Differential and Integral Equations, Volume 5, Number 1, January 1992, pp. 25–42.

NODAL SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS WITH CRITICAL EXPONENT†

GABRIELLA TARANTELLO

Department of Mathematics, Carnegie Mellon University, Pittsburgh, PA 15213

(Submitted by: Haim Brezis)

Abstract: Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with smooth boundary and $p = 2N/(N-2)$ be the critical Sobolev exponent. In this note we extend the results of [10] and [21] concerning nodal solutions (i.e., a solution which changes sign) for the Dirichlet problem: $-\Delta u = |u|^{p-2}u + \lambda u$ on Ω and $u = 0$ on $\partial\Omega$, when $N \geq 6$ and $\lambda \in (0, \lambda_1)$ with λ_1 the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. Similarly, for the problem $-\Delta u = |u|^{p-2}u + \lambda|u|^{q-2}u$ on Ω and $u = 0$ on $\partial\Omega$ we obtain a nodal solution when $\lambda > 0$, $(N+2)/(N-2) < q < 2N/(N-2)$ for $N = 3$, 4, 5 and $2 < q < 2N/(N-2)$ for $N \geq 6$.

Introduction and main results. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with smooth boundary $\partial\Omega$ and $N \geq 3$. Consider the Dirichlet problem:

$$-\Delta u = |u|^{p-2}u + \lambda u \;\text{ on }\; \Omega, \quad u = 0 \;\text{ on }\; \partial\Omega, \tag{*}$$

where $p = 2N/(N-2)$ is the best exponent in the Sobolev embedding and $\lambda > 0$. Define λ_1 to be the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. Using ideas introduced by Aubin ([1]) for the Yamabe problem, Brezis-Nirenberg ([6]) proved that in contrast to the non-existence situation yield by the Pohozaev's identity ([15]) for $\lambda \leq 0$, in case $N \geq 4$ and $\lambda \in (0, \lambda_1)$ then problem $(*)$ always admits a positive solution.

Notice that if $\lambda \geq \lambda_1$, every solution of $(*)$ must change sign in Ω. Existence in such situations has been established in [9]. The dimension $N = 3$ appears more delicate and existence is only possible when $\lambda > \lambda_* > 0$ for a suitable constant λ_* depending on the domain Ω (see [6]). We refer to [4] and [18] for a detailed bibliography related to various interesting aspects of this problem.

In this note, we will be concerned with solutions of $(*)$ which change sign in Ω. Following the notation introduced in [2], we shall refer to these solutions as nodal solutions. The existence of a pair of nodal solutions for $(*)$ has been obtained in [10] and [21] for $N \geq 6$ and $\lambda \in (0, \lambda_1)$.

Here we give a different proof of these results together with a mild extension. As in [10] and [21], we use variational methods. However, our proof relies more on the specific choice of the P.S. sequence then on the appropriate minimax principle. We hope that our point of view will shed some new light on the multiplicity question for problem $(*)$. We have:

Received December 1990.

†This research has been supported in part by NSF grant DMS-9003149.

AMS Subject Classifications: 35J20, 35J25, 35J65.

Theorem 1. *For $N \geq 6$ and $\lambda \in (0, \lambda_1)$, the Dirichlet problem $(*)$ admits a non-trivial solution $u = u(x)$ satisfying*

$$\int_\Omega |u|^{p-2} u v(u) = 0,$$

where $v(u)$ is the first eigenfunction of the weighted eigenvalue problem

$$-(\Delta + \lambda)v = \mu |u|^{p-2} v \text{ on } \Omega, \quad v = 0 \text{ on } \partial\Omega,$$

$\mu \in \mathbb{R}$.

Clearly $u = u(x)$ defines a nodal solution for $(*)$. The dimensions $N = 3, 4, 5, 6$ are more delicate. In fact, it has been pointed out in [2] that for a ball $\Omega \subset \mathbb{R}^N$ with $N = 4, 5, 6$, problem $(*)$ cannot admit radial solutions which change sign when $\lambda > 0$ is sufficiently close to zero. This non-existence result was motivated by an analogous one obtained by Jones in [13] and concerning the related problem

$$-\Delta u = |u|^{p-2} u + \lambda |u|^{q-2} u \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{$(*)_q$}$$

where $q \in (2, \frac{2N}{N-2})$ and $\lambda > 0$. A positive solution for $(*)_q$ was established in [6] for $N \geq 4$. While, for a ball Ω, Jones (see [13]) has obtained infinitely many radial solutions when $q \in (\frac{N+2}{N-2}, \frac{2N}{N-2})$ and $N = 4, 5$ or $q \in (2, \frac{2N}{N-2})$ and $N \geq 6$.

Here we shall prove that, under similar assumptions on q, problem, $(*)_q$ always admits a nodal solution. More precisely, given a smooth function $u \neq 0$, denote by $v_q(u)$ the first eigenfunction for the weighted eigenvalue problem

$$-\Delta v = \mu(|u|^{p-2} + |u|^{q-2}) v \text{ on } \Omega, \quad v = 0 \text{ on } \partial\Omega.$$

We have,

Theorem 2. *Assume that $q \in (\frac{N+2}{N-2}, \frac{2N}{N-2})$ for $N = 3, 4, 5$ and $q \in (2, \frac{2N}{N-2})$ for $N \geq 6$, $\lambda > 0$. Then problem $(*)_q$ admits a non-trivial solution $u = u(x)$ satisfying*

$$\int_\Omega (|u|^{p-2} + \lambda |u|^{q-2}) u v_q(u) = 0.$$

Again, the case where $N = 3, 4, 5$ and $q \in (2, \frac{N+2}{N-2})$ is more delicate, and this type of result may not hold for every given $\lambda > 0$. In this direction, Jones (see [13]) has shown that for a ball Ω, problem $(*)_q$ cannot admit radial solutions which change sign when $q \in (2, \frac{N+2}{N-2})$ and λ is sufficiently close to zero. The same conclusion also holds when $q = \frac{N+2}{N-2}$ (see [14]).

Notice that if we replace p in $(*)$ and $(*)_q$ by r and require $2 < r < \frac{2N}{N-2} = p$ (subcritical case), then by means of the Ljusternik-Schnirelman theory one obtains infinitely many pairs of solutions for $(*)$. However, this approach fails in the limiting case $r = p = \frac{2N}{N-2}$ because of the lack of compactness for the corresponding variational principle; and infinitely many solutions for $(*)$ and $(*)_q$ have been obtained only for special domains with symmetries (cf. [12], [10] and [13]).

So far, the multiplicity question for the critical exponent problem remain as elusive as ever.

Acknowledgements. The author wishes to thank C.V. Coffman for useful discussions.

1. Proof of Theorem 1. As well known, weak solutions of $(*)$ (resp. $(*)_q$) are the critical points for the functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega |u|^2 - \frac{1}{p}\int_\Omega |u|^p$$

resp.

$$F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{q}\int_\Omega |u|^q - \frac{1}{p}\int_\Omega |u|^p, \quad u \in H_0^1(\Omega).$$

A regularity result due to Brezis-Kato (cf. [5]) then guarantees that such weak solutions are, in fact, classical solutions. Denote by $\|\cdot\|$ the standard norm in $H_0^1(\Omega)$. Our first goal is to obtain Theorem 1 in the subcritical case where we replace p with $p-\epsilon$, $\epsilon > 0$ small. To shorten notations set $p_\epsilon = p - \epsilon$, $0 < \epsilon < \frac{4}{N-2}$, and $H = H_0^1(\Omega)$ with scalar product $\langle\, ,\, \rangle$. Define

$$I_\epsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega |u|^2 - \frac{1}{p_\epsilon}\int_\Omega |u|^{p_\epsilon}, \quad u \in H;$$

so $I = I_{\epsilon=0}$. Associated to I_ϵ and I are, respectively, the manifolds

$$\Lambda_\epsilon = \{u \neq 0 : \langle I_\epsilon'(u), u\rangle = 0\}; \quad \Lambda = \{u \neq 0 : \langle I'(u), u\rangle = 0\}.$$

It is easy to see that I_ϵ and I are bounded below, respectively, in Λ_ϵ and Λ. Set

$$c_{1,\epsilon} = \inf_{\Lambda_\epsilon} I_\epsilon \quad \text{and} \quad c_1 = \inf_{\Lambda} I. \tag{1.1}$$

For $\epsilon^* > 0$ small enough, one can provide a constant $\alpha_0 > 0$ so that the following lower bounds holds:

$$c_{1,\epsilon} \geq \alpha_0 \tag{1.2}$$

$\forall \epsilon \in [0, \epsilon^*]$ (we identify $c_{1,\epsilon=0}$ with c_1). It is well known that the minimization problems in (1.1) achieve their infimum, say at $u_{1,\epsilon}$ and u_1 respectively (cf. [16], [6]). Namely,

$$u_{1,\epsilon} \in \Lambda_\epsilon \quad \text{and} \quad I_\epsilon(u_{1,\epsilon}) = c_{1,\epsilon} \tag{1.3$_\epsilon$}$$

and

$$u_1 \in \Lambda \quad \text{and} \quad I(u_1) = c_1. \tag{1.3}$$

Also, we can choose $u_{1,\epsilon} > 0$ and $u_1 > 0$ on Ω.

Lemma 1.1. *$c_{1,\epsilon} \to c_1$ as $\epsilon \to 0$.*

Proof: Easy computations show that

$$c_{1,\epsilon} \leq \frac{1}{N}(|\Omega| + 1)^{\frac{p}{2}}(\lambda_1 - \lambda + 1)^{\frac{p}{p-2}} \tag{1.4}$$

$\forall\, \epsilon \in (0, \frac{4}{N-2})$, where $|\Omega|$ denote the Lebesgue measure of Ω. Consequently, from (1.2) and (1.4), we derive

$$K_1 \leq \|u_{1,\epsilon}\|_{p_\epsilon} \leq K_2 \tag{1.5}$$

$\forall\, \epsilon \in (0, \epsilon^*)$, with K_1 and K_2 positive constants. Set

$$t_\epsilon = \Big(\frac{\|\nabla u_{1,\epsilon}\|_2^2 - \lambda\|u_{1,\epsilon}\|_2^2}{\|u_{1,\epsilon}\|_p^p}\Big)^{\frac{1}{p-2}}.$$

Hence, $t_\epsilon u_{1,\epsilon} \in \Lambda$ and

$$\begin{aligned} c_1 \leq I(t_\epsilon u_{1,\epsilon}) &= \frac{1}{N}\Big(\frac{\|\nabla u_{1,\epsilon}\|_2^2 - \lambda\|u_{1,\epsilon}\|_2^2}{\|u_{1,\epsilon}\|_p^2}\Big)^{\frac{p}{p-2}} \\ &\leq \frac{1}{N}|\Omega|^{\frac{2\epsilon}{p_\epsilon(p-2)}}\Big(\frac{\|\nabla u_{1,\epsilon}\|_2^2 - \lambda\|u_{1,\epsilon}\|_2^2}{\|u_{1,\epsilon}\|_{p_\epsilon}^2}\Big)^{\frac{p}{p-2}} \\ &= \frac{1}{N}|\Omega|^{\frac{2\epsilon}{p_\epsilon(p-2)}}\|u_{1,\epsilon}\|_{p_\epsilon}^{\frac{p(p_\epsilon-2)}{p-2}}. \end{aligned}$$

So, from (1.5) we conclude

$$c_1 \leq (\frac{1}{2} - \frac{1}{p_\epsilon})\|u_{1,\epsilon}\|_{p_\epsilon}^{p_\epsilon} + 0(\epsilon) = c_{1,\epsilon} + 0(\epsilon).$$

Next, we obtain the reverse inequality. Set

$$\tau_\epsilon = \Big(\frac{\|\nabla u_1\|_2^2 - \lambda\|u_1\|_2^2}{\|u_1\|_{p_\epsilon}^{p_\epsilon}}\Big)^{\frac{1}{p_\epsilon-2}};$$

so, $\tau_\epsilon u_1 \in \Lambda_\epsilon$. Therefore,

$$\begin{aligned} c_{1,\epsilon} \leq I_\epsilon(\tau_\epsilon u_1) &= (\frac{1}{2} - \frac{1}{p_\epsilon})\Big(\frac{\|\nabla u_1\|_2^2 - \lambda\|u_1\|_2^2}{\|u_1\|_{p_\epsilon}^{p_\epsilon}}\Big)^{\frac{p_\epsilon}{p_\epsilon-2}} \\ &= (\frac{1}{2} - \frac{1}{p_\epsilon})\Big(\frac{\|u_1\|_p}{\|u_1\|_{p_\epsilon}}\Big)^{\frac{2p_\epsilon}{p_\epsilon-2}}\|u_1\|_p^{\frac{p_\epsilon(p-2)}{p_\epsilon-2}} = c_1 + 0(\epsilon). \end{aligned}$$

This concludes the proof.

Remark 1.1. It can be shown that for a subsequence $\epsilon_n \to 0$ as $n \to +\infty$, it follows

$$\|u_{1,\epsilon_n} - u_1\| \to 0 \quad \text{as} \quad n \to +\infty.$$

Set

$$\psi(u) = \|\nabla u\|_2^2 - \lambda\|u\|_2^2, \quad u \in H.$$

We have,

Lemma 1.2. *For every $r \in (2, \frac{2N}{N-2})$ and $u \in L^r(\Omega)$, $u \neq 0$, there exist a unique $v = v(u) \in H$ such that*

a) $$\int_\Omega |u|^{r-2} v^2 = 1 \quad and \quad v \geq 0;$$

b) $$\psi(v) = \inf\{\psi(w),\ w \in H : \textstyle\int_\Omega |u|^{r-2} w^2 = 1\} := \mu_1(u). \tag{1.6}$$

Furthermore, the map $L^r(\Omega) \to H_0^1(\Omega)$, $u \to v(u)$ is continuous for every $u \neq 0$.

Remark 1.2. Clearly $(\mu_1(u), v(u))$ corresponds to the first eigenpair of the (weighted) eigenvalue problem

$$-(\Delta + \lambda)v = \mu |u|^{r-2} v \ \text{ on } \ \Omega, \qquad v = 0 \ \text{ on } \ \partial\Omega. \tag{\#}$$

Proof: A simple application of the direct method of calculus of variations shows that the infimum in (1.6) is achieved. Also, any point $w \in H$ where such infimum is achieved cannot change sign in Ω. This gives the uniqueness of $v(u)$ and, therefore, its continuity for $u \neq 0$.

Remark 1.3. It is not difficult to show that $v(u)$ is continuous also with respect to the parameter r. Namely, if $\{u_n\}$ is a sequence in $L^p(\Omega)$, $p \in (2, \frac{2N}{N-2})$, which converges to $u^* \in L^p(\Omega)$, $u^* \neq 0$, and $p_n \in (0, p)$ satisfies $p_n \to p$, then $\|\nabla(v_n - v)\|_2 \to 0$ as $n \to +\infty$, where v_n and v are defined according to Lemma 1.2 with $u = u_n$, $r = p_n$ and $u = u^*$, $r = p$, respectively.

To obtain the statement of Theorem 1 in the "subcritical" case, we shall use the Ljusternik-Schnirelman theory for even functionals (cf. [16]). To this purpose, let $A \subset H$ be a closed, bounded set which is $\mathbb{Z}_2$-symmetric (i.e., $u \in A \Rightarrow -u \in A$). As well known, the Krasnselski genus $i(A)$ is well defined for the set A. Fix $\rho > 0$ and let $S_\rho = \{u \in H : \|u\| = \rho\}$. Define

$$H = \{h : H \to H \text{ odd, homeomorphism } \}$$

and set

$$\mathcal{F}_2 = \{A \text{ closed, } \mathbb{Z}_2\text{-symmetric} : \ i(h(A) \cap S_\rho) \geq 2,\ \forall\, h \in H\}.$$

We have

Proposition 1.1. *For every $\epsilon \in (0, \frac{4}{N-2})$, there exists a non-trivial solution u_ϵ of the Dirichlet problem*

$$-\Delta u = |u|^{p_\epsilon - 2} u + \lambda u \ \text{ on } \ \Omega, \qquad u = 0 \ \text{ on } \ \partial\Omega \tag{$*$}_\epsilon$$

satisfying

$$\int_\Omega |u_\epsilon|^{p_\epsilon - 2} u_\epsilon v(u_\epsilon) = 0,$$

where $v(u_\epsilon)$ is defined according to Lemma 1.2 with $u = u_\epsilon$ and $r = p_\epsilon$. Moreover,

$$I_\epsilon(u_\epsilon) = \inf_{A \in \mathcal{F}_2} \sup_A I_\epsilon.$$

Proof: Set

$$c_{2,\epsilon} = \inf_{A \in \mathcal{F}_2} \sup_A I_\epsilon.$$

Since the map $h : u \to \left(\frac{\|\nabla u\|_2^2 - \lambda \|u\|_2^2}{\|u\|_{p_\epsilon}^{p_\epsilon}}\right)^{\frac{1}{p_\epsilon - 2}} u$ defines an odd homomorphism between S_ρ and Λ_ϵ, we have that $i(A \cap \Lambda_\epsilon) \geq 2$, $\forall\, A \in \mathcal{F}_2$. In particular, $c_{2,\epsilon} \geq c_{1,\epsilon}$.

Step 1. For every $A \in \mathcal{F}_2$, there exists $u \in A \cap \Lambda_\epsilon$ such that

$$\int_\Omega |u|^{p_\epsilon - 2} u \; v_\epsilon(u) = 0,$$

where $v_\epsilon(u)$ is defined by Lemma 1.2 with $r = p_\epsilon$. To see this, notice that the map

$$h : A \cap \Lambda_\epsilon \to \mathbb{R}$$

given by

$$h(u) = \int_\Omega |u|^{p_\epsilon - 2} u \; v_\epsilon(u)$$

defines an odd homeomorphism. Since $i(A \cap \Lambda_\epsilon) \geq 2$, necessarily $0 \in h(A \cap \Lambda_\epsilon)$.

Step 2. If $u \in \Lambda_\epsilon$ and $\int_\Omega |u|^{p_\epsilon - 2} u \; v_\epsilon(u) = 0$, then $I_\epsilon(u) \geq c_{2,\epsilon}$. To obtain this, we follow an idea introduced by C.V. Coffman in [11] for an eigenvalue problem of ordinary differential equations.

Set $v_\epsilon = v_\epsilon(u)$ and let $w_\epsilon = w_\epsilon(u) \in H$ be a minimizer for the problem

$$\mu_2 := \inf\{\psi(w),\ \forall\, w \in H : \textstyle\int_\Omega |u|^{p_\epsilon - 2} v_\epsilon w = 0 \text{ and } \int_\Omega |u|^{p_\epsilon - 2} w^2 = 1\}.$$

Since $u \in \Lambda_\epsilon$ and $\int_\Omega |u|^{p_\epsilon - 2} v_\epsilon u = 0$, we obtain

$$\mu_2 \leq \frac{\|\nabla u\|_2^2 - \lambda \|u\|_2^2}{\|u\|_{p_\epsilon}^{p_\epsilon}} = 1.$$

Let $A = \text{span}\{v_\epsilon,\ w_\epsilon\}$. Clearly, $A \in \mathcal{F}_2$ and $\forall\, w \in A$, $w \neq 0$, we have

$$1 \geq \mu_2 \geq \frac{\|\nabla w\|_2^2 - \|w\|_2^2}{\int_\Omega |u|^{p_\epsilon - 2} w^2}.$$

Take $w_0 \in A$ so that $I_\epsilon(w_0) = \max_A I_\epsilon \geq c_{2,\epsilon}$. Since A is a linear space, we derive that $w_0 \in \Lambda_\epsilon$. Furthermore,

$$1 \geq \frac{\|\nabla w_0\|_2^2 - \lambda \|w_0\|_2^2}{\int_\Omega |u|^{p_\epsilon - 2} w^2} \geq \left(\frac{\|w_0\|_{p_\epsilon}}{\|u\|_{p_\epsilon}}\right)^{p_\epsilon - 2}.$$

Consequently,

$$I_\epsilon(u) = \left(\frac{1}{2} - \frac{1}{p_\epsilon}\right)\|u\|_{p_\epsilon}^{p_\epsilon} \geq \left(\frac{1}{2} - \frac{1}{p_\epsilon}\right)\|w_0\|_{p_\epsilon}^{p_\epsilon} = I_\epsilon(w_0) \geq c_{2,\epsilon}.$$

Notice that this also gives

$$c_{2,\epsilon} = \inf\{I_\epsilon(u),\ \forall\, u \in \Lambda_\epsilon : \textstyle\int_\Omega |u|^{p_\epsilon - 2} v_\epsilon u = 0\}.$$

Step 3. There exists a sequence $\{u_n\} \subset H_0^1(\Omega)$ satisfying

a) $c_{2,\epsilon} \leq I_\epsilon(u_n) < c_{2,\epsilon} + \frac{1}{n}$;

b) $\|I'_\epsilon(u_n)\| \leq \frac{1}{n}$;

c) $\int_\Omega |u_n|^{p_\epsilon - 2} v_\epsilon(u_n) u_n = 0$.

To see this, we use the fact that I_ϵ satisfies the (P.S.) condition [cf. [16]) together with a strengthen version of the deformation theorem obtained in [7]. It gives a homeomorphism $\gamma : H \to H$ and a constant $\delta_n \in (0, \frac{1}{n})$ with the property that $I_\epsilon(\gamma(u)) \leq I_\epsilon(u)$ and if $u \in H$ satisfies $I_\epsilon(u) \leq c_{2,\epsilon} + \delta_n$ and $I_\epsilon(\gamma(u)) \geq c_{2,\epsilon} - \delta_n$, then $\|I'_\epsilon \gamma(u))\| < \frac{1}{n}$ (see [7] Corollary 4). Since I'_ϵ is odd, it is possible to choose γ to be odd as well. From the definition of $c_{2,\epsilon}$, we can find $A_n \in \mathcal{F}_2$ such that $\sup_{A_n} \leq c_{2,\epsilon} + \delta_n$. Clearly $\gamma(A_n) \in \mathcal{F}_2$, and from Steps 1 and 2 we can find $u_n \in \gamma(A_n) \cap \Lambda_\epsilon$ such that $\int_\Omega |u_n|^{p_\epsilon - 2} v_\epsilon(u_n) u_n = 0$ and $I(u_n) \geq c_{2,\epsilon}$.

By the result above, we conclude

$$\|I'_\epsilon(u_n)\| < \frac{1}{n}.$$

To conclude the proof of Proposition 1.1, observe that, by the (P.S.) condition, this sequence is relatively compact in H.

Remark 1.4. From Proposition 1.1 it follows that

$$c_{1,\epsilon} < c_{2,\epsilon}, \quad \epsilon \in (0, \frac{4}{N-2}).$$

Moreover, the proof of Step 2 also gives that $\{1, u_{2,\epsilon}\}$ is the second eigenpair for the eigenvalue problem (#) with $u = u_{2,\epsilon}$ and $r = p_\epsilon$.

Our next goal is to provide a crucial estimate for $c_{2,\epsilon}$. We start with the following:

Calculus Lemma. *For every $1 \leq q \leq 3$, there exist a constant C (depending on q) such that for $\alpha, \beta \in \mathbb{R}$ we have*

$$\Big| |\alpha + \beta|^q - |\alpha|^q - |\beta|^q - q\, \alpha\beta(|\alpha|^{q-2} + |\beta|^{q-2}) \Big| \leq \begin{cases} C|\alpha|\ |\beta|^{q-1} & \text{if } |\alpha| \geq |\beta| \\ C|\alpha|^{q-1}|\beta| & \text{if } |\alpha| \leq |\beta| \end{cases}.$$

For $q \geq 3$, there exists a constant C (depending on q) such that for every $\alpha, \beta \in \mathbb{R}$ we have

$$\Big| |\alpha + \beta|^q - |\alpha|^q - |\beta|^q - q\alpha\beta(|\alpha|^{q-2} + |\beta|^{q-2}) \Big| \leq C(|\alpha|^{q-2}\beta^2 + \alpha^2|\beta|^{q-2}).$$

This lemma is proved in [8] (see Lemma 4).

Our assumption of the dimension N, is needed only to provide the following:

Proposition 1.2. *For $N \geq 6$, there exist $\sigma > 0$ and $\epsilon_0 > 0$ such that*

$$c_{2,\epsilon} \leq c_{1,\epsilon} + \frac{1}{N} S^{\frac{N}{2}} - \sigma$$

$\forall \epsilon \in (0, \epsilon_0)$, where S is the best constant in the Sobolev embedding in $\mathbb{R}^N$ (see [1] and [20]).

Proof: To establish the given estimate, we shall use the external functions for the Sobolev embedding in $\mathbb{R}^N$. A similar approach is used in [10] and [21]. Let u_1 be as defined in (1.3). Without loss of generality, assume that $0 \in \Omega$. Let $d > 0$ be such that

$$\overline{B_d(0)} = \{x \in \mathbb{R}^N : |x| \leq d\} \subset \Omega.$$

Given $\delta > 0$, set

$$u_\delta(x) = \delta^{\frac{N-2}{2}}/(\delta^2 + |x|^2)^{\frac{N-2}{2}}$$

and take $\gamma \in C_0^\infty(B_d(0))$ with $\gamma(x) = 1\ \forall x : |x| < \frac{1}{2}d$ and $0 \leq \gamma(x) \leq 1\ \forall x \in \Omega$. Define

$$U_\delta(x) = \gamma(x)u_\delta(x) \in C_0^\infty(\Omega)$$

and set $A_\delta = span\{u_1, U_\delta\}$. Clearly $A_\delta \in \mathcal{F}_2$, so $c_{2,\epsilon} \leq \sup_{A_\delta} I_\epsilon$. For $s, t \in \mathbb{R}$ we have

$$\begin{aligned} I_\epsilon(su_1 + tU_\delta) \leq\ & \frac{1}{2}s^2(\|\nabla u_1\|_2^2 - \lambda\|u_1\|_2^2) - \frac{|s|^{p_\epsilon}}{p_\epsilon}\|u_1\|_{p_\epsilon}^{p_\epsilon} + st\int_\Omega |u_1|^{p_\epsilon - 1}U_\delta \\ & + \frac{1}{2}t^2(\|\nabla u_\delta\|_2^2 - \lambda\|u_\delta\|_2^2) - \frac{|t|^{p_\epsilon}}{p_\epsilon}\|U_\delta\|_{p_\epsilon}^{p_\epsilon} - st\int_\Omega u_1U_\delta(|su_1|^{p_\epsilon-2} + |tU_\delta|^{p_\epsilon-2}) \\ & + C\Big[\int_{\{|su_1|\geq|tU_\delta|\}} |su_1|\ |tU_\delta|^{p_\epsilon-1} + \int_{\{|su_1|\leq|tU_\delta|\}} |su_1|^{p_\epsilon-1}|tU_\delta|\Big], \end{aligned} \tag{1.7}$$

where we have applied the calculus lemma with $\alpha = su_1$, $\beta = tU_\delta$ and $q = p_\epsilon < 3$. In order to estimate the term in the square bracket, observe that $u_1 \in L^\infty(\Omega)$ and

$$\int_{\{|su_1|\geq|tU_\delta|\}} |su_1|\ |tU_\delta|^{p_\epsilon-1} \leq |s|^{p_\epsilon - \frac{N-1}{N-2}}|t|^{\frac{N-1}{N-2}}\delta^{\frac{N-1}{2}}\|u_1\|_\infty^{p_\epsilon - \frac{N-1}{N-2}}\int_\Omega \frac{dx}{|x|^{N-1}}.$$

Hence, for a suitable constant $k_0 > 0$ we obtain

$$\int_{\{|su_1|\geq|tU_\delta|\}} |su_1|\ |tU_\delta|^{p_\epsilon-1} \leq k_0(|s|^{p_\epsilon} + |t|^{p_\epsilon})\delta^{\frac{N-1}{2}}, \quad \forall \epsilon \in (0, \epsilon^*).$$

In a similar way one derives

$$\int_{\{|su_1|\leq|tU_\delta|\}} |su_1|^{p_\epsilon-1}|tU_\delta| \leq k_0(|s|^{p_\epsilon} + |t|^{p_\epsilon})\delta^{\frac{N-1}{2}}, \quad \forall \epsilon \in (0, \epsilon^*).$$

Notice that k_0 can be taken as small as pleased by taking $B_d(0)$ in a small neighborhood of $\partial\Omega$.

Further careful estimates for the external function U_δ yield:

i) $\|\nabla U_\delta\|_2^2 \leq B + a_o\delta^{N-2}$, with $B = \int |\nabla U_{\delta=1}|^2$ (see [6]);

ii) $\|U_\delta\|_{p_\epsilon}^{p_\epsilon} \geq \delta^\epsilon A - a_0\delta^{\frac{N-2}{2}p_\epsilon}$, with $A = \int |U_{\delta=1}|^p$;

and, for $N \geq 5$,

iii) $\|U_\delta\|_2^2 \geq a_1\delta^2 - a_0\delta^{N-2}$ (see [6]), where a_0 and a_1 are suitable positive constants.

Notice that $S = B/A^{2/p}$. Since $\delta^{\frac{N-2}{2}(1+\epsilon)}U_\delta^{p_\epsilon-1} \in L^1(\mathbb{R}^N)$ corresponds to a mollifier, we also obtain

$$\int_\Omega U_\delta^{p_\epsilon-1}u_1 \le c_0\delta^{\frac{N-2}{2}}, \quad \forall\epsilon\in(0,\epsilon^*) \text{ and } \delta\in(0,1)$$

($C_0 > 0$ a constant). A similar estimate follows for the integral $\int_\Omega U_\delta u_1^{p_\epsilon-1}$ since the function $\frac{1}{|x|^{N-2}}$ belongs to $L^1(\Omega)$. So,

$$\int_\Omega U_\delta u_1^{p_\epsilon} \le c_0\delta^{\frac{N-2}{2}}, \quad \forall\epsilon\in(0,\epsilon^*),$$

and the constant c_0 can be assumed as small as needed by taking $B_d(0)$ in a small neighborhood of $\partial\Omega$.

Substituting in (1.7) and using the fact that u_1 solves $(*)$, we derive

$$\begin{aligned} I_\epsilon(su_1+tU_\delta) \le \frac{1}{2}s^2\|u_1\|_p^p - \frac{|s|^{p_\epsilon}}{p_\epsilon}\|u_1\|_{p_\epsilon}^{p_\epsilon} + \frac{t^2}{2}(B-\lambda a_1\delta^2) \\ - \frac{|t|^{p_\epsilon}}{p_\epsilon}\delta^\epsilon A + k_1(s^2+t^2)\delta^{\frac{N-2}{2}} + k_2(|s|^{p_\epsilon}+|t|^{p_\epsilon})\delta^{\frac{N-2}{2}}(1+\sqrt{\delta}), \end{aligned}$$

$\epsilon\in(0,\epsilon^*)$, $\delta\in(0,1)$ and suitable small constants $k_1>0$ and $k_2>0$. Thus for $\delta^*>0$ sufficiently small, we can find constants $R_0>0$ and $\theta>1$ (independent of δ and ϵ) such that

$$I_\epsilon(su_1+tU_\delta)\le 0, \quad \forall\, s,t : \sqrt{s^2+t^2}\ge R_0\delta^{-\epsilon\theta},$$

$\epsilon\in(0,\epsilon^*)$ and $\delta\in(0,\delta^*)$.

On the other hand, if $\sqrt{s^2+t^2}\le R_0\delta^{-\epsilon\theta}$, then

$$\begin{aligned} &I_\epsilon(su_1+tU_\delta) \\ &\le \frac{s^2}{2}\|u_1\|_p^p - \frac{|s|^{p_\epsilon}}{p_\epsilon}\|u_1\|_{p_\epsilon}^{p_\epsilon} + \frac{t^2}{2}(B-\lambda a_1\delta^2) - \frac{1}{p_\epsilon}|t|^{p_\epsilon}\delta^\epsilon A + k_3\delta^{\frac{N-2}{2}-p\theta\epsilon} \\ &\le \Big(\frac{1}{2}-\frac{1}{p_\epsilon}\Big)\Big(\frac{\|u_1\|_p^p}{\|u_1\|_{p_\epsilon}^2}\Big)^{\frac{p_\epsilon}{p_\epsilon-2}} + \Big(\frac{1}{2}-\frac{1}{p_\epsilon}\Big)\frac{(B-\lambda a_1\delta^2)^{\frac{p_\epsilon}{p_\epsilon-2}}}{(\delta^\epsilon A)^{\frac{2}{p_\epsilon-2}}} + k_3\delta^{\frac{N-2}{2}-p\theta\epsilon} \end{aligned}$$

for a suitable small positive constant k_3. Thus, for positive constants α, β and γ, we derive

$$I_\epsilon(su_1+tU_\delta)\le\frac{1}{N}\|u_1\|_p^p+\frac{1}{N}S^{\frac{N}{2}}+\alpha\epsilon(1+\delta^{-\beta\epsilon})+\alpha(\delta^{-\beta\epsilon}-1)-\gamma\delta^2+k_3\delta^{\frac{N-2}{2}-p\theta\epsilon},$$

$\epsilon\in(0,\epsilon^*)$, $\delta\in(0,\delta^*)$. Since $N\ge 6$, and it is always possible to arrange $k_3<\gamma/2$, we obtain that, for a fixed $\delta_0\in(0,\delta^*)$, it is possible to find $\epsilon_0\in(0,\epsilon^*)$ small enough so that

$$\gamma - k_3\delta_0^{\frac{N-6}{2}-p\theta\epsilon}\ge\frac{1}{2}\gamma,$$

for every $\epsilon\in(0,\epsilon_0)$.

Set $\alpha_0 = \alpha(1+\delta_0^{-\beta\epsilon_0})$ and $2\sigma = \frac{1}{2}\gamma\delta_0^2$; from the estimate above we obtain

$$\begin{aligned} c_{2,\epsilon} &\leq \sup_{A_{\delta_0}} I_\epsilon \leq I(u_1) + \frac{1}{N}S^{\frac{N}{2}} + \alpha_0\epsilon + \alpha(\delta_0^{-\beta\epsilon} - 1) - 2\sigma \\ &\leq c_{1,\epsilon} + \frac{1}{N}S^{\frac{N}{2}} + [c_1 - c_{1,\epsilon} + \alpha_0\epsilon + \alpha(\delta_0^{-\beta\epsilon} - 1)] - 2\sigma. \end{aligned}$$

The conclusion now follows since the term in the square bracket tends to zero as $\epsilon \to 0$.

We can finally conclude:

Proof of Theorem 1: It is not difficult to see that $c_{2,\epsilon}$ is bounded uniformly in ϵ. Hence,

$$\|\nabla u_\epsilon\|_2 \leq K, \quad \forall \epsilon \in (0,\epsilon_0), \text{ for a suitable constant } K > 0.$$

For $x \in \Omega$, define $u_\epsilon^+(x) = \max\{u_\epsilon(x), 0\}$ and $u_\epsilon^-(x) = \max\{-u_\epsilon(x), 0\}$. Clearly, $u_\epsilon^\pm \neq 0$ and $u_\epsilon^\pm \in H$. In addition,

$$\|\nabla u_\epsilon^\pm\| \leq K, \quad \forall \epsilon \in (0,\epsilon_0). \tag{1.8}$$

Thus, we can find $\epsilon_n \to 0$ as $n \to +\infty$, u^+, $u^- \in H$ such that

$$u_{\epsilon_n}^\pm \rightharpoonup u^\pm \text{ weakly in } H \text{ as } n \to +\infty.$$

We claim that $u^+ \neq 0$ and $u^- \neq 0$. To shorten notation, set $u_n^\pm = u_{\epsilon_n}^\pm$, $c_{1,n} = c_{1,\epsilon_n}$, $p_n = p_{\epsilon_n}$, $I_n = I_{\epsilon_n}$ and $\Lambda_n = \Lambda_{\epsilon_n}$. Since u_n satisfies $(*)_{\epsilon_n}$, we have that $u_n^\pm \in \Lambda_n$. In particular,

$$I_n(u_n^\pm) \geq c_{1,n}. \tag{1.9}$$

From Proposition 1.2, we also know that

$$I_n(u_n^+) + I_n(u_n^-) = I_n(u_n) = c_{2,\epsilon_n} \leq c_{1,n} + \frac{1}{2}S^{\frac{N}{2}} - \sigma$$

for n large. Necessarily,

$$I_n(u_n^\pm) \leq \frac{1}{2}S^{\frac{N}{2}} - \sigma \tag{1.10}$$

for n large. From (1.8) and the fact that $u_n^\pm \in \Lambda_n$, we derive

$$K_1 \leq \|u_n^\pm\|_p \leq K_2 \tag{1.11}$$

with suitable positive constants K_1 and K_2.

Arguing by contradiction, assume, for example, that $u^+ = 0$. From (1.10) and the fact that $u_n^\pm \in \Lambda_n$, we obtain

$$\frac{1}{2}\|\nabla u_n^+\|_2^2 - \frac{1}{p_n}\|u_n^+\|_{p_n}^{p_n} \leq \frac{1}{N}S^{\frac{N}{2}} - \sigma + o(1) \tag{1.12}$$

and

$$\|\nabla u_n^+\|_2^2 - \|u_n^+\|_{p_n}^{p_n} = o(1). \tag{1.13}$$

Consequently,

$$S\|u_n^+\|_p^2 \le \|\nabla u_n^+\|_2^2 = \|u_n^+\|_{p_n}^{p_n} + o(1) \le \|u_n^+\|_{p_n}^{p_n-2}|\Omega|^{\frac{p-p_n}{p}}\|u_n\|_p^2 + o(1).$$

Since $\|u_n^+\|_p$ is bounded away from zero (see (1.11)), we conclude

$$\|u_n^+\|_{p_n}^{p_n-2} \ge |\Omega|^{\frac{p_n-p}{p}} S + o(1) = S + o(1).$$

That is,

$$\|u_n^+\|_{p_n}^{p_n} \ge S^{\frac{N}{2}} + o(1).$$

This contradicts (1.12) since from (1.13) we have

$$\frac{1}{N}S^{\frac{N}{2}} + o(1) \le \frac{1}{N}\|u_n\|_{p_n}^{p_n} = \frac{1}{2}\|\nabla u_n^+\|_2^2 - \frac{1}{p_n}\|u_n\|_{p_m}^{p_n} + o(1) \le \frac{1}{N}S^{\frac{N}{2}} - \sigma + o(1).$$

Similarly, one shows that $u^- \neq 0$.

Set $u = u^+ - u^-$. Clearly u changes sign in Ω (in particular $u \neq 0$) and

$$u_n := u_{\epsilon_n} \rightharpoonup u \quad \text{weakly in} \quad H.$$

Consequently, $\langle I'(u), w\rangle = 0$, $\forall\, w \in H$; i.e., u is a (weak) nodal solution for $(*)$. In fact, a subsequence of u_n converges strongly to u in H. To see this, notice that $u \in \Lambda$, hence $I(u) \ge c_1$. Set $u_n = u + w_n$, with $w_n \rightharpoonup 0$ weakly in H. We have

$$\begin{aligned} c_{1,n} + \frac{1}{N}S^{\frac{N}{2}} - \sigma &\ge I_n(u + w_n) \\ &= \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p_n}\|u\|_{p_n}^{p_n} - \frac{\lambda}{2}\|u\|_2^2 + \frac{1}{2}\|\nabla w_n\|_2^2 - \frac{1}{p_n}\|w_n\|_{p_n}^{p_n} + o(1) \\ &\ge c_1 + \frac{1}{2}\|\nabla w_n\|_2^2 - \frac{1}{p_n}\|w_n\|_{p_n}^{p_n} + o(1). \end{aligned}$$

Since $|c_{1,n} - c_1| = o(1)$, we derive

$$\frac{1}{2}\|\nabla w_n\|_2^2 - \frac{1}{p_n}\|w_n\|_{p_n}^{p_n} \le \frac{1}{N}S^{\frac{N}{2}} - \sigma + o(1). \tag{1.15}$$

Furthermore,

$$0 = \langle I_n'(u_n),\, u_n\rangle = \langle I'(u),\, u\rangle + \|\nabla w_n\|_2^2 - \|w_n\|_{p_n}^{p_n} + o(1);$$

that is,

$$\|\nabla w_n\|_2^2 - \|w_n\|_{p_n}^{p_n} = o(1). \tag{1.16}$$

As above, one sees how conditions (1.15) and (1.16) imply that the sequence $\|\nabla w_n\|_2$ cannot be bounded away from zero. Hence, $\{w_n\}$ must admit a subsequence which strongly converges to zero.

2. Proof of Theorem 2: Theorem 2 is derived in an analogous way. Define

$$F_\epsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{q}\int_\Omega |u|^q - \frac{1}{p_\epsilon}\int_\Omega |u|^{p_\epsilon}, \quad u \in H,$$

(so $F = F_{\epsilon=0}$) and

$$\Gamma_\epsilon = \{u \neq 0,\ \langle F'_\epsilon(u), u\rangle = 0\}, \quad \Gamma = \{u \neq 0,\ \langle F(u), u\rangle = 0\}.$$

Easy computations show that F_ϵ and F are bounded below in Γ_ϵ and Γ, respectively. Set

$$0 < \gamma_{1,\epsilon} = \inf_{\Gamma_\epsilon} F_\epsilon, \quad 0 < \gamma_1 = \inf_{\Gamma} F. \tag{2.1}$$

Alternatively, one can characterize $\gamma_{1,\epsilon}$ and γ_1 via a mountain-pass principle (notice that both Γ_ϵ and Γ divide H in exactly two components) and obtain that the minimization problems in (2.1) achieve their infimum (cf. [16], [6]). Let

$$u_{1,\epsilon} \in \Gamma_\epsilon \,:\, F_\epsilon(u_{1,\epsilon}) = \gamma_{1,\epsilon} \tag*{(2.2)$_\epsilon$}$$

and

$$u_1 \in \Gamma \,:\, F(u_1) = \gamma_1. \tag{2.2}$$

We can also assume $u_{1,\epsilon} > 0$ and $u_1 > 0$ in Ω.

Lemma 2.1. *$\gamma_{1,\epsilon} \to \gamma_1$ as $\epsilon \to 0$.*

Proof: It is not difficult to show that for $\epsilon > 0$ and suitable constants $C_1 > 0$ and $C_2 > 0$, one has

$$C_1 \leq \|u_{1,\epsilon}\|_{p_\epsilon} \leq C_2.$$

Similar estimates also hold for $\|\nabla u_{1,\epsilon}\|_2$ and $\|u_{1,\epsilon}\|_q$. Notice that for every $u \neq 0$, there exist unique $t_\epsilon(u) > 0$ and $t(u) > 0$ such that

$$t(u)u \in \Gamma \ \text{ and } \ t_\epsilon(u)u \in \Gamma_\epsilon.$$

Furthermore, $t_\epsilon(u) \to t(u)$ as $\epsilon \to 0$. Set $s_\epsilon = t(u_{1,\epsilon})$, so $s_\epsilon u_{1,\epsilon} \in \Gamma$. We have

$$\begin{aligned} \gamma_1 &\leq F(s_\epsilon u_{1,\epsilon}) \\ &= (\frac{1}{2} - \frac{1}{p_\epsilon})s_\epsilon^2\|\nabla u_{1,\epsilon}\|_2^2 + (\frac{1}{p_\epsilon} - \frac{1}{p})s_\epsilon^2\|\nabla u_{1,\epsilon}\|_2^2 + \lambda(\frac{1}{p} - \frac{1}{q})s_\epsilon^q\|u_{1,\epsilon}\|_q^q \\ &= F_\epsilon(u_{1,\epsilon}) + (\frac{1}{2} - \frac{1}{p_\epsilon})(s_\epsilon^2 - 1)\|\nabla u_{1,\epsilon}\|_2^2 + \lambda(\frac{1}{p_\epsilon} - \frac{1}{q})(s_\epsilon^q - 1)\|u_{1,\epsilon}\|_q^q \\ &\quad + (\frac{1}{p_\epsilon} - \frac{1}{p})s_\epsilon^2\|\nabla u_{1,\epsilon}\|_2^2 + \lambda(\frac{1}{p} - \frac{1}{p_\epsilon})s_\epsilon^q\|u_{1,\epsilon}\|_q^q. \end{aligned} \tag{2.3}$$

Since

$$(s^{p_\epsilon} - 1)\|u_{1,\epsilon}\|_{p_\epsilon}^{p_\epsilon} + \lambda(s_\epsilon^q - 1)\|u_{1,\epsilon}\|_q^q = o(1)$$

and $\|u_{1,\epsilon}\|_{p_\epsilon}$, $\|u_{1,\epsilon}\|_q$ are bounded above and below uniformly in ϵ, we conclude

$$|s_\epsilon - 1| = o(1).$$

Hence, from (2.3), we derive

$$\gamma_1 \leq \gamma_{1,\epsilon} + o(1).$$

To obtain the reverse inequality, set $t_\epsilon = t_\epsilon(u_1) > 0$. Thus, $t_\epsilon u_1 \in \Gamma_\epsilon$, $t_\epsilon \to 1$ as $\epsilon \to 0$ and

$$\begin{aligned} \gamma_{1,\epsilon} \leq F_\epsilon(t_\epsilon u_1) &= F(t_\epsilon u_1) + \frac{1}{p}\|t_\epsilon u_1\|_p^p - \frac{1}{p_\epsilon}\|t_\epsilon u_1\|_{p_\epsilon}^{p_\epsilon} \\ &= F(u_1) + o(1) = \gamma_1 + o(1). \end{aligned}$$

As for Lemma 1.3, we obtain:

Lemma 2.2. *For every $2 < q < r \leq \frac{2N}{N-2}$, $\lambda > 0$ and $u \in L^r(\Omega)$, $u \neq 0$, there exists a unique $v_q = v_q(u) \in H_0^1(\Omega)$ such that*

$$\text{a)} \qquad \int_\Omega (|u|^{r-2} + \lambda |u|^{q-2}) v_q^2 = 1, \quad v_q \geq 0;$$

$$\text{b)} \qquad \|\nabla v_q\|_2^2 = \inf \left\{ \|\nabla v\|_2^2 : \int_\Omega (|u|^{r-2} + \lambda |u|^{q-2}) v^2 = 1 \right\}.$$

Furthermore, the map $L^r(\Omega) \to H$, $u \to v_q(u)$ is continuous.

In order to obtain the equivalent of Proposition 1.1, we need the following:

Lemma 2.3. *For $s > 2$ and $x, y \in \mathbb{R}$, we have*

$$\frac{1}{s}(|x|^s - |y|^s) \geq \frac{1}{2}(x^2 - y^2)|y|^{s-2}. \tag{2.4}$$

Proof: Inequality (2.4) is equivalent to

$$\frac{1}{s}(t^s - 1) \geq \frac{1}{2}(t^2 - 1), \quad \forall t \geq 0,$$

which is trivially satisfied.

Proposition 2.1. *For $\epsilon > 0$ small enough, there exists a nontrivial solution u_ϵ of the problem*

$$\begin{cases} -\Delta u = |u|^{p_\epsilon - 2} u + \lambda |u|^{q-2} u & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

satisfying

$$\int_\Omega v_{q,\epsilon}(u_\epsilon) u_\epsilon (|u_\epsilon|^{p_\epsilon - 2} + \lambda |u_\epsilon|^{q-2}) = 0,$$

where $v_{q,\epsilon}(u_\epsilon)$ is defined in Lemma 2.2 with $u = u_\epsilon$ and $r = p_\epsilon$. Furthermore,

$$F_\epsilon(u_\epsilon) = \inf_{A \in \mathcal{F}_2} \sup_A F_\epsilon.$$

Proof: Set

$$\gamma_{2,\epsilon} = \inf_{A \in \mathcal{F}_2} \sup_A F_\epsilon.$$

For $u \neq 0$, let $t_\epsilon(u) > 0$ be the unique value such that $t_\epsilon(u) u \in \Gamma_\epsilon$. Clearly, $t_\epsilon(u) = t_\epsilon(|u|) = t_\epsilon(-u)$ and

$$F_\epsilon(t_\epsilon(u)u) = \max_{t \geq 0} F(tu).$$

The uniqueness of $t_\epsilon(u)$ and its properties give that the map $u \to t_\epsilon(u)$ is continuous for every $u \neq 0$ in H. Consequently, the map $u \to t_\epsilon(u)u$ defines an odd homeomorphism between S_ρ and Γ_ϵ which gives $i(A \cap \Gamma_\epsilon) \geq 2$, $\forall A \in \mathcal{F}_2$. Therefore, $\forall A \in \mathcal{F}_2$, the set $A \cap \Gamma_\epsilon$ must contain an element u satisfying

$$\int_\Omega (|u|^{p_\epsilon - 2} + \lambda |u|^{q-2}) v_{q,\epsilon}(u) u = 0,$$

where $v_{q,\epsilon}(u)$ is given by Lemma 2.2 with $r = p_\epsilon$.

Claim. For every $u \in \Gamma_\epsilon$ satisfying

$$\int_\Omega (|u|^{p_\epsilon - 2} + \lambda |u|^{q-2}) v_{q,\epsilon}(u) u = 0,$$

we have $F_\epsilon(u) \geq \gamma_{2,\epsilon}$. Indeed, let

$$\mu_{2,\epsilon} = \inf\Big\{ \|\nabla w\|^2 : \textstyle\int_\Omega (|u|^{p-2} + \lambda |u|^{q-2}) w^2 = 1, \int_\Omega (|u|^{p_\epsilon - 2} + \lambda |u|^{q-2}) v_{q,\epsilon}(u) w = 0 \Big\}. \tag{2.5}$$

Since $u \in \Gamma_\epsilon$, we have

$$\mu_{2,\epsilon} \leq \frac{\|\nabla u\|_2^2}{\int_\Omega (|u|^p + \lambda |u\ |^q)} = 1.$$

Let $w_\epsilon = w_\epsilon(u)$ be a minimizer for (2.5) and define $A = \operatorname{span}\{v_{q,\epsilon}, w_\epsilon\}$. Clearly, $A \in \mathcal{F}_2$ and

$$1 \geq \mu_{2,\epsilon} \geq \frac{\|\nabla w\|_2^2}{\int_\Omega (|u|^{p-2} + \lambda |u|^{q-2}) w^2}, \quad \forall w \in A,\ w \neq 0. \tag{2.6}$$

For $w_0 \in A$ satisfying $F_\epsilon(w_0) = \sup_A F_\epsilon \geq \gamma_{2,\epsilon}$, we have $w_0 \neq 0$ and $w_0 \in \Gamma_\epsilon$. From (2.6) we derive

$$\int_\Omega (|u|^{p_\epsilon - 2} + \lambda |u|^{q-2}) w_0^2 \geq \|\nabla w_0\|_2^2,$$

which implies

$$\frac{1}{2} \int_\Omega (|u|^{p_\epsilon - 2} + \lambda |u|^{q-2})(w_0^2 - u^2) \geq \frac{\|\nabla w_0\|_2^2}{2} - \frac{\|\nabla u\|_2^2}{2}.$$

Applying Lemma 2.3 with $s = p_\epsilon$ and $s = q$, respectively, we conclude

$$\frac{1}{p_\epsilon} \int_\Omega |w_0|^{p_\epsilon} + \frac{\lambda}{q} \int_\Omega |w_0|^q - \frac{1}{p_\epsilon} \int_\Omega |u|^{p_\epsilon} - \frac{\lambda}{q} \int_\Omega |u|^q \geq \frac{\|\nabla w_0\|_2^2}{2} - \frac{\|\nabla u\|_2^2}{2};$$

that is,

$$F_\epsilon(u) \geq F_\epsilon(w_0) \geq \gamma_{2,\epsilon}.$$

At this point the conclusion follows as in Proposition 1.1. We leave the details to the reader.

Lemma 2.4. *Given $2 < q < \frac{2N}{N-2}$ and a smooth function u_1, there exist positive constant c_0 and θ (depending on q and u_1) such that*

$$\int_\Omega u_1 U_\delta^{q-1} \leq c_0 \delta^{\frac{N+2}{2} + \theta}.$$

Proof: We distinguish the following cases:

i) $2 < q < \frac{2N-2}{N-2}$, in such a situation one easily checks that $U_\delta^{q-1} \in L^1(\Omega)$, therefore

$$\int_\Omega u_1 U_\delta^{q-1} \leq c_1 \delta^{\frac{(N-2)}{2}(q-1)} \quad (c_1 > 0 \text{ constant})\ .$$

ii) $q = \frac{2(N-1)}{N-2}$, we have

$$\int_\Omega u_1 U_\delta^{q-1} = \delta^{\frac{N}{2}} \int_{\{|x|\leq d\}} \frac{u_1}{(\delta^2+|x|^2)^{\frac{N}{2}}},$$

thus, in this case, we obtain

$$\int_\Omega u_1 U_\delta^{q-1} \leq c_2 \delta^{\frac{N}{2}} |\ln \delta| \quad (c_2 > 0 \text{ constant}).$$

iii) $q > \frac{2(N-1)}{N-2}$, easy computations show that, in such a situation, the function $\delta^{\frac{(N-2)}{2}(q-1)-N} U_\delta \in L^1(\mathbb{R}^N)$ define a mollifier. Consequently,

$$\int_\Omega u_1 U_\delta^{q-1} \leq c_3 \delta^{N-\frac{(N-2)}{2}(q-1)} \quad (c_3 > 0 \text{ constant})$$

and $N - \frac{(N-2)}{2}(q-1) > \frac{N-2}{2}$.

Proposition 2.2. *Let q satisfy the assumptions of Theorem 2. There exist $\sigma > 0$ and $\epsilon_0 > 0$ such that*

$$\gamma_{2,\epsilon} \leq \gamma_{1,\epsilon} + \frac{1}{2} S^{\frac{N}{2}} - \sigma \quad \text{for } 0 < \epsilon < \epsilon_0. \tag{2.7}$$

Proof: As for Proposition 1.2, we shall obtain (2.7) by estimating $\sup_{A_\delta} F_\epsilon$ where $A_\delta = span\{u_1, U_\delta\} \in \mathcal{F}_2$ and u_1 is defined in (2.2). Apply the calculus lemma to both terms $\int_\Omega |su_1 + tU_\epsilon|^q$ and $\int_\Omega |su_1 + tU_\epsilon|^{p_\epsilon}$, $s, t \in \mathbb{R}$. For $\epsilon > 0$ and $\delta > 0$ small, the following holds:

$$\begin{aligned} F_\epsilon(su_1 + tU_\delta) \leq & \frac{s^2}{2}\|\nabla u_1\|_2^2 - \frac{|s|^{p_\epsilon}}{p_\epsilon}\|u_1\|_{p_\epsilon}^{p_\epsilon} - \lambda \frac{|s|^q}{q}\|u_1\|_q^q \\ & + \frac{t^2}{2}\|\nabla U_\delta\|_2^2 - \frac{|t|^{p_\epsilon}}{p_\epsilon}\|U_\delta\|_{p_\epsilon}^{p_\epsilon} - \lambda\frac{|t|^q}{q}\|U_\delta\|_q^q + st\int_\Omega \nabla u_1 \cdot \nabla U_\delta \\ & - \lambda\, st \int_\Omega u_1 U_\delta (|su_1|^{q-2} + |tU_\delta|^{q-2}) - st\int_\Omega u_1 U_\delta(|su_1|^{p_\epsilon - 2} + |tU_\delta|^{p_\epsilon-2}) \\ & + R_\delta(|t|^q + |s|^q) + (|t|^{p_\epsilon} + |s|^{p_\epsilon}) S_\delta \end{aligned}$$

and $R_\delta = R_\delta(q)$, $S_\delta = S_\delta(\epsilon)$ are estimated as follows.

In case $N \geq 6$ (i.e., $2 < p$, $q \leq 3$), as for the previous section, we have

$$R_\delta \leq k_0 \delta^{\frac{N-1}{2}} \quad \text{and} \quad S_\delta \leq k_0 \delta^{\frac{N-1}{2}}$$

for a suitable constant $k_0 > 0$.

In case $N = 5$ (i.e., $p_\epsilon > 3$, $\epsilon > 0$ small), then one easily checks that

$$S_\delta \leq k_0 \delta^{2-\epsilon\frac{N-2}{2}}.$$

For R_δ we have estimates as above in case $\frac{N+2}{N-2} \le q < 3$. While for $q > 3$, using a similar analysis of that of Lemma 2.4, one derives

$$R_\delta \le k_0(\delta^{\frac{N-2}{2}\beta} + \delta^2),$$

where $\beta > 1$ is a constant depending on q only.

In case $N = 3,\ 4$, then $p > 3$ and the given assumptions on q also give $q \ge 3$. So,

$$S_\delta \le \begin{cases} K_0 \delta^{2-\epsilon\frac{N-2}{2}} & \text{for } N = 4 \\ k_0\delta & \text{for } N = 3 \end{cases}$$

and

$$R_\delta \le k_0 \delta^{\frac{N-2}{2}\beta} + \begin{cases} K_0\delta^2 \log \delta & \text{for } N = 4 \\ k_0 \delta & \text{for } N = 3 \end{cases}$$

($k_0 > 0$ constant). In conclusion, $S_\delta = o(\delta^{\frac{N-2}{2}})$ and $R_\delta = o(\delta^{\frac{N-2}{2}})$ for $N \ge 3$, q satisfying the given assumptions and $\epsilon > 0$ small.

Next, notice that $\int_\Omega |\nabla u_1 \cdot \nabla U_\delta| \le c_0 \delta^{\frac{N-2}{2}}$ (see [6]) and, as observed in the previous section,

$$\int_\Omega u_1^{q-1} U_\delta \le c_0 \delta^{\frac{N-2}{2}},$$

where the constant $c_0 > 0$ can be arranged as small as needed by taking $B_d(0)$ in a small neighborhood of $\partial\Omega$. Furthermore, under the given assumptions on q, $U_\delta^q \in L^1(\mathbb{R}^N)$ and $\delta^{q\frac{(N-2)}{2}-N} U_\delta^q$ defines a mollifier. Hence,

$$\int_\Omega U_\delta^q \le C\ \delta^{N - \frac{N-2}{2}q}$$

with

$$C = \int_{\mathbb{R}^N} \frac{1}{(1+|x|^2)^{\frac{(N-2)}{2}q}}\,.$$

Set $\tau = N - \frac{(N-2)q}{2}$. This, together with Lemma 2.4 and the estimates above, yields

$$\begin{aligned} F_\epsilon(su_1 + tU_\delta) \le{}& \frac{s^2}{2}\|\nabla u_1\|_2^2 - \frac{|s|^{p_\epsilon}}{p_\epsilon}\|u_1\|_{p_\epsilon}^{p_\epsilon} - \lambda\frac{|s|^q}{q}\|u_1\|_q^q + \frac{t^2}{2}B - \frac{A\delta^\epsilon}{p_\epsilon}|t|^{p_\epsilon} \\ & - \frac{\lambda}{q}C\delta^\tau + k_1(s^2 + t^2 + |s|^q + |t|^q + |s|^{p_\epsilon} + |t|^{p_\epsilon})(\delta^{\frac{N-2}{2}} + o(\delta^{\frac{N-2}{2}})) \end{aligned}$$

($k_1 > 0$ small constant). Therefore, for $\delta_* > 0$ and $\epsilon_* > 0$ sufficiently small, we can find $R_0 > 0$ and $\beta > 0$ such that for $\delta \in (0,\ \delta_*)$ and $\epsilon \in (0,\ \epsilon_*)$, we have

$$F_\epsilon(su_1 + tU_\delta) \le 0, \quad \forall\, s, t : \sqrt{s^2+t^2} \ge R_0\delta^{-\epsilon\beta}.$$

On the other hand, if $\sqrt{s^2+t^2} \le R_0\delta^{-\epsilon\beta}$, then

$$\begin{aligned} F_\epsilon(su_1 + tU_\delta) \le{}& F(u_1) + \frac{t^2}{2}B - \frac{|t|^p}{p}A - \frac{\lambda}{q}C|t|^q\delta^\tau \\ & + (\frac{|s|^p}{p} - \frac{|s|^{p_\epsilon}}{p_\epsilon})\|u_1\|_{p_\epsilon}^{p_\epsilon} + A(\frac{|t|^p}{p} - \frac{|t|_\epsilon^p}{p_\epsilon}) + k_2\delta^{\frac{N-2}{2}-\gamma\epsilon}, \end{aligned}$$

for a suitable $\gamma > 0$ and $k_2 > 0$ an arbitrary small constant. That is,

$$F_\epsilon(su_1 + tU_\delta) \le \gamma_1 + S_* + c_1\epsilon(1+\delta^{-\theta_*\epsilon}) + c_1\delta^{-\theta_*\epsilon}(1-\delta^\epsilon) + k_2\delta^{\frac{N-2}{2}-\gamma\epsilon}, \quad (2.8)$$

where $S_* = \max_{t\ge 0}\{\frac{t^2}{2}B - \frac{At^p}{p} - \frac{\lambda C}{q}t^q\delta^\tau\}$ and $c_1 > 0$, $\theta_* > 0$ are suitable constants. Let $t_\delta > 0$ be the unique point where S_* is achieved. Hence,

$$B - At_\delta^{p-2} = \lambda C\ \delta^\tau t_\delta^{q-2}. \quad (2.9)$$

Set $t_0 = (\frac{B}{A})^{\frac{1}{p-2}}$; we have $0 < t_\delta < t_0$ and $t_\delta \to t_0$ as $\delta \to 0$. Write $t_\delta = t_0(1-c_\delta)$ with $c_\delta > 0$ and $c_\delta \to 0$ as $\delta \to 0$. From (2.9) we obtain

$$B(1-(1-c_\delta)^{p-2}) = \lambda\ Ct_0^{q-2}(1-c_\delta)\delta^\tau;$$

that is,

$$B(p-2)c_\delta(1+o(\delta^\tau)) = \lambda\ Ct_0^{q-2}\delta^\tau + o(\delta^\tau)$$

which gives

$$c_\delta = D\delta^\tau + o(\delta^\tau).$$

with $D = \frac{\lambda Ct_0^{q-2}}{B(p-2)}$. Consequently,

$$S_* = (\frac{1}{2} - \frac{1}{p})t_\delta^2 B - \lambda(\frac{1}{q} - \frac{1}{p})Ct_\delta^q\delta^\tau = \frac{1}{N}S^{\frac{N}{2}} - 2D_*\delta^\tau + o(\delta^\tau)$$

for suitable $D_* > 0$. Hence, for $\delta_* > 0$ sufficiently small, we derive

$$S_* \le \frac{1}{N}S^{\frac{N}{2}} - D_*\delta^\tau,$$

for every $0 < \delta < \delta_*$. Substituting in (2.8), we conclude

$$\sup_{A_\delta} F_\epsilon \le \gamma_1 + \frac{1}{N}S^{\frac{N}{2}} - D_*\delta^\tau + c_1\epsilon(1+\delta^{-\theta_*\epsilon}) + c_1\delta^{-\theta_*\epsilon}(1-\delta^\epsilon) + k_2\delta^{\frac{N-2}{2}-\gamma\epsilon},$$

for $\epsilon > 0$ and $\delta > 0$ small. Since we can always arrange that $k_2 < \frac{1}{3}D_*$ for a fixed $0 < \delta_0 < \delta_*$, we can find a small ϵ_0 so that

$$D_* - k_2\delta_0^{-\gamma\epsilon}\delta_0^{\frac{N-2}{2}-\tau} \ge \frac{1}{2}D_*$$

for every $0 < \epsilon < \epsilon_0$. Notice that our assumptions on q guarantee $\frac{N-2}{2} - \tau \ge 0$. Thus, for $\epsilon \in (0, \epsilon_0)$ and $2\sigma = \frac{1}{2}D_*\delta_0$, we conclude

$$\gamma_{2,\epsilon} \le \sup_{A_{\delta_0}} F_\epsilon \le \gamma_{1,\epsilon} + \frac{1}{2}S^{\frac{N}{2}} - 2\sigma + [\epsilon c_1(1+\delta_0^{-\theta_*\epsilon}) + (1-\delta_0\epsilon)c_1\delta^{-\theta_*\epsilon} + \gamma_1 - \gamma_{1,\epsilon}],$$

and the term in the square bracket tends to zero as $\epsilon \to 0$.

With the obvious modifications, the conclusion of Theorem 2 now follows as in Theorem 1. The details are left to the reader.

REFERENCES

[1] Th. Aubin, "Nonlinear Analysis on Manifold, Monge-Ampere Equation," Springer, Berlin-New York, 1982.

[2] F.V. Atkinson, H. Brezis and L.A. Peletier, *Nodal solutions of elliptic equations with critical Sobolev exponent,* J. Diff. Eqs., 85 (1990), 151–170.

[3] ??? Adimurthi and S.L. Yadava, *Elementary proof of the nonexistence of nodal solutions for the semilinear elliptic equations with critical Sobolev exponent,* Nonlinear Anal., T.M.A., 14 (1990), 785–787.

[4] H. Brezis, *Some variational problems with lack of compactness,* Proc. Symp. Pure Math., Vol. 45, Part 1 (F. Brower Ed.), Amer. Math. Soc. (1986), 165–201.

[5] H. Brezis and T. Kato, *Remarks on the Schrödinger operator with singular complex potential,* J. Math. Pure Appl., 58 (1979), 137–151.

[6] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents,* Comm. Pure Appl. Math., 36 (1983), 437–477.

[7] H. Brezis and L. Nirenberg, *Remarks on finding critical points,* preprint.

[8] H. Brezis and L. Nirenberg, *A minimization problem with critical exponent and non-zero data,* in "Symmetry in Nature", Scuola Norm. Sup. Pisa (1989), 129–140.

[9] A. Capozzi, D. Fortunato and G. Palmiere, *An existence result for nonlinear elliptic problems involving critical Sobolev exponent,* Ann. I.H.P. Analyse Non Lineaire, 2 (1985), 463–470.

[10] G. Cerami, S. Solimini and M. Struwe, *Some existence results for superlinear elliptic boundary value problems involving critical exponents,* J. Funct. Anal., 69 (1986), 289–309.

[11] C.V. Coffman, *Lyusternik-Schnirelman theory: Complementary principles and Morse index,* Nonlinear Anal., T.M.A., 12 (1988), 507–529.

[12] D. Fortunato and E. Jannelli, *Infinitely many solutions for some nonlinear elliptic problems in symmetrical domains,* Proc. Roy. Soc. Edinburg, 105A (1987), 205–213.

[13] C. Jones, *Radial solutions of a semilinear elliptic equation at a critical exponent,* Archive Rat. Mech. Anal., in press.

[14] M.C. Knapp, in preparation.

[15] S. Pohozaev, *Eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl., 6 (1965), 1408–1411.

[16] R.H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations,* CBMS Regional Conference Series Math., 65, Amer. Math. Soc., Providence (1986).

[17] S. Solimini, *On the existence of infinitely many radial solutions for some elliptic problems,* Rev. Math. Appl., in press.

[18] M. Struwe, *Variational methods and their applications to nonlinear partial differential equations and Hamiltonian systems,* Lecture Notes, E.T.H., Zurich, (1989).

[19] M. Struwe, *A global compactness result for elliptic boundary value problems involving limiting nonlinearities,* Math. Z., 187 (1984), 511–517.

[20] G. Talenti, *Best constant in Sobolev inequalities,* Ann. Math. Pure. Appl., 110 (1976), 353–372.

[21] D. Zhang, *On multiple solutions of* $\Delta u + \lambda u + |u|^{\frac{4}{n-2}} = 0$, Nonlinear Anal., T.M.A. 13 (1989), 353–372.