TROTTER PRODUCT FORMULAE FOR HAMILTON-JACOBI EQUATIONS IN INFINITE DIMENSIONS

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Abstract. Two Trotter formulae for Hamilton-Jacobi equations associated with optimal control problems governed by variational inequalities are proposed. The main results of this paper state the convergence of these formulae. An important tool in our proofs is the nonlinear version of the Chernoff formula. An application to Hamilton-Jacobi equations arising in control of parabolic obstacle problem is also given.

1. Introduction. We shall present here two Trotter product formulae for the solution of the following Hamilton-Jacobi equation

\[ \varphi_t(t,y) + h^*(-B^\star \varphi_y(t,y)) + (Ay + \partial I_K(y), \varphi_y(t,y)) = g(y) \quad \text{in } [0,T] \times H \]

with the initial condition

\[ \varphi(0,y) = \varphi_0(y) \quad \text{for } y \in H. \]

The convergence of these formulae represents our main interest in this paper.

It is well known that (1) (together with (2)) is the dynamic programming equation for the distributed optimal control problem: minimize

\[ \int_0^T (h(u(t)) + g(y(t))) \, dt + \varphi_0(y(T)) \]

over all \( u \in L^2(0,T;U), y \in C([0,T];H) \) satisfying the variational inequality

\[ y'(t) + Ay(t) + \partial I_K(y(t)) \ni Bu(t) \]

with the initial condition

\[ y(0) = y_0. \]

Here \( H \) and \( U \) are two Hilbert spaces whose scalar products and norms are denoted by the same symbols; \( \langle \cdot, \cdot \rangle \) and \( | \cdot | \), respectively.

The data of the Cauchy problem (1), (2) satisfy the following hypotheses:

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(i) \( h : U \to (-\infty, +\infty] \) is convex, lower semicontinuous, not identically +\( \infty \) and
\[
h(u) \geq c_1 |u|^2 - c_2 \quad \text{for any} \quad u \in U,
\]
where \( c_1 > 0 \) and \( c_2 \in \mathbb{R} \). We have denoted by \( h^* \) the conjugate function of \( h \).

(ii) \( g, \varphi_0 : H \to \mathbb{R} \) are Lipschitz continuous on bounded subsets and bounded from below by affine functions.

(iii) Let \( V \) be a real Hilbert space continuously and densely imbedded in \( H \) and let \( V' \) be its dual space. Identifying \( H \) with its own dual, we have \( V \subset H \subset V' \).
We denote by \((\cdot, \cdot)\) the pairing between \( V \) and \( V' \) and by \( \| \cdot \| \) the norm of \( V \). Assume in addition that the injection of \( V \) into \( H \) is compact. In this context \( A \) is a linear continuous and symmetric operator from \( V \) to \( V' \) satisfying
\[
(Ay, y) \geq \omega \|y\|^2 \quad \text{for all} \quad y \in V,
\]
where \( \omega > 0 \).

(iv) Let \( K \) be a closed convex subset of \( H \) such that \( 0 \in K \) and let \( I_K \) be its indicator function. Denote by \( \partial I_K \) the subdifferential of \( I_K \); i.e.,
\[
\partial I_K(y) = \{p \in H : (p, y - z) \geq 0 \quad \text{for all} \quad z \in K\}.
\]
Assume that the operator \( A + \partial I_K \) is maximal monotone in \( H \times H \) and \( D(A + \partial I_K) = K \).

(v) \( B : U \to H \) is a linear continuous operator with \( B^* \) its adjoint.

To assure the maximal monotonicity of \( A + \partial I_K \) and the equality \( D(A) \cap K = K \), it suffices to require that for every \( \epsilon > 0 \),
\[
(Ay, (\partial I_K)_\epsilon(y)) \geq -c(1 + ||(\partial I_K)_\epsilon||)(1 + |y|)
\]
for all \( y \in D(A) = \{y \in V: Ay \in H\} \),
where \( c > 0 \) and \( (\partial I_K)_\epsilon \) is the Yosida approximation of \( \partial I_K \); i.e., \( (\partial I_K)_\epsilon = \epsilon^{-1}(I - (I + \epsilon \partial I_K)^{-1}) \), \( I \) being the identity operator on \( H \) (see for instance [1, p. 19]).

By a standard existence result, for every \( y_0 \in K \) the variational inequality (4) has a unique weak solution. Moreover, if \( y_0 \in K \cap V \), then (4) has a unique solution \( y \in W^{1,2}([0,T];H) \cap L^2(0,T;D(A)) \cap C([0,T];V) \) (see [1, p. 131]).

As solution of the Hamilton-Jacobi equation (1) (with the initial condition (2)), we may consider the optimal value function associated with control problem (3), (4), (5):
\[
\varphi(t,y) = \inf \left\{ \int_0^t (h((s)) + g(y(s))) \, ds + \varphi_0(y(t)) : y' + Ay + \partial I_K(y) \exists Bu, \ y(0) = y, \ u \in L^2(0,t;U) \right\},
\]
\((t,y) \in [0,T] \times K \). As a matter of fact, under appropriate hypotheses, the dynamic programming principle shows that \( \varphi \) satisfies (1) in a certain "good" sense. For
example, \( \varphi \) given by (10) is the unique solution of (1), (2) in the viscosity sense (see [6]).

Now, to motivate our approximation schemes and to visualize them as Trotter schemes, we shall do some short heuristic considerations. In order to obtain an approximate solution to the Cauchy problem (1), (2) on small intervals \([0, \varepsilon]\), we decompose this problem into two simpler problems, the second being

\[
\varphi(t, y) + h^*(-B^*\varphi_y(t, y)) = 0 \quad \text{in} \quad (0, \varepsilon) \times H, \\
\varphi(0, y) = \varphi_0((I + \varepsilon \partial I_K)^{-1}(I + \varepsilon A)^{-1}y) + \varepsilon g((I + \varepsilon \partial I_K)^{-1}(I + \varepsilon A)^{-1}y) \quad \text{for} \quad y \in H.
\]

Here \( y \mapsto \varphi_0((I + \varepsilon \partial I_K)^{-1}(I + \varepsilon A)^{-1}y) + \varepsilon g((I + \varepsilon \partial I_K)^{-1}(I + \varepsilon A)^{-1}y) \) is an approximate solution of the decomposition (containing the unbounded term and the right-hand side of (1)) with \( \varphi_0 \) as initial datum. Using now the Hopf representation formula, we obtain the exact solution of the second problem:

\[
\varphi(t, y) = \inf \left\{ \varphi_0((I + \varepsilon \partial I_K)^{-1}(I + \varepsilon A)^{-1}(y + \varepsilon Bu)) + \varphi_0((I + \varepsilon \partial I_K)^{-1}(I + \varepsilon A)^{-1}(y + \varepsilon Bu)) : u \in U \right\},
\]

\((t, y) \in (0, \varepsilon) \times H\). Let \( \varepsilon = T/N \) (\( N \) being sufficiently large). We have \((I + \varepsilon \partial I_K)^{-1} = P_K\), where \( P = P_K \) is the projection operator of \( H \) into \( K \). The above formula suggests to us that the following iterative scheme gives an approximate solution to (1), (2):

\[
\varphi^\varepsilon(t, y) = \begin{cases} 
\inf \left\{ \varepsilon h(u) + \varphi_0((I + \varepsilon \partial I_K)^{-1}(I + \varepsilon A)^{-1}(y + \varepsilon Bu)) + \varphi_0((I + \varepsilon \partial I_K)^{-1}(I + \varepsilon A)^{-1}(y + \varepsilon Bu)) : u \in U \right\} \\
\quad \text{for} \quad (t, y) \in (\varepsilon, T] \times H,
\end{cases}
\]

\( \varphi^\varepsilon(0, y) = \varphi_0(y) \quad \text{for} \quad y \in H. \)

Similar considerations lead to the alternative scheme:

\[
\psi^\varepsilon(t, y) = \begin{cases} 
\inf \left\{ \varepsilon h(u) + \varphi_0((I + \varepsilon \partial I_K)^{-1}(y + \varepsilon Bu)) + \psi_0((I + \varepsilon \partial I_K)^{-1}(y + \varepsilon Bu)) : u \in U \right\} \\
\quad \text{for} \quad (t, y) \in (\varepsilon, T] \times H,
\end{cases}
\]

\( \psi^\varepsilon(0, y) = \varphi_0(y) \quad \text{for} \quad y \in H. \)

The purpose of this paper is to prove the convergence of \( \varphi^\varepsilon \) and \( \psi^\varepsilon \) (defined above) to \( \varphi \) (given by (10)) as \( \varepsilon \to 0 \). We shall illustrate the results obtained (Theorem 1 and Theorem 2 in Section 2) by a specific example where the state equation is the parabolic obstacle problem (see Section 3).
Trotter product formulae for infinite-dimensional Hamilton-Jacobi equations have been considered for the first time by V. Barbu in [2] and [3]. The main point in our approach is the use of the Hopf representation formula which leads to Trotter schemes where minimization on $L^2(0,T;U)$ (as in [2, 3]) is replaced by minimization on $U$. Also, in our formulae the resolvent $(I + \varepsilon A)^{-1}$ replaces the exponential $e^{-\varepsilon A}$ (i.e., the semigroup generated by $-A$ at $\varepsilon$). This is convenient from the point of view of the numerical treatment of (11) or (12): it is easier to compute the resolvent than the semigroup.

To end this introduction, we mention that, in general, for infinite-dimensional Hamilton-Jacobi equations, even the standard Galerkin approximation scheme fails to converge (see Example II.1 in [5]).

2. Convergence of the Trotter formulae. This section contains the whole substance of the paper.

Theorem 1. In addition to the hypotheses (i)-(v) we suppose that $P$ maps $V$ into itself and

$$(APy, Py) \leq (Ay, y) \text{ for all } y \in V.$$ \hspace{1cm} (13)

Then

$$\lim_{\varepsilon \to 0} \varphi^\varepsilon(t, y) = \varphi(t, y) \text{ for any } (t, y) \in [0, T] \times K.$$ \hspace{1cm} (14)

Proof: Let $\varepsilon = T/N$. Fix $(t, y) \in [0, T] \times K$. Let us suppose that $t/\varepsilon$ is not an integer. If we successively apply (11) on the intervals $(t - \varepsilon, t], (t - 2\varepsilon, t - \varepsilon], \ldots, (t - \lfloor t/\varepsilon \rfloor \varepsilon, t - (\lfloor t/\varepsilon \rfloor - 1)\varepsilon], (0, t - \lfloor t/\varepsilon \rfloor \varepsilon)$, one obtains

$$\varphi^\varepsilon(t, y) = \inf_{u_1 \in U} \inf_{u_2 \in U} \ldots \inf_{u_{\lfloor t/\varepsilon \rfloor + 1}} \inf_{u \in U} \{\varepsilon h(u_1) + \varepsilon h(u_2) + \ldots + \varepsilon h(u_{\lfloor t/\varepsilon \rfloor})$$

$$+ (t - \lfloor t/\varepsilon \rfloor \varepsilon) h(u_{\lfloor t/\varepsilon \rfloor + 1}) + \varepsilon g((P + \varepsilon A)^{-1}(y + \varepsilon Bu_1))$$

$$+ \varepsilon g((P + \varepsilon A)^{-1}(P + \varepsilon A)^{-1}(y + \varepsilon Bu_1) + \varepsilon Bu_2)) + \ldots$$

$$+ \varepsilon g((P + \varepsilon A)^{-1}(P + \varepsilon A)^{-1}(y + \varepsilon Bu_1) + \varepsilon Bu_2)$$

$$+ \ldots + \varepsilon Bu_{\lfloor t/\varepsilon \rfloor}) + \varepsilon g((P + \varepsilon A)^{-1}(P + \varepsilon A)^{-1}(y + \varepsilon Bu_1) + \varepsilon Bu_2)$$

$$+ \ldots + \varepsilon Bu_{\lfloor t/\varepsilon \rfloor + 1}) + (t - \lfloor t/\varepsilon \rfloor \varepsilon) Bu_{\lfloor t/\varepsilon \rfloor + 1})$$

$$+ \varphi_0((P + \varepsilon A)^{-1}(P + \varepsilon A)^{-1}$$

$$+ \ldots (P + \varepsilon A)^{-1}(P + \varepsilon A)^{-1}(y + \varepsilon Bu_1) + \varepsilon Bu_2) + \ldots + \varepsilon Bu_{\lfloor t/\varepsilon \rfloor})$$

$$+ (t - \lfloor t/\varepsilon \rfloor \varepsilon) Bu_{\lfloor t/\varepsilon \rfloor + 1})) \}.$$}

Clearly, if $t/\varepsilon$ is an integer, we do not take into account the last interval $(0, t - [t/\varepsilon] \varepsilon)$ and the modifications we have to do in (14) are obvious. So, we explicitly treat only
the case when \( t/\epsilon \) is not an integer (the other case being completely similar). By (14) we are able to write

\[
\varphi^\epsilon(t, y) = \inf \left\{ \sum_{i=1}^{\lfloor t/\epsilon \rfloor} \epsilon h(u(i\epsilon)) + (t - \lfloor t/\epsilon \rfloor) \epsilon h(u((\lfloor t/\epsilon \rfloor + 1)\epsilon)) + \sum_{i=1}^{\lfloor t/\epsilon \rfloor} \epsilon g(y^\epsilon(i\epsilon)) + \epsilon g(y^\epsilon(t)) + \varphi_0(y^\epsilon(t)) : u \text{ is a step function from } [0, T] \text{ to } U \right\},
\]

where, for a step function \( u : [0, T] \to U \), \( y^\epsilon \) is defined by

\[
y^\epsilon(t) = y^\epsilon_c(t) = P(I + \epsilon A)^{-1}(y^\epsilon((i-1)\epsilon) + (t - (i-1)\epsilon)Bu(i\epsilon))
\]

for \( t \in ((i-1)\epsilon, i\epsilon], \ i = 1, 2, \ldots, N \) \( y^\epsilon(0) = y \).

**Lemma 1.** We have

\[
\varphi(t, y) = \inf \left\{ \int_0^t (h(u(s)) + g(y(s))) ds + \varphi_0(y(t)) : y' + Ay + \partial I_K(y) \ni Bu, \ y(0) = y, \ u \text{ is a step function from } [0, T] \text{ to } U \text{ which takes constant values on } (0, \epsilon], \right\}
\]

\( (\epsilon, 2\epsilon, \ldots, (T-\epsilon, T], \ \epsilon > 0 \) \( \text{ for } (t, y) \in [0, T] \times K. \)

**Proof:** Clearly, it suffices to show that, for any given \( u \in L^2(0, t; U) \) with \( h(u) \in L^1(0, t) \), there exists a sequence of step functions (as in the statement of Lemma 1) denoted by \( \{u_n\} \) such that

\[
u_n \to u \text{ in } L^2(0, t; U) \text{ and } \lim_{n \to \infty} \int_0^t h(u_n(s)) ds = \int_0^t h(u(s)) ds.
\]

To this aim, we start with a sequence of step functions \( \{v_n\} \) which only satisfies \( v_n \to u \) in \( L^2(0, t; U) \). Let \( \lambda_n = (\int_0^t |v_n - u| ds)^{1/2} > 0 \). Define

\[
u_n(s) = (I + \lambda_n \partial h)^{-1} v_n(s) \text{ for every } s \in [0, t].
\]

Obviously, \( u_n \) is a step function and \( u_n \to u \) almost everywhere on \( [0, t] \); therefore, by Lebesgue dominated convergence theorem, \( u_n \to u \) in \( L^2(0, t; U) \). Noting that \( v_n(s) - u_n(s) \in \lambda_n \partial h(u_n(s)) \), we have

\[
h(u_n(s)) \leq h(u(s)) + \frac{1}{\lambda_n} (v_n(s) - u_n(s), u_n(s) - u(s))
\]

\[
\leq h(u(s)) + \frac{1}{\lambda_n} (v_n(s) - u_n(s), v_n(s) - u(s)).
\]
Integrating the above inequality from 0 to t, we obtain
\[
\int_0^t h(u_n(s)) \, ds \leq \int_0^t h(u(s)) \, ds + \left( \int_0^t |v_n(s) - u_n(s)| \, ds \right)^{1/2}
\]
and hence
\[
\limsup_{n \to \infty} \int_0^t h(u_n(s)) \, ds \leq \int_0^t h(u(s)) \, ds.
\]
On the other hand, by the lower semicontinuity in \(L^2(0, t; U)\) of the functional \(u \mapsto \int_0^t h(u(s)) \, ds\), we have
\[
\liminf_{n \to \infty} \int_0^t h(u_n(s)) \, ds \geq \int_0^t h(u(s)) \, ds
\]
which completes the proof of the lemma.

Now, let \(u\) be a step function from \([0, T]\) to \(U\) which takes constant values on 
\((0, \epsilon_0], (\epsilon_0, 2\epsilon_0], \ldots, ((N_0 - 1)\epsilon_0, N_0\epsilon_0] = (T - \epsilon_0, T]\). Consider the unique weak solution \(y = y_u\) of the Cauchy problem
\[
y' + Ay + \partial I_K(y) \ni Bu, \quad y(0) = y. \tag{17}
\]

**Lemma 2.** We have
\[
\lim_{\epsilon \to 0} y_{u_\epsilon}(t) = y_u(t) \quad \text{strongly in } H \quad \text{for all } t \in [0, T].
\]

**Proof:** We shall apply the nonlinear version of the Chernoff formula (see [4], Theorem 4.3) to families of nonlinear contractions \(T(\epsilon)\) of the form
\[
T(\epsilon)z = P(I + \epsilon A)^{-1}(z + \epsilon Bu) \quad \text{for all } z \in H,
\]
where \(u \in U\). So, we have to verify the condition
\[
\lim_{\epsilon \to 0} (I + \frac{\lambda}{\epsilon}(I - T(\epsilon)))^{-1} z = (I + \lambda(A + \partial I_K - Bu))^{-1} z \tag{18}
\]
for all \(z \in K\) and any \(\lambda > 0\).

Let us observe that
\[
T(\epsilon)z = (I + \epsilon \partial I_K)^{-1}(I + \epsilon A)^{-1}(I + \epsilon C)^{-1} z,
\]
where
\[
Cz = -Bu \quad \text{for all } z \in H.
\]
As \(C\) and \(A + \partial I_K + C\) are maximal monotone operators (because \(C\) is constant), we can easily adapt for three operators the proof of a well known result (for two operators) of Brézis and Pazy (see for instance Proposition 4.3 in [4]) to deduce that (18) is valid. We emphasize that an important point in this adaptation is the use of constancy of \(C\).
Fix \( t \in [0,T] \), arbitrary, and let \( k_0 = [t/\epsilon_0] \). Assume that \( t/\epsilon_0 \) is not an integer (the other case being simpler). Consider \( 0 = t_0^\epsilon < t_1^\epsilon < \cdots < t_k^\epsilon < t_{k+1}^\epsilon < t \) defined by

\[
t_k^\epsilon = t_{k-1}^\epsilon + n_k^\epsilon \epsilon \quad \text{such that } t_k^\epsilon \leq k \epsilon_0 < t_k^\epsilon + \epsilon, \quad k = 1, 2, \ldots, k_0,
\]

\[
t_{k+1}^\epsilon = \lceil t/\epsilon \rceil \epsilon = t_{k_0}^\epsilon + n_{k_0+1}^\epsilon \epsilon,
\]

where \( \epsilon = T/N \) is sufficiently small and \( n_1^\epsilon, n_2^\epsilon, \ldots, n_{k_0}^\epsilon, n_{k_0+1}^\epsilon \in \mathbb{N} \).

Define the operators \( T_1(\epsilon), T_2(\epsilon), \ldots, T_{N_0}(\epsilon) \) and \( T(\epsilon; t) \) by

\[
T_k(\epsilon)z = P(I + \epsilon A)^{-1}(z + \epsilon Bu(k_0 \epsilon_0)) \quad \text{for all } z \in H, \quad k = 1, 2, \ldots, N_0,
\]

\[
T(\epsilon; t)z = P(I + \epsilon A)^{-1}(z + (t - \lfloor t/\epsilon \rfloor \epsilon) Bu(\lfloor t/\epsilon \rfloor + 1 \epsilon)) \quad \text{for all } z \in H.
\]

Denote also by \( S_k \) the semigroup of nonlinear contractions generated by

\[
A + \partial I K - Bu(k_0 \epsilon_0), \quad k = 1, 2, \ldots, N_0 \quad \text{and} \quad S_0 = I.
\]

We have

\[
y_u'(t) - y_u(t) = T(\epsilon; t)T_{k_0+1} \left( \frac{t_{k_0+1}^\epsilon - t_{k_0}^\epsilon}{n_{k_0+1}^\epsilon} \right) n_{k_0+1}^\epsilon \cdots T_{k+1} \left( \frac{t_k^\epsilon - t_{k-1}^\epsilon}{n_k^\epsilon} \right) n_k^\epsilon \cdots T_1 \left( \frac{t_1^\epsilon}{n_1^\epsilon} \right) y
\]

\[
- S_{k_0+1} \left( t - \frac{t}{\epsilon_0} \epsilon_0 \right) S_{k_0} \epsilon_0 \cdots S_k \epsilon_0 \cdots S_1 \epsilon_0 \epsilon_0 y
\]

\[
= \sum_{k=1}^{k_0} \left( T(\epsilon; t)T_{k_0+1} \left( \frac{t_{k_0+1}^\epsilon - t_{k_0}^\epsilon}{n_{k_0+1}^\epsilon} \right) n_{k_0+1}^\epsilon \cdots T_{k+1} \left( \frac{t_k^\epsilon - t_{k-1}^\epsilon}{n_k^\epsilon} \right) n_k^\epsilon \cdots T_1 \left( \frac{t_1^\epsilon}{n_1^\epsilon} \right) y
\]

\[
- T(\epsilon; t)T_{k_0+1} \left( \frac{t_{k_0+1}^\epsilon - t_{k_0}^\epsilon}{n_{k_0+1}^\epsilon} \right) n_{k_0+1}^\epsilon \cdots T_{k+1} \left( \frac{t_k^\epsilon - t_{k-1}^\epsilon}{n_k^\epsilon} \right) n_k^\epsilon \cdots T_1 \left( \frac{t_1^\epsilon}{n_1^\epsilon} \right) y
\]

\[
- S_k \left( t_k^\epsilon - t_{k-1}^\epsilon \right) S_{k-1} \epsilon_0 \cdots S_0 \epsilon_0 y
\]

\[
+ T(\epsilon; t)T_{k_0+1} \left( \frac{t_{k_0+1}^\epsilon - t_{k_0}^\epsilon}{n_{k_0+1}^\epsilon} \right) n_{k_0+1}^\epsilon \cdots T_{k+1} \left( \frac{t_k^\epsilon - t_{k-1}^\epsilon}{n_k^\epsilon} \right) n_k^\epsilon \cdots T_1 \left( \frac{t_1^\epsilon}{n_1^\epsilon} \right) y
\]

\[
- T(\epsilon; t)T_{k_0+1} \left( \frac{t_{k_0+1}^\epsilon - t_{k_0}^\epsilon}{n_{k_0+1}^\epsilon} \right) n_{k_0+1}^\epsilon \cdots T_{k+1} \left( \frac{t_k^\epsilon - t_{k-1}^\epsilon}{n_k^\epsilon} \right) n_k^\epsilon \cdots T_1 \left( \frac{t_1^\epsilon}{n_1^\epsilon} \right) y
\]

\[
- S_k \epsilon_0 \cdots S_0 \epsilon_0 y
\]

\[
+ T(\epsilon; t)T_{k_0+1} \left( \frac{t_{k_0+1}^\epsilon - t_{k_0}^\epsilon}{n_{k_0+1}^\epsilon} \right) n_{k_0+1}^\epsilon S_k \epsilon_0 \cdots S_1 \epsilon_0 y
\]

\[
- T(\epsilon; t)S_{k_0+1}(t_{k_0+1}^\epsilon - t_{k_0}^\epsilon) S_k \epsilon_0 \cdots S_1 \epsilon_0 y
\]

\[
+ T(\epsilon; t)S_{k_0+1}(t_{k_0+1}^\epsilon - t_{k_0}^\epsilon) S_k \epsilon_0 \cdots S_1 \epsilon_0 y
\]

\[
- T(\epsilon; t)S_{k_0+1}(t - k_0 \epsilon_0) S_k \epsilon_0 \cdots S_1 \epsilon_0 y
\]

\[
+ T(\epsilon; t)S_{k_0+1}(t - k_0 \epsilon_0) S_k \epsilon_0 \cdots S_1 \epsilon_0 y
\]

\[
- S_{k_0+1}(t - \frac{t}{\epsilon_0} \epsilon_0) S_k \epsilon_0 \cdots S_1 \epsilon_0 y.
\]
Obviously, \( \lim_{\epsilon \to 0} t_k^\epsilon = k \epsilon_0 \) for \( k = 1, 2, \ldots, k_0 \), \( \lim_{\epsilon \to 0} t_{k_0+1}^\epsilon = t \) and \( \lim_{\epsilon \to 0} n_k^\epsilon = \infty \) for \( k = 1, 2, \ldots, k_0, k_0 + 1 \). Since the operators \( T_2, T_3, \ldots, T_{k_0+1}, T(\epsilon; t) \) are contractions, applying the nonlinear Chernoff formula for \( T_1, T_2, \ldots, T_{k_0+1} \) in the points \( y, S_1(\epsilon_0)y, \ldots, S_{k_0}(\epsilon_0) \cdots S_1(\epsilon_0)y \), respectively, and using also the continuity of the semigroups \( S_1, S_2, \ldots, S_{k_0+1} \), we obtain the statement of Lemma 2 and the proof is complete.

Let us continue the proof of Theorem 1. For a step function \( u \) from \([0, T]\) to \( U \), define

\[
J_\epsilon(u) = \sum_{i=1}^{[t/\epsilon]} e(t - \lfloor \frac{t}{\epsilon} \rfloor \epsilon) h(u([\frac{t}{\epsilon}] + 1)\epsilon)) \\
+ \sum_{i=1}^{[t/\epsilon]} e(t - \lfloor \frac{t}{\epsilon} \rfloor \epsilon) h(u([\frac{t}{\epsilon}] + 1)\epsilon)) \\
J(u) = \int_0^t (h(u(s)) + g(u(s))) ds + \varphi_0(y_u(t)),
\]

where \( y_u \) and \( y_u \) are solutions of (16) and (17), respectively. Define also

\[
J_\epsilon = \inf \{ J_\epsilon(u) : u \text{ is a step function from } [0, T] \text{ to } U \},
\]

\[
J = \inf \{ J(u) : u \text{ is a step function from } [0, T] \text{ to } U \text{ which takes constant values on } (0, \epsilon], (\epsilon, 2\epsilon], \ldots, (T - \epsilon, T], \epsilon > 0 \}.
\]

Clearly, \( \varphi^\epsilon(t, y) = J_\epsilon \) and by Lemma 1, \( \varphi(t, y) = J \).

Let \( \eta > 0 \), arbitrary. By the definition of \( J \), there exists a step function \( u = u_\eta \) which takes constant values on \((0, \eta], (\eta, 2\eta], \ldots, (T - \eta, T] \) such that

\[
J \leq J(u_\eta) < J + \eta.
\]

But one can write

\[
J_\epsilon(u) = \int_0^{[t/\epsilon]} (h(\tilde{u}_\epsilon(s)) + g(\tilde{y}_u^\epsilon(s))) ds + \varphi_0(y_u^\epsilon(t)) + (t - \lfloor \frac{t}{\epsilon} \rfloor \epsilon) h(u([\frac{t}{\epsilon}] + 1)\epsilon)) + e(t - \lfloor \frac{t}{\epsilon} \rfloor \epsilon) h(u([\frac{t}{\epsilon}] + 1)\epsilon)) + e(t - \lfloor \frac{t}{\epsilon} \rfloor \epsilon) h(u([\frac{t}{\epsilon}] + 1)\epsilon)),
\]

where

\[
\tilde{y}^\epsilon(s) = y^\epsilon(i\epsilon), \quad \tilde{u}_\epsilon(s) = u(i\epsilon) \quad \text{for } s \in ((i - 1)\epsilon, i\epsilon], \quad i = 1, 2, \ldots, \lfloor \frac{t}{\epsilon} \rfloor.
\]

Using the following obvious estimate

\[
|\tilde{y}_u^\epsilon(s) - y_u^\epsilon(s)| \leq \epsilon |Bu(\lfloor \frac{s}{\epsilon} \rfloor + 1)\epsilon)|
\]

together with Lemma 2, we easily infer that \( \lim_{\epsilon \to 0} \tilde{y}_u^\epsilon(s) = y_u(s) \) for every \( s \in [0, t] \).

Also, \( \lim_{\epsilon \to 0} \tilde{u}_\epsilon(s) = u(s) \) except the possible discontinuity points \( k\epsilon_\eta \) of \( u \). Besides one readily ascertains that \( \tilde{y}_u^\epsilon(s) (s \in [0, t], \epsilon > 0) \) is bounded in \( H \) (for we have
\[|P(I + \epsilon A)^{-1} z| \leq |z| \text{ for every } z \in H, \text{ because } 0 \in K.\]

From these, by Lebesgue dominated convergence theorem, we obtain

\[\lim_{\epsilon \to 0} J_\epsilon(u_\eta) = J(u_\eta).\]

Thus we can find \(\delta_\eta > 0\) such that

\[J_\epsilon(u_\eta) < J + 2\eta \quad \text{for all } \epsilon < \delta_\eta\]

and hence

\[J_\epsilon < J + 2\eta \quad \text{for } \epsilon < \delta_\eta.\]

We have obtained \((\eta > 0 \text{ being arbitrary})\)

\[\limsup_{\epsilon \to 0} J_\epsilon \leq J.\]

**Remark 1.** This first part of the proof also works if we replace in (10), (11) the operators \(A\) and \(\partial I_K\) by two maximal monotone operators \(A_1 : D(A_1) \subset H \to 2^H\) and \(A_2 : D(A_2) \subset H \to 2^H\) such that \(A_1 + A_2\) is maximal monotone in \(H \times H\).

Consequently, in this wider context, it remains valid that

\[\limsup_{\epsilon \to 0} \varphi^\epsilon(t, y) \leq \varphi(t, y) \quad \text{for any } (t, y) \in [0, T] \times \overline{D(A_1) \cap D(A_2)}.\]

Now we shall prove that

\[\liminf_{\epsilon \to 0} J_\epsilon \geq J.\]

To this end, observe that we can choose a subsequence of \(\{\epsilon\}\), again denoted \(\{\epsilon\}\), and a corresponding sequence of step functions \(u^\epsilon\), each of them taking constant values on \((0, \epsilon], (\epsilon, 2\epsilon], \ldots, (T - \epsilon, T]\), so that

\[\lim_{\epsilon \to 0} J_\epsilon(u^\epsilon) = \liminf_{\epsilon \to 0} J_\epsilon \leq J.\]

Hence, by (6) and assumption (ii), using also the estimate

\[|y_{u^\epsilon}^s(s)| \leq |y| + \sum_{i=1}^{[s/\epsilon]} \epsilon |Bu^\epsilon(i\epsilon)| + (s - \lfloor s/\epsilon \rfloor \epsilon)|Bu^\epsilon((\lfloor s/\epsilon \rfloor + 1)\epsilon)|\]

\[= |y| + \int_0^s |Bu^\epsilon(\tau)| d\tau,\]

we deduce that \(\{u^\epsilon\}\) is bounded in \(L^2(0, t; U)\); consequently, on a subsequence, \(u^\epsilon \to u^0\) weakly in \(L^2(0, t; U)\) as \(\epsilon \to 0\).

**Lemma 3.** Let \(\{u^\epsilon\}\) be a sequence of step functions like above and \(y \in K \cap D(A)\). If \(u^\epsilon \to u^0\) weakly in \(L^2(0, t; U)\), then

\[y_{u^\epsilon}^s(s) \to y_{u^0}(s) \quad \text{strongly in } H \quad \text{for all } s \in [0, t].\]
Proof: First we shall prove that \( y^\varepsilon(s) \) converges strongly in \( H \) (possibly on a certain subsequence) for every \( s \in [0, t] \). To this purpose consider

\[
z^\varepsilon_u(s) = (I + \varepsilon A)^{-1}(y^\varepsilon_u((i-1)\varepsilon) + (s - (i-1)\varepsilon)Bu^\varepsilon(i\varepsilon))
\]

for \( s \in ((i-1)\varepsilon, i\varepsilon] \), \( i = 1, 2, \cdots, [\frac{t}{\varepsilon}] + 1 \). Obviously, we have \( y^\varepsilon_u(s) = Pz^\varepsilon_u(s) \). For simplicity we set \( z^\varepsilon_u = z^\varepsilon \) and \( y^\varepsilon_u = y^\varepsilon \). We can write

\[
z^\varepsilon(s) = (I + \varepsilon A)^{-1}(Pz^\varepsilon((i-1)\varepsilon) + (s - (i-1)\varepsilon)Bu^\varepsilon(i\varepsilon))
\]

for \( s \in ((i-1)\varepsilon, i\varepsilon] \), \( i = 1, 2, \cdots, [\frac{t}{\varepsilon}] + 1 \). It is easy to see that \( z^\varepsilon \) satisfies the difference scheme

\[
\frac{1}{\varepsilon}(z^\varepsilon(i\varepsilon) - z^\varepsilon((i-1)\varepsilon)) + Az^\varepsilon(i\varepsilon) + \partial I^\varepsilon_K(z^\varepsilon((i-1)\varepsilon)) = Bu^\varepsilon(i\varepsilon),
\]

\( i = 2, 3, \cdots, [\frac{t}{\varepsilon}] \). Here \( I^\varepsilon_K \) is the regularization of the convex function \( I_K \). (Note that \( \partial I^\varepsilon_K = \partial I^\varepsilon_K \)). Rearranging suitably, we have

\[
\frac{1}{\varepsilon}(Pz^\varepsilon(i\varepsilon) - Pz^\varepsilon((i-1)\varepsilon)) + Az^\varepsilon(i\varepsilon) + \partial I^\varepsilon_K(z^\varepsilon(i\varepsilon)) = Bu^\varepsilon(i\varepsilon),
\]

\( i = 2, 3, \cdots, [\frac{t}{\varepsilon}] \). Multiplying (scalarly in \( H \)) by \( z^\varepsilon(i\varepsilon) - z^\varepsilon((i-1)\varepsilon) \), after some calculation we obtain

\[
\frac{1}{\varepsilon}|y^\varepsilon(i\varepsilon) - y^\varepsilon((i-1)\varepsilon)|^2 + \frac{1}{2}(Az^\varepsilon(i\varepsilon), z^\varepsilon(i\varepsilon)) - \frac{1}{2}(Az^\varepsilon((i-1)\varepsilon), z^\varepsilon((i-1)\varepsilon)) \leq 2\varepsilon|Bu^\varepsilon(i\varepsilon)|^2 + \frac{\varepsilon}{2}|y^\varepsilon((i-1)\varepsilon) - y^\varepsilon((i-2)\varepsilon)|^2,
\]

\( i = 2, 3, \cdots, [\frac{t}{\varepsilon}] \). If \( s/\varepsilon \) is not an integer, we similarly get

\[
\frac{1}{\varepsilon}|y^\varepsilon(s) - y^\varepsilon([\frac{s}{\varepsilon}]\varepsilon)|^2 + \frac{1}{2}(Az^\varepsilon(s), z^\varepsilon(s)) - \frac{1}{2}(Az^\varepsilon([\frac{s}{\varepsilon}]\varepsilon), z^\varepsilon([\frac{s}{\varepsilon}]\varepsilon)) + I^\varepsilon_K(z^\varepsilon(s)) - I^\varepsilon_K(z^\varepsilon([\frac{s}{\varepsilon}]\varepsilon)) \leq 2\varepsilon|Bu^\varepsilon([\frac{s}{\varepsilon}]\varepsilon + 1)\varepsilon)|^2
\]

\[
+ \frac{\varepsilon}{2}|Bu^\varepsilon([\frac{s}{\varepsilon}]\varepsilon)|^2 + \frac{1}{2\varepsilon}|y^\varepsilon([\frac{s}{\varepsilon}]\varepsilon) - y^\varepsilon((\frac{s}{\varepsilon} - 1)\varepsilon)|^2.
\]

Next we have

\[
z^\varepsilon(s) - y + \varepsilon Az^\varepsilon(s) = \varepsilon Bu^\varepsilon(s).
\]

A scalar multiplication by \( z^\varepsilon(s) - y \) and an appropriate estimate of the right-hand side yield

\[
\frac{1}{2}(Az^\varepsilon(s), z^\varepsilon(s)) - \frac{1}{2}(Ay, y) \leq \frac{\varepsilon}{4}|Bu^\varepsilon(s)|^2.
\]

As regards \( I^\varepsilon_K(z^\varepsilon(s)) \), since \( I^\varepsilon_K(z^\varepsilon(s)) \leq \frac{1}{2\varepsilon}|z^\varepsilon(s) - y|^2 \) (by definition of \( I^\varepsilon_K \)), we easily get

\[
I^\varepsilon_K(z^\varepsilon(s)) \leq \varepsilon|Ay|^2 + \varepsilon|Bu^\varepsilon(s)|^2.
\]
Adding now the inequalities (19)-(22) and using also the inequality

\[ I^\epsilon_K(z^\epsilon(s)) \geq I_K(Pz^\epsilon(s)) = 0 \]

as well as the assumption (13) along with the coercivity condition (7), we deduce that the sequence \( \{y^\epsilon(s)\} \) is bounded in \( V \) for all \( s \in [0,t] \). Taking in sum \( s = t \), we also obtain

\[ \frac{1}{[t/\epsilon]} \sum_{i=1}^{[t/\epsilon]} |y^\epsilon(i\epsilon) - y^\epsilon((i - 1)\epsilon)|^2 + \frac{1}{\epsilon} |y^\epsilon(t) - y^\epsilon([t/\epsilon]\epsilon)|^2 \leq \text{const.} \]

and hence

\[ \frac{1}{[t/\epsilon]} \sum_{i=1}^{[t/\epsilon]} |y^\epsilon(i\epsilon) - y^\epsilon((i - 1)\epsilon)| + |y^\epsilon(t) - y^\epsilon([t/\epsilon]\epsilon)| \leq \text{const.} \]

This last inequality together with the fact (easily to check) that

\[ |y^\epsilon(s) - y^\epsilon(s')| \leq |s - s'| |Bu^\epsilon(i\epsilon)| \text{ for } s, s' \in ((i - 1)\epsilon, i\epsilon], \ i = 1, 2, \ldots, [t/\epsilon] + 1 \]

shows that the functions \( y^\epsilon \) are of bounded variation on \([0,t]\) and the sequence of the variations of \( y^\epsilon \) is bounded. Since the injection \( V \subset H \) is compact, using a strong version of Helly theorem (due to Foias), we conclude that there exists a function \( y^0 : [0,t] \to H \) which is of bounded variation on \([0,t]\) such that, on a subsequence, again denoted \( \{y^\epsilon\} \), we have

\[ y^\epsilon_{u^\epsilon}(s) \to y^0(s) \text{ strongly in } H \text{ for every } s \in [0,t]. \]  \hspace{1cm} (23)

Now we shall show that \( y^\epsilon_{u^\epsilon}(s) \to y^0_{u^\epsilon}(s) \) strongly in \( H \) for all \( s \in [0,t] \). Let \( \eta > 0 \), arbitrary. Choose a step function \( u = u_\eta \) which takes constant values on \((0, \epsilon\eta], (\epsilon\eta, 2\epsilon\eta], \ldots, (T - \epsilon\eta, T] \) (where \( \epsilon\eta = T/N_\eta \)) such that

\[ \left( \int_0^t |u_{\eta}(s) - u^0(s)|^2 ds \right)^{1/2} < \eta \text{ and } |y_{u_{\eta}}(s) - y_{u^0}(s)| < \eta \text{ for all } s \in [0,t]. \] \hspace{1cm} (24)

One easily checks that \( y^\epsilon_{u^\epsilon} \) and \( y^\epsilon_u \) satisfy the following difference schemes

\[ \frac{1}{\epsilon} (y^\epsilon_{u^\epsilon}(i\epsilon) - y^\epsilon_{u^\epsilon}((i - 1)\epsilon)) + A_{\epsilon} y^\epsilon_{u^\epsilon}((i - 1)\epsilon) + \partial I_K(y^\epsilon_{u^\epsilon}(i\epsilon)) \ni (I + \epsilon A)^{-1} Bu^\epsilon(i\epsilon), \]

\[ \frac{1}{\epsilon} (y^\epsilon_u(i\epsilon) - y^\epsilon_u((i - 1)\epsilon)) + A_{\epsilon} y^\epsilon_u((i - 1)\epsilon) + \partial I_K(y^\epsilon_u(i\epsilon)) \ni (I + \epsilon A)^{-1} Bu(i\epsilon), \]

\( (i = 1, 2, \ldots, [t/\epsilon]) \), respectively, \( A_{\epsilon} \) being the Yosida approximation of \( A \). Rearranging above, we can write

\[ \frac{1}{\epsilon} (I + \epsilon A)^{-1} y^\epsilon_{u^\epsilon}(i\epsilon) - (I + \epsilon A)^{-1} y^\epsilon_{u^\epsilon}((i - 1)\epsilon)) + A_{\epsilon} y^\epsilon_{u^\epsilon}(i\epsilon) + \partial I_K(y^\epsilon_{u^\epsilon}(i\epsilon)) \ni (I + \epsilon A)^{-1} Bu^\epsilon(i\epsilon), \]
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\[ \frac{1}{\epsilon} ((I + \epsilon A)^{-1} y_u^\epsilon(i\epsilon) - (I + \epsilon A)^{-1} y_u^\epsilon((i - 1)\epsilon)) + A_\epsilon y_u^\epsilon(i\epsilon) + \partial I_K(y_u^\epsilon(i\epsilon)) \]

\[ \exists (I + \epsilon A)^{-1} Bu(i\epsilon), \quad i = 1, 2, \ldots, \lfloor \frac{t}{\epsilon} \rfloor. \]

Let \( s \in [0, t] \), arbitrary. Suppose \( s \neq [s/\epsilon] \epsilon \). We subtract the previous two equations and multiply the difference by \( y_u^\epsilon(i\epsilon) - y_u^\epsilon((i - 1)\epsilon) \). Using then the monotonicity of \( A_\epsilon \), \( \partial I_K \) and adding the obtained inequalities with respect to \( i \) from 1 to \( [s/\epsilon] \), we get

\[ \frac{1}{2} \left| (I + \epsilon A)^{-1} y_u^\epsilon([s/\epsilon] \epsilon) - (I + \epsilon A)^{-1} y_u^\epsilon([s/\epsilon] \epsilon) \right|^2 \]

\[ \leq \sum_{i=1}^{[s/\epsilon]} \epsilon (Bu^\epsilon(i\epsilon) - Bu(i\epsilon), (I + \epsilon A)^{-1} y_u^\epsilon(i\epsilon) - (I + \epsilon A)^{-1} y_u^\epsilon(i\epsilon)). \]

Hence

\[ \frac{1}{2} \left| y_u^\epsilon(s) - y_u^\epsilon(s) \right|^2 \leq 3 \sum_{i=1}^{[s/\epsilon]} \epsilon (u^\epsilon(i\epsilon) - u(i\epsilon), B^* (I + \epsilon A)^{-1} y_u^\epsilon(i\epsilon)) \]

\[ - B^* (I + \epsilon A)^{-1} y_u^\epsilon(i\epsilon)) + \frac{3}{2} \epsilon^2 |Bu^\epsilon([s/\epsilon] + 1)\epsilon)|^2 + \frac{3}{2} \epsilon^2 |Bu([s/\epsilon] + 1)\epsilon)|^2. \]

In the case when \( s = [s/\epsilon] \epsilon \), we add the same inequalities from 1 to \( [s/\epsilon] - 1 \), the computations being similar. We can write

\[ \sum_{i=1}^{[s/\epsilon]} \epsilon (u^\epsilon(i\epsilon) - u(i\epsilon), B^* (I + \epsilon A)^{-1} y_u^\epsilon(i\epsilon) - B^* (I + \epsilon A)^{-1} y_u^\epsilon(i\epsilon)) \]

\[ = \int_0^{[s/\epsilon] \epsilon} (u^\epsilon(\tau) - \bar{u}_\epsilon(\tau), B^* (I + \epsilon A)^{-1} \bar{y}_u^\epsilon(\tau) - B^* (I + \epsilon A)^{-1} \bar{y}_u^\epsilon(\tau)) d\tau, \]

where

\[ \bar{y}_u^\epsilon(\tau) = y_u^\epsilon(i\epsilon), \quad \bar{y}_u^\epsilon(\tau) = y_u^\epsilon(i\epsilon) \quad \text{and} \quad \bar{u}_\epsilon(\tau) = u(i\epsilon) \]

for \( \tau \in ((i - 1)\epsilon, i\epsilon], i = 1, 2, \ldots, [\frac{s}{\epsilon}] \). Since \( |\bar{y}_u^\epsilon(\tau) - y_u^\epsilon(\tau)| \leq \epsilon |Bu^\epsilon(((\tau/\epsilon) + 1)\epsilon)| \) (do not forget that \( \{u^\epsilon\} \) is bounded in \( L^2(0, t; U) \)), (23) and Lemma 2 we have \( \bar{y}_u^\epsilon(\tau) \to y^0(\tau) \) (on a certain subsequence) and \( \bar{y}_u^\epsilon(\tau) \to y_u(\tau) \) strongly in \( H \) for all \( \tau \in [0, s] \). We also have \( \bar{u}_\epsilon(\tau) \to u(\tau) \) for \( \tau \in [0, s] \) excepting a finite number of \( \tau \). Now letting \( \epsilon \) tend to zero in (25) (on a certain subsequence), we obtain

\[ \frac{1}{2} |y^0(s) - y_u(s)|^2 \leq 3 \int_0^s (Bu^0(\tau) - Bu(\tau), y^0(\tau) - y_u(\tau)) d\tau. \]

Hence, taking (24) into account, \( (\eta > 0 \text{ is arbitrary}) \) the statement of Lemma 3 follows.

To finish the proof of Theorem 1 consider first \( y \in K \cap D(A) \). We write (as in the first part of the proof)

\[ J_\epsilon(u^\epsilon) = \int_0^t h(u^\epsilon(s)) ds + \int_0^{[t/\epsilon]} g(\bar{y}_u^\epsilon(s)) ds + \varphi_0(y_u^\epsilon(t)) + \epsilon g(y_u^\epsilon(t)), \]
where \( \tilde{y}_{0t} \) has been defined above. Using the fact that \( u \mapsto \int_0^t h(u(s)) ds \) is weakly lower semicontinuous in \( L^2(0, t; U) \) and the Lebesgue theorem together with Lemma 3, we get

\[
\lim_{\epsilon \to 0} J_\epsilon(u^\epsilon) \geq J(u^0) \geq J.
\]

Here \( J(u^0) \) is defined as before. If \( u \in K \), a density argument based on the Lipschitz continuity on bounded subsets of the functions \( u \mapsto \varphi(t, y) \), \( u \mapsto \varphi^\epsilon(t, y) \) (uniformly with respect to \( \epsilon \)) completes the proof of Theorem 1.

Now, for reader's convenience, we outline the proof of the fact that \( y \mapsto \varphi^\epsilon(t, y) \) is Lipschitz continuous on bounded subsets, uniformly with respect to \( \epsilon \).

Observe first that the infimum in the definition of \( \varphi^\epsilon(t, y) \) is attained at a certain step function \( u^\epsilon \). Indeed, let \( \{u_n\} \) be a minimizing sequence for \( \varphi^\epsilon(t, y) \). By use of (6) we obtain that the sequences \( \{u_n(\epsilon)\}, \{u_n(2\epsilon)\}, \ldots, \{u_n([t/\epsilon] + 1)\} \) are bounded in \( U \), so, on a certain subsequence of \( \{n\} \), they converge to \( u^\epsilon(\epsilon), u^\epsilon(2\epsilon), \ldots, u^\epsilon([t/\epsilon] + 1)\), respectively, where \( u^\epsilon \) is a certain step function. If \( z_{u_n}^\epsilon \) is defined as in Lemma 3, some calculation gives

\[
(Az_{u_n}^\epsilon(\epsilon), z_{u_n}^\epsilon(\epsilon)) \leq \frac{1}{2\epsilon} |y_{u_n}^\epsilon((i-1)\epsilon)|^2 + \frac{\epsilon}{2} |Bu_n(\epsilon)|^2, \quad i = 1, 2, \ldots, [t/\epsilon],
\]

\[
(Az_{u_n}^\epsilon(t), z_{u_n}^\epsilon(t)) \leq \frac{1}{2\epsilon} |y_{u_n}^\epsilon([t/\epsilon])|^2 + \frac{\epsilon}{2} |Bu_n([t/\epsilon] + 1)\epsilon)|^2,
\]

so, by (13) and (7), we obtain that \( \{y_{u_n}^\epsilon(\epsilon)\}, i = 1, 2, \ldots, [t/\epsilon], \{y_{u_n}^\epsilon(t)\} \) are bounded in \( V \) (do not forget that \( y_{u_n}^\epsilon(0) = y \)). But the inclusion \( V \subset H \) is compact, therefore, on a subsequence of \( \{n\} \), the previous sequences converge to \( y^\epsilon(\epsilon), i = 1, 2, \ldots, [t/\epsilon], y^\epsilon(t), \) respectively. Using the fact that \( \partial I_K \) is demiclosed in \( H \times H \), we infer that \( y^\epsilon(\epsilon) = y^\epsilon(\epsilon), i = 1, 2, \ldots, [t/\epsilon], y^\epsilon(t) = y^\epsilon(t) \). Letting \( n \) tend to \( \infty \) and taking into account the lower semicontinuity of \( h \), we conclude that the infimum is attained at \( u^\epsilon \).

Now let \( y, z \in H \), arbitrary, such that \( |y|, |z| \leq r \). We have

\[
\varphi^\epsilon(t, z) - \varphi^\epsilon(t, y) \leq \sum_{i=1}^{[t/\epsilon]} \epsilon (g(z^\epsilon_{u^\epsilon}(\epsilon)) - g(y^\epsilon_{u^\epsilon}(\epsilon)))
\]

\[
+ \epsilon (g(z^\epsilon_{u^\epsilon}(t)) - g(y^\epsilon_{u^\epsilon}(t))) + \varphi_0(z^\epsilon_{u^\epsilon}(t)) - \varphi_0(y^\epsilon_{u^\epsilon}(t)),
\]

where \( u^\epsilon \) is a step function at which the infimum in the definition of \( \varphi^\epsilon(t, y) \) is attained. Here \( z^\epsilon_{u^\epsilon} \) is defined as in (16) with \( z \) instead of \( y \). Taking (6) into account, we easily get

\[
\sum_{i=1}^{[t/\epsilon]} \epsilon|u^\epsilon(i\epsilon)|^2 + (t - [t/\epsilon])\epsilon|u^\epsilon([t/\epsilon] + 1)\epsilon|^2 \leq \text{const.},
\]

where the above constant depends only on \( r \). Consequently \( y^\epsilon_{u^\epsilon}(\epsilon), y^\epsilon_{u^\epsilon}(t), z^\epsilon_{u^\epsilon}(\epsilon) \) and \( z^\epsilon_{u^\epsilon}(t) \) are bounded by a constant which depends only on \( r \). We can now use (ii) to ascertain the Lipschitz continuity of \( y \mapsto \varphi^\epsilon(t, y) \) on bounded subsets, uniformly with respect to \( \epsilon \).

As regards the scheme (12), its convergence is stated in the following theorem.
Theorem 2. If the hypotheses (i)-(v) are satisfied, then
\[
\lim_{\epsilon \to 0} \psi^\epsilon(t, y) = \varphi(t, y) \quad \text{for any } (t, y) \in [0, T] \times K.
\]

The proof of Theorem 2 is completely similar to that of the previous theorem except maybe the proof of Lemma 3. In the present case, for a step function \( u : [0, T] \to U \), \( y^\epsilon \) is defined by
\[
\begin{cases}
y_u^\epsilon(t) = (I + \epsilon A)^{-1} P(y_u^\epsilon((i-1)\epsilon) + (t - (i-1)\epsilon)Bu(i\epsilon)), \\
t \in ((i-1)\epsilon, i\epsilon], \quad i = 1, 2, \ldots, N,
\end{cases}
y_u^\epsilon(0) = y.
\]

Lemma 4. Let \( \{u^\epsilon\} \) be a sequence of step functions from \([0, T]\) to \( U \) such that each \( u^\epsilon \) takes constant values on \((0, \epsilon], (\epsilon, 2\epsilon], \ldots, (T-\epsilon, T]\) and let \( y \in K \cap D(A) \). If \( u^\epsilon \to u^0 \) weakly in \( L^2(0, t; U) \), then
\[
y_u^\epsilon(s) \to y_{u^0}(s) \quad \text{strongly in } H \text{ for all } s \in [0, t],
\]
where \( y_{u^0} \) is the solution of (17) corresponding to \( u = u^0 \).

Proof: The arguments are similar to those of Lemma 3 but with some differences. One easily verifies that \( y^\epsilon = y_u^\epsilon \), satisfies the difference scheme
\[
\frac{1}{\epsilon} (y^\epsilon(i\epsilon) - y^\epsilon((i-1)\epsilon)) + Ay^\epsilon(i\epsilon) + I_K(y^\epsilon((i-1)\epsilon))
\]
\[
= \frac{1}{\epsilon} (P(y^\epsilon((i-1)\epsilon) + \epsilon Bu^\epsilon(i\epsilon)) - Py^\epsilon((i-1)\epsilon)),
\]
i = 1, 2, \ldots, \lfloor \frac{t}{\epsilon} \rfloor, s \in [0, t]. Multiplying the previous equation by \( y^\epsilon(i\epsilon) - y^\epsilon((i-1)\epsilon) \), after some calculation we get
\[
\frac{1}{\epsilon} |Py^\epsilon(i\epsilon) - Py^\epsilon((i-1)\epsilon)|^2 + \frac{1}{2} (Ay^\epsilon(i\epsilon), y^\epsilon(i\epsilon))
\]
\[
- \frac{1}{2} (Ay^\epsilon((i-1)\epsilon), y^\epsilon((i-1)\epsilon)) + I_K(y^\epsilon(i\epsilon)) - I_K(y^\epsilon((i-1)\epsilon))
\]
\[
\leq |Bu^\epsilon(i\epsilon)||y^\epsilon(i\epsilon) - y^\epsilon((i-1)\epsilon)|, \quad i = 1, 2, \ldots, \lfloor \frac{t}{\epsilon} \rfloor.
\]

Similarly, when \( s/\epsilon \) is not an integer, we have
\[
\frac{1}{\epsilon} |Py^\epsilon(s) - Py^\epsilon([s/\epsilon]\epsilon)|^2 + \frac{1}{2} (Ay^\epsilon(s), y^\epsilon(s))
\]
\[
- \frac{1}{2} (Ay^\epsilon([s/\epsilon]\epsilon), y^\epsilon([s/\epsilon]\epsilon)) + I_K(y^\epsilon(s)) - I_K(y^\epsilon([s/\epsilon]\epsilon))
\]
\[
\leq (\frac{s}{\epsilon} - [\frac{s}{\epsilon}]) |Bu^\epsilon([\frac{s}{\epsilon}]\epsilon + 1)\epsilon)||y^\epsilon(s) - y^\epsilon([\frac{s}{\epsilon}]\epsilon)|.
We now add this inequalities by taking into account the following:

\[ |y^\epsilon((i+1)\epsilon) - y^\epsilon(i\epsilon)|^2 \leq 3|Py^\epsilon(i\epsilon) - Py^\epsilon((i - 1)\epsilon)|^2 + 3\epsilon^2|Bu^\epsilon((i+1)\epsilon)|^2 + 3\epsilon^2|Bu^\epsilon(i\epsilon)|^2, \quad i = 1, 2, \ldots, \left\lfloor \frac{s}{\epsilon} \right\rfloor - 1, \]

\[ |y^\epsilon(s) - y^\epsilon(\left\lfloor \frac{s}{\epsilon} \right\rfloor \epsilon)|^2 \leq 3|Py^\epsilon(\left\lfloor \frac{s}{\epsilon} \right\rfloor \epsilon) - Py^\epsilon(\left\lfloor \frac{s}{\epsilon} \right\rfloor - 1)\epsilon)|^2 + 3\epsilon^2|Bu^\epsilon(\left\lfloor \frac{s}{\epsilon} \right\rfloor + 1)\epsilon)|^2 + 3\epsilon^2|Bu^\epsilon(\left\lfloor \frac{s}{\epsilon} \right\rfloor \epsilon)|^2. \]

Next, as in the proof of Lemma 3, applying the infinite-dimensional Helly theorem, we find a function of bounded variation \(y^0: [0, t] \rightarrow H\) such that, on a subsequence, we have

\[ y^\epsilon_u(s) \rightarrow y^0(s) \quad \text{strongly in } H \text{ for every } s \in [0, t]. \tag{26} \]

Further, for \(\eta > 0\), arbitrary, choose a step function \(u = u_\eta\) (which takes constant values on the subintervals \(((i - 1)\epsilon, i\epsilon_\eta], i = 1, 2, \ldots, N_\eta\) satisfying (24). Consider the functions \(z^\epsilon_u\) and \(z^\epsilon_u\):

\[ z^\epsilon_u(s) = P(y^\epsilon_u((i - 1)\epsilon) + (s - (i - 1)\epsilon)Bu^\epsilon(i\epsilon)), \]

\[ z^\epsilon_u(s) = P(y^\epsilon_u((i - 1)\epsilon) + (s - (i - 1)\epsilon)Bu^\epsilon(i\epsilon)) \]

for \(s \in ((i - 1)\epsilon, i\epsilon], i = 1, 2, \ldots, \left\lfloor \frac{s}{\epsilon} \right\rfloor + 1.\) Clearly, \(y^\epsilon_u(s) = (I + \epsilon A)^{-1}z^\epsilon_u(s)\) and \(y^\epsilon_u(s) = (I + \epsilon A)^{-1}z^\epsilon_u(s), s \in (0, t].\) It is easy to see that \(z^\epsilon_u\) and \(z^\epsilon_u\) satisfy

\[ \frac{1}{\epsilon}(z^\epsilon_u((i - 1)\epsilon) - z^\epsilon_u((i - 1)\epsilon)) + A\epsilon z^\epsilon_u((i - 1)\epsilon) + \partial I_K(z^\epsilon_u((i - 1)\epsilon)) \ni Bu^\epsilon(i\epsilon), \]

\[ \frac{1}{\epsilon}(z^\epsilon_u((i - 1)\epsilon) - z^\epsilon_u((i - 1)\epsilon)) + A\epsilon z^\epsilon_u((i - 1)\epsilon) + \partial I_K(z^\epsilon_u((i - 1)\epsilon)) \ni Bu(i\epsilon), \]

for \(i = 2, 3, \ldots, \left\lfloor \frac{s}{\epsilon} \right\rfloor\) and hence

\[ \frac{1}{2}(y^\epsilon_u(i\epsilon) - y^\epsilon_u(i\epsilon), z^\epsilon_u(i\epsilon) - z^\epsilon_u(i\epsilon)) \]

\[ + \frac{1}{2}(y^\epsilon_u((i - 1)\epsilon) - y^\epsilon_u((i - 1)\epsilon), z^\epsilon_u((i - 1)\epsilon) - z^\epsilon_u((i - 1)\epsilon)) \]

\[ \leq \epsilon(u^\epsilon(i\epsilon) - u(i\epsilon), B^*(z^\epsilon_u(i\epsilon) - z^\epsilon_u(i\epsilon))), \quad i = 2, 3, \ldots, \left\lfloor \frac{s}{\epsilon} \right\rfloor. \tag{27} \]

In the case when \(s/\epsilon\) is not an integer, we similarly obtain

\[ \frac{1}{2}(y^\epsilon_u(s) - y^\epsilon_u(s), z^\epsilon_u(s) - z^\epsilon_u(s)) - \frac{1}{2}(y^\epsilon_u(\left\lfloor \frac{s}{\epsilon} \right\rfloor \epsilon)) \]

\[ - y^\epsilon_u(\left\lfloor \frac{s}{\epsilon} \right\rfloor \epsilon) = z^\epsilon_u(\left\lfloor \frac{s}{\epsilon} \right\rfloor \epsilon) - z^\epsilon_u(\left\lfloor \frac{s}{\epsilon} \right\rfloor \epsilon)) \]

\[ \leq (s - \left\lfloor \frac{s}{\epsilon} \right\rfloor \epsilon)(u^\epsilon(\left\lfloor \frac{s}{\epsilon} \right\rfloor + 1)\epsilon) - u(\left\lfloor \frac{s}{\epsilon} \right\rfloor + 1)\epsilon), B^*(z^\epsilon_u(s) - z^\epsilon_u(s))). \tag{28} \]
Adding the inequalities (27), (28), we have
\[\frac{1}{2}|y_u^\epsilon(s) - y_u^\epsilon(s)|^2 - \frac{1}{2}(y_u^\epsilon(\epsilon) - y_u^\epsilon(\epsilon), z_u^\epsilon(\epsilon) - z_u^\epsilon(\epsilon))\]
\[\leq \sum_{i=2}^{[s/\epsilon]} \epsilon(u^\epsilon(ie) - u(ie), B^*(z_u^\epsilon(ie) - z_u^\epsilon(ie)))\]
\[+ \epsilon(u^\epsilon(\lfloor \frac{s}{\epsilon} \rfloor + 1)\epsilon - u(\lfloor \frac{s}{\epsilon} \rfloor + 1)\epsilon), B^*(z_u^\epsilon(s) - z_u^\epsilon(s))).\]
If \(s/\epsilon\) is an integer, we add the inequalities (27) only \((i = 2, 3, \ldots, [s/\epsilon])\), the rest being completely similar. One can write
\[\frac{1}{2}|y_u^\epsilon(s) - y_u^\epsilon(s)|^2 \leq \frac{1}{2}(y_u^\epsilon(\epsilon) - y_u^\epsilon(\epsilon), z_u^\epsilon(\epsilon) - z_u^\epsilon(\epsilon))\]
\[+ \int_0^{[\lfloor s/\epsilon \rfloor - 1]\epsilon} (u^\epsilon(\tau + \epsilon) - \tilde{u}_\epsilon(\tau), B^*(\tilde{z}_u^\epsilon(\tau) - \tilde{z}_u^\epsilon(\tau))) \, d\tau\]
\[+ \epsilon(u^\epsilon(\lfloor \frac{s}{\epsilon} \rfloor + 1)\epsilon - u(\lfloor \frac{s}{\epsilon} \rfloor + 1)\epsilon), B^*(z_u^\epsilon(s) - z_u^\epsilon(s))),\]
where
\[\tilde{z}_u^\epsilon(\tau) = z_u^\epsilon((i + 1)\epsilon), \quad \tilde{z}_u^\epsilon(\tau) = z_u^\epsilon((i + 1)\epsilon) \quad \text{and} \quad \tilde{u}_\epsilon(\tau) = u((i + 1)\epsilon)\]
for \(\tau \in ((i - 1)\epsilon, i\epsilon]\), \(i = 1, 2, \ldots, [\frac{s}{\epsilon}] - 1\).

Obvious estimates for \(|\tilde{z}_u^\epsilon(\tau) - Py_u^\epsilon(\tau)|\) and \(|\tilde{z}_u^\epsilon(\tau) - Py_u^\epsilon(\tau)|\) along with (26) and Lemma 2 yield \(\tilde{z}_u^\epsilon(\tau) \rightarrow Py_0(\tau)\) (on a certain subsequence) and \(\tilde{z}_u^\epsilon(\tau) \rightarrow Py_0(\tau)\), strongly in \(H\) for all \(\tau \in [0, s]\). Also it is easy to show that \((y_u^\epsilon(\epsilon) - y_u^\epsilon(\epsilon), z_u^\epsilon(\epsilon) - z_u^\epsilon(\epsilon)) \rightarrow 0\). Letting \(\epsilon\) tend to zero in (29), we get
\[\frac{1}{2}|y_0(s) - y_u(s)|^2 \leq \int_0^s (Bu^0(\tau) - Bu(\tau), Py^0(\tau) - Py_u(\tau)) \, d\tau\]
\[\leq \int_0^s |Bu^0(\tau) - Bu(\tau)||y^0(\tau) - y_u(\tau)| \, d\tau,\]
which together with (24) \((\eta > 0\) being arbitrary) lead to the conclusion of Lemma 4.

**Remark 2.** Let us point out that the proof of Theorem 2 does not use the additional assumption (13) which is required in the proof of Theorem 1. However, this assumption (which we can also find in [2, 3]) is satisfied in many relevant situations.

**3. An example.** The approximation schemes (11), (12) may be applied to Hamilton-Jacobi equations arising in control of parabolic obstacle problem.

Let \(\Omega\) be an open and bounded subset of \(\mathbb{R}^n\) having sufficiently smooth boundary. Consider the problem
\[\frac{\partial y}{\partial t} - \Delta y \geq Bu, \quad y \geq 0 \quad \text{a.e. in } [0, T] \times \Omega,\]
\[\left(\frac{\partial y}{\partial t} - \Delta y - Bu\right)y = 0 \quad \text{a.e. in } [0, T] \times \Omega,\]
\[y = 0 \quad \text{on } [0, T] \times \partial \Omega,\]
\[y(0, x) = y_0(x) \quad \text{on } \Omega,\]
\[y(0, x) = y_0(x) \quad \text{on } \Omega,\]
where \( y_0(x) \geq 0 \) for almost every \( x \in \Omega \).

It is well known (see for instance [1]) that this problem is of the form (4), (5), where \( H = L^2(\Omega), V = H^1_0(\Omega), A = -\Delta, K = \{ y \in L^2(\Omega) : y(x) \geq 0 \text{ a.e. } x \in \Omega \} \). We have \( D(A) = H^1_0(\Omega) \cap H^2(\Omega) \) and \( (Py)(x) = \max\{y(x), 0\} = y^+(x) \text{ a.e. } x \in \Omega \).

The assumption (23) becomes

\[
\int_\Omega |\nabla y^+|^2 \, dx \leq \int_\Omega |\nabla y|^2 \, dx, \quad y \in H^1_0(\Omega),
\]

which is easy to verify. The scheme (11) looks in this case like this:

\[
\varphi^\varepsilon(t, y) = \left\{ \begin{array}{ll}
\inf\{\varepsilon h(u) + \varepsilon g(\{(I - \varepsilon \Delta)^{-1}(y + \varepsilon Bu)\}_+) \\
+ \varphi^\varepsilon(t - \varepsilon, (I - \varepsilon \Delta)^{-1}(y + \varepsilon Bu)\}_+) : u \in U\}
\end{array} \right.
\]

for \((t, y) \in (\varepsilon, T] \times L^2(\Omega),\)

\[
\inf\{\varepsilon h(u) + \varepsilon g(\{(I - \varepsilon \Delta)^{-1}(y + t Bu)\}_+) \\
+ \varphi_0((I - \varepsilon \Delta)^{-1}(y + t Bu)\}_+) : u \in U\}
\]

for \((t, y) \in (0, \varepsilon] \times L^2(\Omega),\)

\[
\varphi^\varepsilon(0, y) = \varphi_0(y) \quad \text{for } y \in L^2(\Omega).
\]

To be more specific, define the function \( h : L^2(\Omega) \rightarrow (-\infty, +\infty) \) by

\[
h(u) = \int_\Omega h_0(u(x)) \, dx \quad \text{for all } u \in L^2(\Omega),
\]

where

\[
h_0(u) = \left\{ \begin{array}{ll}
0 & \text{if } |u| \leq 1, \\
+\infty & \text{if } |u| > 1.
\end{array} \right.
\]

Minimize

\[
\int_0^T \int_\Omega h_0(u(t, x)) \, dx \, dt + \frac{1}{2} \int_\Omega |y(T, x) - y_T(x)|^2 \, dx
\]

over all \( u \in L^2(0, T; L^2(\Omega)) \) and \( y \in C([0, T]; L^2(\Omega)) \) which satisfy the initial boundary value problem (30) with \( B = I \). Then, in this case Theorem 1 asserts that the following scheme

\[
\varphi^\varepsilon(t, y) = \left\{ \begin{array}{ll}
\inf\{\varphi^\varepsilon(t - \varepsilon, ((I - \varepsilon \Delta)^{-1}(y + \varepsilon u))_+) \quad u \in L^2(\Omega), \\
|u(x)| \leq 1 \text{ a.e. } x \in \Omega \}
\end{array} \right.
\]

for \((t, y) \in (\varepsilon, T] \times L^2(\Omega),\)

\[
\inf\left\{ \frac{1}{2} \int_\Omega \left| ((I - \varepsilon \Delta)^{-1}(y + tu))_+(x) - y_T(x) \right|^2 \, dx : u \in L^2(\Omega), \\
|u(x)| \leq 1 \text{ a.e. } x \in \Omega \}
\end{array} \right.
\]

for \((t, y) \in (0, \varepsilon] \times L^2(\Omega),\)

converges to the solution

\[
\varphi(t, y) = \inf\left\{ \frac{1}{2} \int_\Omega |y(t, x) - y_T(x)|^2 \, dx : y' - \Delta y + \partial I_K(y) \ni u, \quad y(0) = y_0, \\
u \in L^2(0, t; L^2(\Omega)), \ |u(s, x)| \leq 1 \text{ a.e. } (s, x) \in [0, t] \times \Omega \right\}
\]
of the corresponding Hamilton-Jacobi equation (4), (5).

To obtain a more explicit formula, we express the resolvent of Δ with the aid of an appropriate Green function. For example, if \( n = 1 \) and \( \Omega = (0,1) \), we have

\[
((I - \epsilon \Delta)^{-1}(y + \epsilon u))(x) = \int_0^1 G(x, \xi)(y(\xi) + \epsilon u(\xi)) \, d\xi,
\]

where

\[
G(x, \xi) = -\frac{1}{c} \begin{cases} (e^{\xi/\sqrt{\epsilon}} - e^{-x/\sqrt{\epsilon}})(e^{(\xi-1)/\sqrt{\epsilon}} - e^{-(x-1)/\sqrt{\epsilon}}) & \text{for } x \leq \xi, \\ (e^{\xi/\sqrt{\epsilon}} - e^{-x/\sqrt{\epsilon}})(e^{(x-1)/\sqrt{\epsilon}} - e^{-(\xi-1)/\sqrt{\epsilon}}) & \text{for } x \geq \xi, \end{cases}
\]

\[
c = \frac{2}{\sqrt{\epsilon}}(e^{1/\sqrt{\epsilon}} - e^{-1/\sqrt{\epsilon}}).
\]

REFERENCES