ in $L^2_{loc}(\mathbb{R}^+_+; L^2(\Gamma))$ as $\lambda \downarrow 0$. Also let $\beta_\lambda(r) := \beta(r) + \rho(r, r \in \mathbb{R}$, for each $\lambda > 0$. Now, for any $T' > 0$, consider the problem

$$\begin{align*}
\begin{cases}
\; u_{i\lambda,t} - \Delta \tilde{\beta}_{i\lambda} + \lambda \tilde{\beta}_{i\lambda} = f_\lambda, & \tilde{\beta}_{i\lambda} \in \beta_\lambda(u_{i\lambda}), \quad \text{in } (0, T') \times \Omega, \\
\; \partial_n \tilde{\beta}_{i\lambda} = h_\lambda & \text{on } (0, T') \times \Gamma, \\
\; u_{i\lambda}(0, \cdot) = u_{0i} & \text{in } \Omega.
\end{cases}
\end{align*}
$$

(6.1)

Then, by slightly modified arguments in Sections 4 and 6 of [14] we see that (a) for any $\lambda > 0$ problem (6.1) has one and only one solution $u_{i\lambda}$ in a similar sense to Definition 3.1; (b) if $T' > 0$ is small, then $u_{i\lambda}$ tends to the solution $u_i$ of $CP(\beta; f, h; u_{0i})$ on $[0, T']$ in the sense that

$$u_{i\lambda}(t) \rightharpoonup u_i(t) \text{ weakly in } H \text{ and uniformly in } t \in [0, T'] \text{ as } \lambda \downarrow 0.$$ 

Moreover, since $f_\lambda$ and $h_\lambda$ are regular in $t$, it follows (cf. [15; Theorem 1.1]) that $u_{i\lambda} \in W^{1,2}_{loc}((0, T']; H)$. Therefore, since $u_{01} \leq u_{02}$ almost everywhere in $\Omega$, the standard $L^1$-space technique (cf. [1]) yields that $u_{1\lambda} \leq u_{2\lambda}$ almost everywhere in $(0, T') \times \Omega$. This together with (b) implies that $u_1 \leq u_2$ almost everywhere on
$$(0, T') \times \Omega$$ for small $T' > 0$. By repeating the same argument as above we con­sequently obtain $u_1 \leq u_2$ on $\mathbb{R}_+ \times \Omega$.

**Proof of Proposition 6.1:** Since $c_{01} < c_{02}$, there are functions $u_{0i}$, $i = 1, 2$, in $H$ satisfying the properties in the statement of Lemma 6.1. Now, let $u_i$ be the solution of $CP(\beta; f, h; u_{0i})$ on $\mathbb{R}_+$, $i = 1, 2$. Then, by Lemma 6.1, we have

$$u_1 \leq u_2 \quad \text{a.e. on } \mathbb{R}_+ \times \Omega. \quad (6.2)$$

Here, apply Theorem 2.3 to see that there are $\omega_i \in \mathcal{P}(c_{0i})$, $i = 1, 2$, such that $u_i(t) - \omega_i(t) \to 0$ weakly in $H$ as $t \to \infty$. From this, with (6.2), it results that $\omega_1 \leq \omega_2$ almost everywhere on $\mathbb{R} \times \Omega$.

Finally we see that Corollary 1 is an immediate consequence of the above proposition and Remark 5.1. Also, if $\beta$ is singlevalued, then $\beta_R[\omega]$ is a singleton for any function $\omega$, so that Corollary 2 is derived easily from Proposition 6.1 and (2.13) of Theorem 2.2.

**Acknowledgements.** The author wishes to express his gratitude to the referee who kindly gave him some constructive suggestions.

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