PERRON’S METHOD FOR MONOTONE SYSTEMS
OF SECOND-ORDER ELLIPTIC
PARTIAL DIFFERENTIAL EQUATIONS

HITOSHI ISHII
Department of Mathematics, Chuo University, Bunkyo-ku, Tokyo 112, Japan

(Submitted by: M.G. Crandall)

1. Introduction. Recently the study of systems of fully nonlinear, degenerate elliptic PDEs has been undertaken by several authors [1, 6, 9, 10, 14, 16, 20] in the framework of viscosity solution. It is now known [10] that Perron’s method extends to quasi-monotone systems and it, together with fixed point theorems, produces continuous viscosity solutions for monotone systems. The adaptation of Perron’s method to quasi-monotone systems is rather straightforward and, indeed, not more than a componentwise application of Perron’s method to scalar equations as explained implicitly in [9].

The objective here is to introduce a new adaptation of Perron’s method to systems which are not necessarily quasi-monotone. The new method directly yields continuous viscosity solutions for monotone systems, without appealing to any fixed point theorem, when uniqueness of viscosity solutions is available.

This paper is organized as follows. In Section 2, we give the definition of viscosity solutions for systems of fully nonlinear, second-order, elliptic PDEs in diagonal form. In Section 3, we generalize Perron’s method so as to apply to such systems. In Section 4, we establish uniqueness results for multi-valued viscosity solutions and an existence result for continuous viscosity solutions under a monotonicity and a regularity assumption on the systems. In Section 5, we reformulate the monotonicity and the regularity assumption and generalize the uniqueness and existence results of Section 4. Section 6 concerns two examples of systems which illustrate the applicability of our uniqueness and existence results.

2. Viscosity solutions for systems. We are concerned with the system of nonlinear second-order elliptic PDEs

\[
\begin{align*}
F_1(x,u(x),Du_1(x),D^2u_1(x)) &= 0 & \text{in } \Omega, \\
F_2(x,u(x),Du_2(x),D^2u_2(x)) &= 0 & \text{in } \Omega, \\
\vdots \\
F_m(x,u(x),Du_m(x),D^2u_m(x)) &= 0 & \text{in } \Omega.
\end{align*}
\]
Here $\Omega$ is an open subset of $\mathbb{R}^N$, $u = (u_1, \ldots, u_m) : \Omega \to \mathbb{R}^m$ represents the unknown function, and $F = (F_1, \ldots, F_m) : \Omega \times \mathbb{R}^m \times \mathbb{R}^N \times \mathbb{S}^N \to \mathbb{R}^m$ is a given function, where $\mathbb{S}^N$ denotes the set of real symmetric matrices of order $N$. We will use the notation $A = \{1, \ldots, m\}$.

We begin by recalling the definition \cite{5, 17, 1, 10} of viscosity solutions for system (2.1).

For a function $v : \overline{\Omega} \to \mathbb{R}$, we define

$$v^*(x) = \limsup_{\epsilon \downarrow 0} \{v(y) : |x - y| < \epsilon, y \in \overline{\Omega}\}$$

and

$$v_*(x) = \liminf_{\epsilon \downarrow 0} \{v(y) : |x - y| < \epsilon, y \in \overline{\Omega}\}.$$

We note that $v^*$ and $v_*$ are upper and lower semicontinuous functions, respectively, on $\overline{\Omega}$ with values in $\mathbb{R} \cup \{\pm \infty\}$ and $v_* \leq v \leq v^*$ in $\overline{\Omega}$.

For a function $u : \overline{\Omega} \to \mathbb{R}^m$, we write $u^* = (u_1^*, \ldots, u_m^*)$ and $u_* = (u_1^*, \ldots, u_m^*)$. Also, we define $\bar{u} : \overline{\Omega} \to 2^{\mathbb{R}^m}$, the graph closure of $u$, by

$$\bar{u}(x) = \{r \in \mathbb{R}^m : \text{there is a sequence } \{x^n\} \subset \overline{\Omega} \text{ such that } x^n \to x \text{ and } u(x^n) \to r\}.$$  

It should be remarked that $\bar{u}$ is closed; i.e., if sequences $\{x^n\} \subset \overline{\Omega}$ and $\{r^n\} \subset \mathbb{R}^m$ satisfy $x^n \to x$ and $r^n \to r$ for some $x \in \overline{\Omega}$ and $r \in \mathbb{R}^m$ and $r^n \in \bar{u}(x^n)$ for all $n$, then $r \in \bar{u}(x)$.

**Definition 2.1.** Let $u : \overline{\Omega} \to \mathbb{R}^m$ be a locally bounded function.

(i) We call $u$ a viscosity subsolution of (2.1) if whenever $\varphi \in C^2(\Omega)$, $k \in A$ and $u^*_k - \varphi$ attains its local maximum at $x \in \Omega$ then

$$\min\{F_k(x, r, D\varphi(x), D^2\varphi(x)) : r \in \bar{u}(x), \ r_k = u^*_k(x)\} \leq 0.$$ 

(ii) Similarly, we call $u$ a viscosity supersolution of (2.1) if whenever $\varphi \in C^2(\Omega)$, $k \in A$ and $u_* - \varphi$ attains its local minimum at $x \in \Omega$ then

$$\max\{F_k(x, r, D\varphi(x), D^2\varphi(x)) : r \in \bar{u}(x), \ r_k = u_k(x)\} \geq 0.$$ 

(iii) Finally, we call $u$ a viscosity solution of (2.1) if it is both a viscosity sub- and supersolution of (2.1).

It should be remarked that the minimum of $F_{k,*}$ and the maximum of $F_{k,*}$ in the above definition are attained since the sets $\{r \in \mathbb{R}^m : r \in \bar{u}(x), \ r_k = u_{k,*}(x)\}$ and $\{r \in \mathbb{R}^m : r \in \bar{u}(x), \ r_k = u_{k,*}(x)\}$ are non-empty and compact and $F_{k,*}$ and $F_{k,*}$ are lower semicontinuous and upper semicontinuous, respectively.

Since we will be concerned mainly with viscosity sub-, super- and solutions, we call them simply sub-, super- and solutions, respectively, in what follows.

For a function $v : \Omega \to \mathbb{R}$ and $x \in \Omega$, we define the semijets $J^{2,+}v(x)$ and $J^{2,-}v(x)$ of $v$ of second order at $x$, respectively, by

$$J^{2,+}v(x) = \{(p, X) \in \mathbb{R}^N \times \mathbb{S}^N : v(x + h) \leq v(x) + \langle p, h \rangle + \frac{1}{2} \langle Xh, h \rangle + o(|h|^2) \text{ as } h \to 0\}.$$
and
\[
J^{2,-} v(x) = \{(p, X) \in \mathbb{R}^N \times \mathcal{S}^N : v(x + h) \geq v(x) + \langle p, h \rangle + \frac{1}{2} \langle Xh, h \rangle + o(|h|^2) \text{ as } h \to 0 \}.
\]

Their closures are defined by
\[
\overline{J}^{2,+} v(x) = \{(p, X) \in \mathbb{R}^N \times \mathcal{S}^N : \text{there is a sequence } (x^n, p^n, X^n) \in \Omega \times \mathbb{R}^N \times \mathcal{S}^N \text{ such that } (p^n, X^n) \in J^{2,+} v(x^n) \text{ and } (x^n, v(x^n), p^n, X^n) \to (x, v(x), p, X) \text{ as } n \to \infty \}\]
and
\[
\overline{J}^{2,-} v(x) = \{(p, X) \in \mathbb{R}^N \times \mathcal{S}^N : \text{there is a sequence } (x^n, p^n, X^n) \in \Omega \times \mathbb{R}^N \times \mathcal{S}^N \text{ such that } (p^n, X^n) \in J^{2,-} v(x^n) \text{ and } (x^n, v(x^n), p^n, X^n) \to (x, v(x), p, X) \text{ as } n \to \infty \}\]
for \( x \in \Omega \).

**Proposition 2.2.** Let \( u : \overline{\Omega} \to \mathbb{R}^n \) be locally bounded. Then we have
(i) \( u \) is a subsolution (resp., a supersolution) of (2.1) if and only if
\[
\min \{ F_k(x, r, p, X) : r \in \bar{u}(x), r_k = u^*_k(x) \} \leq 0 \tag{2.2}
\]
for all \( x \in \Omega \) and \((p, X) \in J^{2,+} u^*_k(x) \) (resp.,
\[
\max \{ F_k(x, r, p, X) : r \in \bar{u}(x), r_k = u^*_k(x) \} \geq 0
\]
for all \( x \in \Omega \) and \((p, X) \in J^{2,-} u^*_k(x) \).
(ii) Also, \( u \) is a subsolution (resp., a supersolution) of (2.1) if and only if
\[
\min \{ F_k(x, r, p, X) : r \in \bar{u}(x), r_k = u^*_k(x) \} \leq 0 \tag{2.3}
\]
for all \( x \in \Omega \) and \((p, X) \in \overline{J}^{2,+} u^*_k(x) \) (resp.,
\[
\max \{ F_k(x, r, p, X) : r \in \bar{u}(x), r_k = u^*_k(x) \} \geq 0
\]
for all \( x \in \Omega \) and \((p, X) \in \overline{J}^{2,-} u^*_k(x) \).

**Proof:** The assertion (i) is standard, and so we omit its proof.

Now we prove (ii). Let \( u \) be a subsolution of (2.1). Fix \( k \in A, x \in \Omega \) and \((p, X) \in \overline{J}^{2,+} u^*_k(x) \). By definition, there is a sequence \((x^n, p^n, X^n) \in \Omega \times \mathbb{R}^N \times \mathcal{S}^N \) such that
\[
(x^n, u^*_k(x^n), p^n, X^n) \to (x, u^*_k(x), p, X) \text{ as } n \to \infty,
\]
and
\[
(p^n, X^n) \in J^{2,+} u^*_k(x^n).
\]
Since $u$ is a subsolution of (2.1) (i.e., (2.2) holds), there is a sequence \( \{r^n\} \subset \mathbb{R}^m \) such that \( r^n \in \bar{u}(x^n) \), \( r^n_k = u^*_k(x^n) \) and

\[
F_k^*(x^n, r^n, p^n, X^n) \leq 0
\]

for all $n$. Noting that $u$ is locally bounded, we can choose a subsequence \( \{n_l\} \) so that $r^{n_l}$ converges to some point $r \in \mathbb{R}^m$. It is clear that $r_k = u^*_k(x)$ and $r \in \bar{u}(x)$. Sending $l \to \infty$, we have

\[
F_k^*(x, r, p, X) \leq 0,
\]

from which we conclude that (2.3) holds.

The opposite implication follows from the assertion (i) and the obvious inclusion $J_{2,+}^+ u_k^*(x) \subset J_{2,+}^+ u_k^*(x)$ for all $k \in A$.

The proof of the remainder can be done similarly and is left to the reader.

Now we generalize the definition of viscosity solution to multi-valued functions. Let $u : \overline{\Omega} \to 2^\mathbb{R}^m$. We define $\overline{u} : \overline{\Omega} \to 2^\mathbb{R}^m$, the graph closure of $u$, by

\[
\overline{u}(x) = \{r \in \mathbb{R}^m : \text{there are sequences } \{x^n\} \subset \overline{\Omega} \text{ and } \{r^n\} \subset \mathbb{R}^m \text{ such that } r^n \in \bar{u}(x^n) \text{ and } r^n \to r\}.
\]

We say that $u$ is strict if $u(x) \neq \emptyset$ for all $x \in \overline{\Omega}$ and that $u$ is locally bounded if the subset $\bigcup\{u(y) : y \in \overline{\Omega}, |y - x| \leq a\}$ of $\mathbb{R}^m$ is bounded for any $x \in \overline{\Omega}$ and $a > 0$. We assume that $u$ is strict and locally bounded and define $u^* = (u^*_1, \ldots, u^*_m)$ and $u_* = (u_{1*}, \ldots, u_{m*}) : \Omega \to \mathbb{R}^m$ by

\[
u^*_k(x) = \max\{r_k : r \in \overline{u}(x)\} \text{ and } u_{k*}(x) = \min\{r_k : r \in \overline{u}(x)\}.
\]

**Definition 2.3.** Let $u : \overline{\Omega} \to 2^\mathbb{R}^m$ be strict and locally bounded.

(i) We call $u$ a multi-valued (viscosity) subsolution of (2.1) if whenever $\varphi \in C^2(\Omega)$, $k \in A$ and $x \in \Omega$ is a local maximum point of $u_k^* - \varphi$ then

\[
\min\{F_k^*(x, r, D\varphi(x), D^2\varphi(x)) : r \in \overline{u}(x), r_k = u_k^*(x)\} \leq 0.
\]

(ii) We call $u$ a multi-valued (viscosity) supersolution of (2.1) if whenever $\varphi \in C^2(\Omega)$, $k \in A$ and $x \in \Omega$ is a local minimum point of $u_*^* - \varphi$ then

\[
\max\{F_k^*(x, r, D\varphi(x), D^2\varphi(x)) : r \in \overline{u}(x), r_k = u_{k*}(x)\} \geq 0.
\]

(iii) Finally, we call $u$ a multi-valued (viscosity) solution of (2.1) if it is both a multi-valued sub- and supersolution of (2.1).

We will omit mentioning the adjective "multi-valued" when no confusion may occur.

We should remark that the above definition is a generalization of Definition 2.1 in the following sense. Let $u : \overline{\Omega} \to \mathbb{R}^m$ be locally bounded. Define $U : \overline{\Omega} \to 2^\mathbb{R}^m$ by $U(x) = \{u(x)\}$. Then, $U$ is strict and locally bounded and if $u$ is a subsolution (resp., supersolution, solution) of (2.1), then $U$ is a multi-valued subsolution (resp., supersolution, solution) of (2.1). Now let $U : \overline{\Omega} \to 2^\mathbb{R}^m$ be locally bounded. Assume that $U$ is single-valued; i.e., for each $x \in \overline{\Omega}$ there is a $u(x) \in \mathbb{R}^m$ so that $U(x) = \{u(x)\}$. The relation that $u(x) \in U(x)$ for all $x \in \overline{\Omega}$ uniquely defines a function $u : \overline{\Omega} \to \mathbb{R}^m$. Then, $u$ is locally bounded and if $U$ is a subsolution (resp., supersolution, solution) of (2.1), then $u$ is a subsolution (resp., supersolution, solution) of (2.1).

We have an analogue of Proposition 2.2 for multi-valued functions.
Proposition 2.4. Let $u : \Omega \to \mathbb{R}^m$ be strict and locally bounded. Then we have

(i) $u$ is a subsolution (resp., supersolution) of (2.1) if and only if

$$\min \{ F_k^*(x, r, p, X) : r \in \bar{u}(x), r_k = u_k^*(x) \} \leq 0$$

for all $x \in \Omega$ and $(p, X) \in J^{2,+} u_k^*(x)$ (resp.,

$$\max \{ F_k^*(x, r, p, X) : r \in \bar{u}(x), r_k = u_k^*(x) \} \geq 0$$

for all $x \in \Omega$ and $(p, X) \in J^{2,-} u_k^*(x)$).

(ii) Also, $u$ is a subsolution (resp., supersolution) of (2.1) if and only if

$$\min \{ F_k^*(x, r, p, X) : r \in \bar{u}(x), r_k = u_k^*(x) \} \leq 0$$

for all $x \in \Omega$ and $(p, X) \in \overline{J}^{2,+} u_k^*(x)$ (resp.,

$$\max \{ F_k^*(x, r, p, X) : r \in \bar{u}(x), r_k = u_k^*(x) \} \geq 0$$

for all $x \in \Omega$ and $(p, X) \in \overline{J}^{2,-} u_k^*(x)$).

Since the proof of this proposition is similar to that of Proposition 2.2, we omit giving it here.

3. Perron’s method for systems. We begin with the next lemma.

Lemma 3.1. Let $\mathcal{S}$ be a nonempty set of multi-valued subsolutions (resp., supersolutions) of (2.1). Define $u(x) = \cup \{ v(x) : v \in \mathcal{S} \}$ for $x \in \bar{\Omega}$. Assume that $u$ is locally bounded. Then $u$ is a multi-valued subsolution (resp., supersolution) of (2.1).

Proof: It is clear that $u$ is strict. We prove the subsolution part and leave the proof of the supersolution part to the reader. Fix any $\varphi \in C^2(\Omega)$ and $k \in A$. Let $x \in \Omega$ be a maximum point of $u_k^* - \varphi$. We may assume that $x$ is a strict maximum point of $u_k^* - \varphi$. By definition, there is an $r \in \bar{u}(x)$ for which $r_k = u_k^*(x)$. Also, there are sequences $\{x^n\} \subset \Omega$ and $\{r^n\} \subset \mathbb{R}^m$ such that $r^n \in u(x^n)$ for all $n$ and $x^n \to x$ and $r^n \to r$ as $n \to \infty$. Moreover, there is a sequence $\{v^n\} \subset \mathcal{S}$ such that $r^n \in v^n(x^n)$ for all $n$. In conclusion, we can choose sequences $\{x^n\} \subset \Omega$, $\{r^n\} \subset \mathbb{R}^m$ and $\{v^n\} \subset \mathcal{S}$ so that $r^n \in v^n(x^n)$ for all $n$ and $x^n \to x$, $r^n_k \to u_k^*(x)$ as $n \to \infty$. It is clear that $v^n_k^* \leq u_k^*$ and $r^n_k \leq v^n_k^*(x^n)$ for all $n$. It is now easily seen that $v^n_k^* - \varphi$ attains a maximum at a point near $x$ for $n$ large enough and such maximum points converge to $x$ as $n \to \infty$. We may assume that $v^n_k^* - \varphi$ attains a maximum at a point in $\Omega$ for any $n$. Let $y^n \in \Omega$ be a maximum point of $v^n_k^* - \varphi$. Since $v^n$ is a multi-valued subsolution of (2.1), there is a point $s^n \in \overline{v^n(y^n)}$ for which $s^n_k = v^n_k^*(y^n)$ and

$$F_k^*(y^n, s^n, D\varphi(y^n), D^2\varphi(y^n)) \leq 0.$$

Sending $n \to \infty$ along a subsequence for which $\{s^n\}$ is convergent, we see that

$$F_k^*(x, s, D\varphi(x), D^2\varphi(x)) \leq 0$$
for some $s \in \bar{u}(x)$. Noting that
\[ r_k^n - \varphi(x^n) \leq (v_k^n - \varphi)(x^n) \leq (v_k^n* - \varphi)(y^n) = s_k^n - \varphi(y^n), \]
we see that $s_k = u_k^*(x)$. Thus, we conclude that $u$ is a multi-valued subsolution of (2.1).

Ellipticity is needed to ensure that classical sub-, super- and solutions are also viscosity sub-, super- and solutions, respectively.

**Definition 3.2.** A function $F = (F_1, \ldots, F_m) : \Omega \times \mathbb{R}^m \times \mathbb{R}^N \times \mathbb{S}^N \to \mathbb{R}^m$ is called degenerate elliptic if whenever $X, Y \in \mathbb{S}^N$, $X \leq Y$, $k \in A$ and $(x, r, p) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^N$ then
\[ F_k^*(x, r, p, X) \geq F_k^*(x, r, p, Y). \]

We use the following notation. For $r = (r_1, \ldots, r_m)$, $s = (s_1, \ldots, s_m) \in \mathbb{R}^m$, we write $r \leq s$ if $r_k \leq s_k$ for all $k \in A$. For functions $u, v : \bar{\Omega} \to \mathbb{R}^m$, we also write $u \leq v$ in $\Omega$ if $u_k \leq v_k$ in $\bar{\Omega}$ for all $k \in A$.

**Definition 3.3.** Let $f, g : \bar{\Omega} \to \mathbb{R}^m$. We call the pair $(f, g)$ a pair of a subsolution and a supersolution of (2.1) in the strong sense if the following three conditions hold:
(i) $f, g$ are locally bounded and lower semicontinuous and upper semicontinuous on $\bar{\Omega}$, respectively;
(ii) $f \leq g$ on $\bar{\Omega}$;
(iii) With the notations
\[ G_k(x, t, p, X) = \sup\{F_k(x, r, p, X) : f(x) \leq r \leq g(x), r_k = t\}, \]
and
\[ H_k(x, t, p, X) = \inf\{F_k(x, r, p, X) : f(x) \leq r \leq g(x), r_k = t\} \]
for $k \in A$ and $(x, t, p, X) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N$, if $\varphi \in C^2(\Omega)$, $k \in A$ and $x \in \Omega$ is a local maximum point of $f_k^* - \varphi$, then
\[ G_k^*(x, f_k^*(x), D\varphi(x), D^2\varphi(x)) \leq 0, \]
and if $\varphi \in C^2(\Omega)$, $k \in A$ and $x \in \Omega$ is a local minimum point of $g_k^* - \varphi$, then
\[ H_k^*(x, g_k^*(x), D\varphi(x), D^2\varphi(x)) \geq 0. \]

The last condition (iii) is equivalent to saying that $f$ is a subsolution of
\[
\begin{align*}
\left\{ \begin{array}{l}
G_1(x, u_1(x), Du_1(x), D^2u_1(x)) = 0 \quad \text{in } \Omega, \\
\vdots \\
G_m(x, u_m(x), Du_m(x), D^2u_m(x)) = 0 \quad \text{in } \Omega,
\end{array} \right.
\]
and $g$ is a supersolution of
\[
\begin{align*}
\left\{ \begin{array}{l}
H_1(x, u_1(x), Du_1(x), D^2u_1(x)) = 0 \quad \text{in } \Omega, \\
\vdots \\
H_m(x, u_m(x), Du_m(x), D^2u_m(x)) = 0 \quad \text{in } \Omega.
\end{array} \right.
\]
We also note that if $(f, g)$ is a pair of a subsolution and a supersolution of (2.1) in the strong sense, then $f$ and $g$ are a subsolution and a supersolution of (2.1), respectively.

**Remark.** We borrowed the above idea from [6].
Theorem 3.4. Assume that $F$ is degenerate elliptic and that there is a pair $(f, g)$ of a subsolution and a supersolution of (2.1) in the strong sense. Then there is a multi-valued solution $u$ such that $f(x) \leq r \leq g(x)$ for all $x \in \overline{\Omega}$ and $r \in u(x)$.

The proof will be done by using a variant of Perron's method. Generalizations of Perron's method [7] to systems have already been obtained in [6, 9, 10].

Proof: We intend to use Zorn's lemma to obtain a multi-valued solution of (2.1).

First, we introduce a structure of partial order to the set $\mathcal{P}$ of all pairs of a subsolution and a supersolution of (2.1) in the strong sense. For $(f_1, g_1), (f_2, g_2) \in \mathcal{P}$, if

$$f_1 \leq f_2 \quad \text{and} \quad g_2 \leq g_1 \quad \text{on} \quad \overline{\Omega},$$

then we write $(f_1, g_1) \prec (f_2, g_2)$. It is easily checked that the relation $\prec$ defines a partial order in $\mathcal{P}$.

Let $\mathcal{L}$ be a nonempty linearly ordered subset of $\mathcal{P}$ under the ordering $\prec$. We want to show that $\mathcal{L}$ has an upper bound in $\mathcal{P}$. We define $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_m), \tilde{g} = (\tilde{g}_1, \ldots, \tilde{g}_m) : \overline{\Omega} \to \mathbb{R}^m$ by

$$\tilde{f}_k(x) = \sup \{ v_k(x) : (v, w) \in \mathcal{L} \} \quad \text{and} \quad \tilde{g}_k(x) = \inf \{ w_k(x) : (v, w) \in \mathcal{L} \}$$

and claim that $(\tilde{f}, \tilde{g}) \in \mathcal{P}$ and that $(\tilde{f}, \tilde{g})$ is an upper bound of $\mathcal{L}$.

It is clear that $\tilde{f}$ and $\tilde{g}$ are lower semicontinuous and upper semicontinuous on $\overline{\Omega}$, respectively. Let $(f^1, g^1), (f^2, g^2) \in \mathcal{L}$. Since $\mathcal{L}$ is linearly ordered, we have either

$$f^1 \leq f^2 \leq g^2 \leq g^1 \quad \text{on} \quad \overline{\Omega}$$

or

$$f^2 \leq f^1 \leq g^1 \leq g^2 \quad \text{on} \quad \overline{\Omega}.$$ 

In either case, we have

$$f^1 \leq g^2 \quad \text{on} \quad \overline{\Omega}.$$ 

Therefore,

$$v \leq \tilde{f} \leq \tilde{g} \leq w \quad \text{on} \quad \overline{\Omega}$$

for all $(v, w) \in \mathcal{L}$. Since for any $(v, w) \in \mathcal{L}$, $v$ and $w$ are locally bounded, the above chain of inequalities implies that $\tilde{f}$ and $\tilde{g}$ are locally bounded.

In order to conclude $(\tilde{f}, \tilde{g}) \in \mathcal{P}$, it remains to show that $(\tilde{f}, \tilde{g})$ satisfies the third condition of Definition 3.3. Set

$$\tilde{G}_k(x, t, p, X) = \sup \{ F_k(x, r, p, X) : \tilde{f}(x) \leq r \leq \tilde{g}(x), \ r_k = t \}$$

and

$$\tilde{H}_k(x, t, p, X) = \inf \{ F_k(x, r, p, X) : \tilde{f}(x) \leq r \leq \tilde{g}(x), \ r_k = t \}$$

for $k \in A$ and $(x, t, p, X) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N$. Noting that if $(v, w) \in \mathcal{L}$, then

$$\{ r : \tilde{f}(x) \leq r \leq \tilde{g}(x) \} \subset \{ r : v(x) \leq r \leq w(x) \},$$
we easily deduce that, for any \((v, w) \in \mathcal{L}\), \(v\) is a subsolution of

\[
\begin{aligned}
\tilde{G}_1(x, u_1(x), Du_1(x), D^2u_1(x)) &= 0 \quad \text{in } \Omega, \\
\vdots \\
\tilde{G}_m(x, u_m(x), Du_m(x), D^2u_m(x)) &= 0 \quad \text{in } \Omega.
\end{aligned}
\] (3.1)

By Lemma 3.1, we see that \(\tilde{f}\) is a subsolution of (3.1). Similarly, we deduce that \(\tilde{g}\) is a supersolution of

\[
\begin{aligned}
\tilde{H}_1(x, u_1(x), Du_1(x), D^2u_1(x)) &= 0 \quad \text{in } \Omega, \\
\vdots \\
\tilde{H}_m(x, u_m(x), Du_m(x), D^2u_m(x)) &= 0 \quad \text{in } \Omega.
\end{aligned}
\] (3.2)

Therefore, \((\tilde{f}, \tilde{g})\) satisfies the condition (iii) of Definition 3.3. Thus, we have proved that \((\tilde{f}, \tilde{g}) \in \mathcal{P}\), i.e., \((\tilde{f}, \tilde{g})\) is a pair of a subsolution and a supersolution of (2.1) in the strong sense.

Now, we apply Zorn’s lemma to conclude that there is a maximal element of \(\mathcal{P}\). Fix a maximal element \((\tilde{f}, \tilde{g})\) of \(\mathcal{P}\). We set \(u(x) = \{r \in \mathbb{R}^m : \tilde{f}(x) \leq r \leq \tilde{g}(x)\}\) for \(x \in \overline{\Omega}\) and want to show that \(u\) is a multi-valued solution of (2.1). It is clear that \(u\) is strict. Because of the semicontinuity and the local boundedness of \(\tilde{f}\) and \(\tilde{g}\), \(u\) is closed and locally bounded. Notice that \(\bar{u} = u, \bar{v} = \tilde{g}\) and \(\bar{u} = \tilde{f}\).

To see that \(u\) is a multi-valued subsolution of (2.1), we argue by contradiction. Suppose that \(u\) is not a multi-valued subsolution of (2.1). Then there is a \(k \in A\), a function \(\varphi \in C^2(\overline{\Omega})\) and a maximum point \(z \in \Omega\) of \(\hat{g}_k - \varphi\) for which we have

\[
\min\{F_{k*} (z, r, D\varphi(z), D^2\varphi(z)) : r \in u(z), r_k = \hat{g}_k(z)\} > 0. \quad (3.3)
\]

We may assume, without loss of generality, that \(\hat{g}_k(z) = \varphi(z), \varphi \in C^2(\overline{\Omega})\), and

\[
(\hat{g}_k - \varphi)(x) \leq -|x - z|^4 \quad \text{for } x \in \overline{\Omega}. \quad (3.4)
\]

We claim now that

\[
\hat{f}^*_k(z) < \varphi(z). \quad (3.5)
\]

To see this, suppose to the contrary that \(\hat{f}^*_k(z) \geq \varphi(z)\). Combining this with the inequality \(\hat{f}^*_k(z) \leq \hat{g}_k(z) \leq \varphi(z)\), we have \(\hat{f}^*_k(z) = \varphi(z)\) and, hence, \(z\) is a maximum point of \(\hat{f}^*_k - \varphi\). Hence, we have

\[
\hat{G}_{k*}(z, \varphi(z), D\varphi(z), D^2\varphi(z)) \leq 0,
\]

where \(\hat{G}_k : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N \to \mathbb{R} \cup \{-\infty\}\) is defined by

\[
\hat{G}_k(x, t, p, X) = \sup\{F_k(x, r, p, X) : r \in u(x), r_k = t\}.
\]

This yields

\[
\min\{F_{k*} (z, r, D\varphi(z), D^2\varphi(z)) : r \in u(z), r_k = \varphi(z)\} \leq 0,
\]
which contradicts (3.3). This proves our claim.

It is easily seen that the function

\[(t, x) \mapsto \min \{ F_k(x, r, D\varphi(x), D^2\varphi(x)) : r \in u(x), \ r_k = \varphi(x) - t \}\]

is lower semicontinuous on \(\mathbb{R} \times \Omega\). In view of (3.3) and (3.4), we can choose a \(\delta > 0\) so that

\[
\min \{ F_k(x, r, D\varphi(x), D^2\varphi(x)) : r \in u(x), \ r_k = \varphi(x) - \delta^4 \} > 0
\]

(3.6)

and

\[
\hat{f}_k^*(x) \leq \varphi(x) - \delta^4
\]

(3.7)

for all \(x \in B(z, 2\delta)\). We define \(h = (h_1, \ldots, h_m) : \Omega \rightarrow \mathbb{R}^m\) by

\[h_k(x) = \min \{ \hat{g}_k(x), \varphi(x) - \delta^4 \} \quad \text{and} \quad h_j(x) = \hat{g}_j(x) \text{ for } j \neq k.
\]

By (3.4), we have

\[
\hat{g}_k(x) \leq \varphi(x) - \delta^4 \quad \text{for } x \in \overline{\Omega} \setminus B(z, \delta)
\]

and, therefore,

\[h(x) = \hat{g}(x) \text{ for } x \in \overline{\Omega} \setminus B(z, \delta). \quad (3.8)
\]

We now assert that \((\hat{f}, h) \in \mathcal{P}\), i.e., \((\hat{f}, h)\) is a pair of a subsolution and a supersolution of (2.1) in the strong sense. In view of (3.7) and (3.8), we have \(\hat{f} \leq h \leq \hat{g}\) on \(\overline{\Omega}\). Also, it is clear that \(h\) is upper semicontinuous on \(\overline{\Omega}\). Let \(\psi \in C^2(\Omega)\) and \(j \in A\). Let \(y \in \Omega\) be a maximum point of \(\hat{f}_j - \psi\). Because of the inclusion \(\{ r : \hat{f}(y) \leq r \leq h(y) \} \subset u(y)\), we immediately have

\[
G_j(y, f_j^*(y), D\psi(y), D^2\psi(y)) \leq 0, \quad (3.9)
\]

where \(G_j : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N \rightarrow \mathbb{R} \cup \{ \pm \infty \}\) is given by

\[
G_j(x, t, p, X) = \sup \{ F_j(x, r, p, X) : \hat{f}(y) \leq r \leq h(y), r_j = t \}.
\]

Now let \(y \in \Omega\) be a minimum point of \(h_j - \psi\). If \(h_j(y) \geq \hat{g}_j(y)\), then \(h_j(y) = \hat{g}_j(y)\) and hence \(y\) is a minimum point of \(\hat{g}_j - \psi\). Therefore, in this case, we have as above

\[
H_j(y, h_j(y), D\psi(y), D^2\psi(y)) \geq 0, \quad (3.10)
\]

where \(H_j : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N \rightarrow \mathbb{R} \cup \{ \pm \infty \}\) is defined by

\[
H_j(x, t, p, X) = \inf \{ F_j(x, r, p, X) : \hat{f}(y) \leq r \leq h(y), r_j = t \}.
\]

Therefore, we have

\[D\varphi(y) = D\psi(y) \quad \text{and} \quad D^2\varphi(y) \geq D^2\psi(y).
\]
Consider the case when \( h_{j*}(y) < g_{j*}(y) \). We then have \( j = k, y \in B(z, \delta) \) by (3.8), and \( h_j(x) = \varphi(x) - \delta^4 \) near the point \( y \); so that \( \varphi - \psi \) attains a local minimum at \( y \). Using the degenerate ellipticity of \( F \), we deduce from (3.6) that for some \( r \in \mathbb{R}^m \) satisfying \( \hat{f}(y) \leq r \leq h(y) \) and \( r_j = h_{j*}(y)(= h_j(y)) \),

\[
0 < F_{j*}(y, r, D\varphi(y), D^2\varphi(y)) \leq F_j^*(y, r, D\psi(y), D^2\psi(y)) \\
\leq H_j^*(y, h_{j*}(y), D\psi(y), D^2\psi(y)).
\]

Hence, we have (3.10) also in the case when \( h_{j*}(y) < g_{j*}(y) \). Thus, \((\hat{f}, h)\) is a pair of a subsolution and a supersolution of (2.1) in the strong sense.

We now know that \((\hat{f}, h) \in \mathcal{P} \) and \((\hat{f}, \hat{g}) < (\hat{f}, h)\). By the maximality of \((\hat{f}, \hat{g})\), we must have \( \hat{g} = h \). However, we have \( h_k^*(z) \leq \varphi(z) - \delta^4 \leq \hat{g}_k^*(z) \), a contradiction. This shows that \( u \) is a multi-valued subsolution of (2.1).

Arguments similar to the above prove that \( u \) is a multi-valued supersolution of (2.1); the details are left to the reader.

4. Uniqueness and existence results for monotone systems. We will establish uniqueness and existence results for system (2.1) under appropriate hypotheses. Here existence results are those for continuous solutions.

We list below some notations which will be convenient to state our monotonicity assumption on \( F \). For \( s, t \in \mathbb{R}^m \), \( \|s - t\| \) denotes \( \max_{k \in A} |s_k - t_k| \), the distance between \( s \) and \( t \). Let \( U, V \) be bounded subsets of \( \mathbb{R}^m \). For \( k \in A \), we write

\[
U_k^* = \max \{r_k : r \in U\} \quad \text{and} \quad U_{k*} = \min \{r_k : r \in U\}.
\]

We define

\[
d(U, V) = \sup \{\|s - t\| : s \in U, t \in V\}.
\]

It is easily checked that

\[
d(U, V) = \max \{U_k^* - V_{k*}, V_k^* - U_{k*} : k \in A\}
\]

and that if \( s \in U, t \in V, j \in A \) and \( s_j - t_j = d(U, V) \), then \( s_j = U_j^* \) and \( t_j = V_j^* \).

We write

\[
A^+(U, V) = \{k \in A : U_k^* - V_{k*} = d(U, V)\},
\]

\[
A^-(U, V) = \{k \in A : V_k^* - U_{k*} = d(U, V)\},
\]

and \( A(U, V) = A^+(U, V) \cup A^-(U, V) \).

(F.1) There is a positive number \( \lambda \) such that if \( U, V \) are compact subsets of \( \mathbb{R}^m \) and \( d(U, V) > 0 \), then for each \( \alpha > 1 \) and \( x, y \in \Omega \) there exists a \( j = j(U, V, \alpha, x, y) \in A(U, V) \) for which if \( j \in A^+(U, V) \), then

\[
\min \{F_j(x, r, \alpha(x-y), X) : r \in U, r_j = U_j^*\} \\
\geq \max \{F_j^*(x, r, \alpha(x-y), X) : r \in V, r_j = V_j^*\} + \lambda(U_j^* - V_j^*)
\]

for all \( X \in \mathcal{S}^N \) and if \( j \in A^-(U, V) \), then

\[
\min \{F_j(x, r, \alpha(x-y), X) : r \in V, r_j = V_j^*\} \\
\geq \max \{F_j^*(x, r, \alpha(x-y), X) : r \in U, r_j = U_j^*\} + \lambda(V_j^* - U_j^*)
\]
for all \( X \in \mathbb{S}^N \).

We also need uniform continuity of \( F \) in the variable \( x \).

(F.2) There is a continuous function \( \omega : [0, \infty) \rightarrow [0, \infty) \) with \( \omega(0) = 0 \) such that if \( X, Y \in \mathbb{S}^N, \alpha > 1 \) and

\[
-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},
\]

then

\[
F_{k\alpha}(y, r, \alpha(x-y), -Y) - F_{k\alpha}^*(x, r, \alpha(x-y), X) \leq \omega(\alpha|x-y|^2 + 1/\alpha)
\]

for all \( k \in A, x, y \in \Omega \) and \( r \in \mathbb{R}^m \).

This assumption (F.2) is satisfied in a wide class of degenerate elliptic PDEs including first-order PDEs. See for this [8, 11, 4].

Theorem 4.1. Let \( \Omega \) be bounded, and let (F.1) and (F.2) hold. Let \( u, v : \overline{\Omega} \rightarrow 2^\mathbb{R}^m \) be solutions of (2.1). Assume \( d(\bar{u}(x), \bar{v}(x)) = 0 \) on \( \partial\Omega \). Then \( d(\bar{u}(x), \bar{v}(x)) = 0 \) on \( \overline{\Omega} \).

It should be noticed that for \( U, V \subset \mathbb{R}^m \), \( d(U, V) = 0 \) if and only if \( U = V = \{ r \} \) for some \( r \in \mathbb{R}^m \).

Theorem 4.2. Let \( \Omega \) be bounded and let (F.1) and (F.2) hold. Assume that there is a pair \((f, g)\) of a subsolution and a supersolution of (2.1) in the strong sense such that \( f = g \) on \( \partial\Omega \). Then there is a solution \( u \in C(\overline{\Omega})^m \) of (2.1) satisfying \( f \leq u \leq g \) on \( \overline{\Omega} \).

Theorem 4.2 is a consequence of Theorems 3.4 and 4.1. We first prove Theorem 4.2, taking for granted that Theorem 4.1 is true.

Proof of Theorem 4.2: By Theorem 3.4, there is a multi-valued solution \( U \) of (2.1) such that \( f(x) \leq r \leq g(x) \) for all \( x \in \overline{\Omega} \) and \( r \in U(x) \). This inequality guarantees that \( d(\bar{U}(x), \bar{U}(x)) = 0 \) for all \( x \in \partial\Omega \). Therefore, by Theorem 4.1, for all \( x \in \overline{\Omega} \) we have \( d(\bar{U}(x), \bar{U}(x)) = 0 \) and hence \( \bar{U}(x) = \{ u(x) \} \) for some \( u(x) \in \mathbb{R}^m \).

As we have already seen, the function \( u : x \rightarrow u(x) \) is a solution of (2.1). Also, it is clear that \( U_*(x) = u(x) = U^*(x) \) for all \( x \in \overline{\Omega} \). This shows that \( u \in C(\overline{\Omega})^m \) and that \( f \leq u \leq g \) on \( \overline{\Omega} \).

We use the following lemma, which integrates the recent developments of comparison arguments in the theory of viscosity solutions of second-order PDEs (see [2, 3]).

Lemma 4.3. Let \( u \) and \( -v \) be real-valued upper semicontinuous functions on \( \Omega \). Assume that the function \( u(x) - v(y) - \alpha \frac{1}{2}|x-y|^2 \), with \( \alpha > 0 \), on \( \Omega \times \Omega \) attains a local maximum at \((\bar{x}, \bar{y})\). Then there are \( X, Y \in \mathbb{S}^n \) such that

\[
(\alpha(\bar{x} - \bar{y}), X) \in \overline{\mathbb{S}^n}^2 u(\bar{x}), \quad (\alpha(\bar{x} - \bar{y}), -Y) \in \overline{\mathbb{S}^n}^2 v(\bar{y})
\]

and

\[
-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.
\]
Lemma 4.4. Assume (F.1) and (F.2). Let \( \alpha > 1 \) and \( X, Y \in S^N \) satisfy (4.1). Then
\[
F_k^*(y, r, \alpha(x-y), -Y) - F_k^*(x, r, \alpha(x-y), X) \leq \omega(\alpha|x-y|^2 + 1/\alpha)
\]
for all \( k \in A, x, y \in \Omega \) and \( r \in \mathbb{R}^m \), where \( \omega \) is the function from (F.2).

Proof: Let \( \alpha > 1 \) and \( X, Y \in \mathbb{R}^N \) satisfy (4.1). Fix \( k \in A, x, y \in \Omega \) and \( r \in \mathbb{R}^m \). Let \( \epsilon_k \) denote the unit vector in \( \mathbb{R}^m \) with unity as its k-th component. Fix any \( \epsilon > 0 \). From (F.1), it follows that if \( \xi, \eta \in \Omega \) and \( \beta > 1 \), then
\[
F_k(\xi, r + \epsilon \epsilon_k, \beta(\xi - \eta), -Y) \geq F_k^*(\xi, r, \beta(\xi - \eta), -Y).
\]
(Here we have taken \( U = \{r + \epsilon \epsilon_k\} \) and \( V = \{r\} \).) We take \( \xi = y, \eta = y - \frac{\alpha}{\beta}(x-y) \), so that \( \beta(\xi - \eta) = \alpha(x-y) \) and choose \( \beta \) large enough so that \( \eta \in \Omega \). We then have
\[
F_k^*(y, r + \epsilon \epsilon_k, \alpha(x-y), -Y) \geq F_k^*(y, r, \alpha(x-y), -Y).
\]
Now, using (F.2), we have
\[
F_k^*(y, r, \alpha(x-y), -Y) \leq F_k^*(x, r + \epsilon \epsilon_k, \alpha(x-y), -Y) + \omega(\alpha|x-y|^2 + 1/\alpha).
\]
Sending \( \epsilon \downarrow 0 \), we conclude the proof.

Proof of Theorem 4.1: We argue by contradiction and thus suppose that \( \theta = \sup_{x \in \Omega} d(\bar{u}(x), \bar{v}(x)) > 0 \).

We first observe that, since \( \bar{u} \) and \( \bar{v} \) are locally bounded and closed, the function \( (x, y) \to d(\bar{u}(x), \bar{v}(y)) \) is bounded and upper semicontinuous on \( \overline{\Omega} \times \overline{\Omega} \).

Now, define \( \Psi : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R} \) by
\[
\Psi(x, y) = d(\bar{u}(x), \bar{v}(y)) - \frac{\alpha}{2}|x-y|^2
\]
for \( \alpha > 1 \). Since \( \Psi \) is bounded and upper semicontinuous on \( \overline{\Omega} \times \overline{\Omega} \), it attains a maximum. Let \( (x_\alpha, y_\alpha) \in \overline{\Omega} \times \overline{\Omega} \) be a maximum point of \( \Psi \). It is clear that
\[
\theta \leq \max_{x \in \overline{\Omega}} \Psi \leq \Psi(x_\alpha, y_\alpha),
\]
from which it follows that
\[
\frac{\alpha}{2}|x_\alpha - y_\alpha|^2 \leq d(\bar{u}(x_\alpha), \bar{v}(y_\alpha)) - \theta.
\]
From this, we see that \( |x_\alpha - y_\alpha| \to 0 \) as \( \alpha \to \infty \). Let \( \{\alpha_j\} \subset (1, \infty) \) be a sequence such that \( \alpha_j \to \infty \) and \( x_{\alpha_j} \to z \) for some \( z \in \overline{\Omega} \) as \( j \to \infty \). Noting that \( y_{\alpha_j} \to z \) as \( j \to \infty \) and using the upper semicontinuity of \( d(\bar{u}(x), \bar{v}(y)) \), we see that
\[
\limsup_{j \to \infty} \frac{\alpha_j}{2}|x_{\alpha_j} - y_{\alpha_j}|^2 \leq d(\bar{u}(z), \bar{v}(z)) - \theta \leq 0.
\]
From this observation and the assumption that \( d(\bar{u}(x), \bar{v}(x)) = 0 \) on \( \partial \Omega \), we deduce that
\[
\lim_{\alpha \to \infty} \alpha|x_\alpha - y_\alpha|^2 = 0 \text{ and } x_\alpha, y_\alpha \in \Omega \text{ for } \alpha \text{ large enough.}
\]
Henceforth, we assume \( \alpha \) so large that \( x_\alpha, y_\alpha \in \Omega \).

We want to show that
\[
\lambda \theta \leq \omega(\alpha |x_\alpha - y_\alpha|^2 + 1/\alpha).
\]  
\( \text{(4.3)} \)

To see this, we first set \( U = \bar{u}(x_\alpha) \) and \( V = \bar{v}(y_\alpha) \). Since \( d(U, V) > 0 \), using (F.1) with \( x_\alpha \) and \( y_\alpha \) in place of \( x \) and \( y \), we may choose a \( j \in A(U, V) \) such that if \( j \in A^+(U, V) \) and \( X \in S^N \), then
\[
\min \{ F_{j^*}(x_\alpha, r, \alpha(x_\alpha - y_\alpha), X) : r \in U, r_j = U_j^* \} \geq \max \{ F_{j^*}(x_\alpha, r, \alpha(x_\alpha - y_\alpha), X) : r \in V, r_j = V_j^* \} + \lambda(U_j^* - V_j^*)
\]  
\( \text{(4.4)} \)

and if \( j \in A^-(U, V) \) and \( X \in S^N \), then
\[
\min \{ F_{j^*}(y_\alpha, r, \alpha(y_\alpha - x_\alpha), X) : r \in V, r_j = V_j^* \} \geq \max \{ F_{j^*}(y_\alpha, r, \alpha(y_\alpha - x_\alpha), X) : r \in U, r_j = U_j^* \} + \lambda(V_j^* - U_j^*)
\]  
\( \text{(4.5)} \)

Consider first the case when \( j \in A^+(U, V) \). We then have
\[
d(U, V) = U_j^* - V_j^* = u_j^*(x_\alpha) - v_j^*(y_\alpha).
\]

On the other hand, it is clear that for all \( x, y \in \overline{\Omega} \),
\[
u_j^*(x) - v_j^*(y) \leq d(\bar{u}(x), \bar{v}(y)) \leq d(U, V).
\]

These observations show that the function
\[
(x, y) \rightarrow u_j^*(x) - v_j^*(y) - \frac{\alpha}{2}|x - y|^2
\]
has a maximum at \( (x_\alpha, y_\alpha) \). Henceforth, we put \( p = \alpha(x_\alpha - y_\alpha) \). By Lemma 4.3, we may choose \( X, Y \in S^N \) so that
\[
(p, X) \in \overline{J}^{2,+} u_j^*(x_\alpha), \quad (p, -Y) \in \overline{J}^{2,-} v_j^*(y_\alpha)
\]
and
\[
-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.
\]

Because \( u \) and \( v \) are solutions of \( (2.1) \), we have
\[
\min \{ F_{j^*}(x_\alpha, r, p, X) : r \in U, r_j = U_j^* \} \leq 0
\]
and
\[
\max \{ F_{j^*}(y_\alpha, r, p, -Y) : r \in V, r_j = V_j^* \} \geq 0.
\]

From the first inequality, using (4.4), we have
\[
0 \geq \max \{ F_{j^*}(x_\alpha, r, p, X) : r \in V, r_j = V_j^* \} + \lambda(U_j^* - V_j^*).}
\]
Therefore, in view of Lemma 4.4, we have

\[ 0 \geq \max \{ F_j^*(x_\alpha), r, p, X : r \in V, r_j = V_j^* \} \]
\[ - \max \{ F_j^*(y_\alpha), r, p, -Y : r \in V, r_j = V_j^* \} \]
\[ + \lambda (U_j^* - V_j^*) \geq \lambda \theta - \omega (\alpha |x_\alpha - y_\alpha|^2 + \frac{1}{\alpha}) \],

which proves (4.3). Similar arguments also prove (4.3) in the case when \( j \in A^-(U, V) \).

Sending \( \alpha \to \infty \) in (4.3) yields a contradiction, which proves our assertion.

Let us formulate a uniqueness result for unbounded domains. We assume that there is a function \( h \in C(\overline{\Omega}) \) and a collection \( \{ h_\epsilon \}_{\epsilon > 0} \) of \( C^2 \) functions on \( \overline{\Omega} \) having the properties (4.6)-(4.9)

\( h(x) > 0 \) for \( x \in \overline{\Omega} \) \hspace{1cm} (4.6)

\( h_\epsilon(x) \to 0 \) as \( \epsilon \downarrow 0 \) for every \( x \in \overline{\Omega} \) \hspace{1cm} (4.7)

\( h(x) - h_\epsilon(x) \to \infty \) as \( x \in \overline{\Omega} \) and \( |x| \to \infty \) \hspace{1cm} (4.8)

\( F_k(x, r, p - D h_\epsilon(x), X - D^2 h_\epsilon(x)) \leq F_k(x, r, p, X) \) \hspace{1cm} (4.9)

for all \( k \in A \) and \( (x, r, p, X) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^N \times S^N \).

For \( U \subset \mathbb{R}^m \), we write \( \| U \| \) for \( \sup \{ \| r \| : r \in U \} \).

**Theorem 4.5.** Assume (F.1), (F.2) and the existence of \( h \in C(\overline{\Omega}) \) and a collection \( \{ h_\epsilon \}_{\epsilon > 0} \) \( C^2 \) functions on \( \overline{\Omega} \) having the properties (4.6)-(4.9). Let \( u \) and \( v \) be multi-valued solutions of (2.1).

Suppose that

\[ \sup \frac{\| u(x) \|}{h(x)} < \infty \text{ and } \sup \frac{\| v(x) \|}{h(x)} < \infty \]

and that \( d(\bar{u}(x), \bar{v}(x)) = 0 \) for \( x \in \partial \Omega \). Then \( d(\bar{u}(x), \bar{v}(x)) = 0 \) for all \( x \in \Omega \).

**Proof:** By multiplying \( h \) and the \( h_\epsilon \) by a positive constant, we may assume that

\[ \frac{\| u(x) \|}{h(x)} \leq 1 \text{ and } \frac{\| v(x) \|}{h(x)} \leq 1, \]

so that

\[ \lim_{x \in \overline{\Omega}, |x| \to \infty} (\| u(x) \| - h_\epsilon(x)) = -\infty \]

and

\[ \lim_{x \in \overline{\Omega}, |x| \to \infty} (\| v(x) \| - h_\epsilon(x)) = -\infty. \]

For \( \alpha > 1 \) and \( \epsilon > 0 \), we define \( \Psi : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R} \) by

\[ \Psi(x, y) = d(\bar{u}(x), \bar{v}(y)) - h_\epsilon(x) - h_\epsilon(y) - \frac{\alpha}{2} |x - y|^2. \]

Noting that \( d(\bar{u}(x), \bar{v}(y)) \leq \| \bar{u}(x) \| + \| \bar{v}(y) \| \), we see that there is a positive \( R \) depending only on \( \epsilon \) such that

\[
\begin{cases}
\Psi(x, x) \leq 0 & \text{if } x \in \partial B(0, R) \cap \overline{\Omega}, \\
\theta_\epsilon \equiv \max \{ d(\bar{u}(x), \bar{v}(x)) - 2 h_\epsilon(x) : x \in \overline{\Omega} \cap B(0, R) \} > 0.
\end{cases}
\]
Fix such an \( R \). Let \( \Omega_{R} = \Omega \cap B(0, R)^{0} \). By the assumption that \( d(\bar{u}(x), \bar{v}(x)) = 0 \) for \( x \in \partial \Omega \), we see that
\[
\Psi(x, x) \leq 0 \text{ on } \partial \Omega_{R}.
\]

Now, let \((x_{\alpha}, y_{\alpha})\) be a maximum point of \( \Psi \) over \( \overline{\Omega}_{R} \times \overline{\Omega}_{R} \), so that
\[
\Psi(x, y) \leq \Psi(x_{\alpha}, y_{\alpha}) \text{ for all } (x, y) \in \overline{\Omega}_{R} \times \overline{\Omega}_{R}.
\]

From this, it follows that
\[
\frac{\alpha}{2} |x_{\alpha} - y_{\alpha}|^{2} \leq d(\bar{u}(x_{\alpha}), \bar{v}(y_{\alpha})) - h_{\alpha}(x_{\alpha}) - h_{\alpha}(y_{\alpha}) - \theta_{c}.
\]

From this, it is seen, as in the proof of Theorem 4.1, that \( \alpha |x_{\alpha} - y_{\alpha}|^{2} \to 0 \) as \( \alpha \to \infty \) and that \( x_{\alpha}, y_{\alpha} \in \Omega_{R} \) for \( \alpha \) sufficiently large. Henceforth, we assume \( \alpha \) is large enough, so that \( x_{\alpha}, y_{\alpha} \in \Omega_{R} \).

Set \( U = \bar{u}(x_{\alpha}) \) and \( V = \bar{v}(y_{\alpha}) \). We have \( d(U, V) > 0 \). By assumption (F.1), we may choose a \( j \in A(U, V) \) for which we have (4.4) if \( j \in A^{+}(U, V) \) and \( X \in \mathbb{S}^{N} \) and (4.5) if \( j \in A^{-}(U, V) \) and \( X \in \mathbb{S}^{N} \).

Consider the case when \( j \in A^{+}(U, V) \). The function
\[
(x, y) \to u_{j}^{*}(x) - v_{j}^{*}(y) - h_{\epsilon}(x) - h_{\epsilon}(y) - \frac{\alpha}{2} |x - y|^{2}
\]
on \( \overline{\Omega}_{R} \times \overline{\Omega}_{R} \) has a maximum at \((x_{\alpha}, y_{\alpha})\). We set
\[
p = \alpha(x_{\alpha} - y_{\alpha}), \bar{u}(x) = u_{j}^{*}(x) - h_{\epsilon}(x) \text{ and } \bar{v}(x) = v_{j}^{*}(x) + h_{\epsilon}(x).
\]

By Lemma 4.3, we may choose \( X, Y \in \mathbb{S}^{N} \) so that
\[
(p, X) \in \overline{J}^{2,+} \bar{u}(x_{\alpha}), (p, -Y) \in \overline{J}^{2,-} \bar{v}(y_{\alpha})
\]
and
\[
-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.
\]

Now that \( u \) and \( v \) are solutions of (2.1), noting that
\[
\overline{J}^{2,+} \bar{u}(x) = \overline{J}^{2,+} u_{j}^{*}(x) - (Dh_{\epsilon}(x), D^{2}h_{\epsilon}(x))
\]
and
\[
\overline{J}^{2,-} \bar{v}(x) = \overline{J}^{2,+} v_{j}^{*}(x) + (Dh_{\epsilon}(x), D^{2}h_{\epsilon}(x)),
\]
we have
\[
\min \{ F_{j}(x, r, p + Dh_{\epsilon}(x_{\alpha}), X + D^{2}h_{\epsilon}(x_{\alpha})) : r \in U, \ r_{j} = U_{j_{\alpha}}^{*} \} \leq 0
\]
and
\[
\max \{ F_{j}^{*}(y_{\alpha}, r, p - Dh_{\epsilon}(x_{\alpha}), -Y - D^{2}h_{\epsilon}(x_{\alpha})) : r \in V, \ r_{j} = V_{j_{\alpha}}^{*} \} \geq 0.
\]
From these together with (4.9), we have
\[ \min\{F_j^*(x_\alpha, r, p, X) : r \in U, \ r_j = U_j^* \} \leq 0 \]
and
\[ \max\{F_j^*(y_\alpha, r, p, -Y) : r \in V, \ r_j = V_j^* \} \geq 0. \]
We now proceed as in the proof of Theorem 4.1 to obtain
\[ \lambda \theta_\epsilon \leq \omega(\alpha|x_\alpha - y_\alpha|^2 + 1/\alpha). \quad (4.11) \]
Similarly, we have (4.11) in the case when \( j \in A^-(U, V) \). Passing to the limit as \( \alpha \to \infty \), we obtain a contradiction, which completes the proof.

5. Remarks concerning conditions (F.1) and (F.2). In this section, we will give generalizations of Theorem 4.1 by reformulating (F.1) and (F.2).

We begin by unifying conditions (F.1) and (F.2).

(F.3) Let \( R \geq 1 \). Let \( \{U_\alpha\} \alpha > R \) and \( \{V_\alpha\} \alpha > R \) be collections of compact subsets of \( \mathbb{R}^m \) such that \( \cup_{\alpha > R} U_\alpha \) and \( \cup_{\alpha > R} V_\alpha \) are bounded and
\[ 0 < \lim_{\alpha \to \infty} d(U_\alpha, V_\alpha) < \infty. \quad (5.1) \]
Let \( \{x_\alpha\} \alpha > R \) and \( \{y_\alpha\} \alpha > R \) be sequences of points in \( \Omega \) such that
\[ \lim_{\alpha \to \infty} \alpha|x_\alpha - y_\alpha|^2 = 0. \quad (5.2) \]
Let \( \{X_\alpha\} \alpha > R, \{Y_\alpha\} \alpha > R \) be sequences of matrices \( X_\alpha, Y_\alpha \in \mathbb{S}^N \) satisfying
\[ -3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X_\alpha & 0 \\ 0 & Y_\alpha \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \quad (5.3) \]
Then there is an \( \alpha > R \) and a \( j \in A(U_\alpha, V_\alpha) \) for which either the following (5.4) and (5.5) or (5.6) and (5.7) hold:

\[ j \in A^+(U_\alpha, V_\alpha); \]
\[ \begin{cases} 
\min\{F_j^*(x_\alpha, r, \alpha(x_\alpha - y_\alpha), X_\alpha) : r \in U_\alpha, r_j = U_j^* \} > 0 \\
\text{or} \\
\max\{F_j^*(y_\alpha, r, \alpha(x_\alpha - y_\alpha), -Y_\alpha) : r \in V_\alpha, r_j = V_j^* \} < 0;
\end{cases} \quad (5.4) \]
\[ j \in A^-(U_\alpha, V_\alpha); \]
\[ \begin{cases} 
\min\{F_j^*(y_\alpha, r, \alpha(y_\alpha - x_\alpha), Y_\alpha) : r \in V_\alpha, r_j = V_j^* \} > 0 \\
\text{or} \\
\max\{F_j^*(x_\alpha, r, \alpha(y_\alpha - x_\alpha), -X_\alpha) : r \in U_\alpha, r_j = U_j^* \} < 0.
\end{cases} \quad (5.5) \]

A careful review of the proof of Theorem 4.1 shows that the next uniqueness theorem is valid (see also the proof of Theorem 5.2 below).
Theorem 5.1. Assume (F.3) and that $\Omega$ is bounded. Let $u$ and $v$ be multi-valued solutions of (2.1). Assume $d(\bar{u}(x), \bar{v}(x)) = 0$ on $\partial \Omega$. Then $d(\bar{u}(x), \bar{v}(x)) = 0$ on $\bar{\Omega}$.

We should remark that (F.1) and (F.2) together imply (F.3) and, hence, that Theorem 5.1 is a generalization of Theorem 4.1.

The next condition is satisfied with systems from optimal control with switching cost, which we will see in Section 6. For simplicity of notation we denote by $C^{+}_{k, R}(U)$, where $k \in K$, $R \geq 1$ and $U$ is a compact subset of $\mathbb{R}^m$, the set of those collections $\{U_{\alpha}\}_{\alpha > R}$ of compact subsets of $\mathbb{R}^m$ which have the following properties: (i) $\lim_{\alpha \to \infty} U_{\alpha k} = U_k^*$, and (ii) for any neighborhood $O$ of $U$ there is an $M > R$ such that $U_{\alpha} \subset O$ for all $\alpha \geq M$. Similarly, for $k \in K$, $R \geq 1$ and a compact subset $U$ of $\mathbb{R}^m$, $C^{-}_{k, R}(U)$ denotes the set of those collections $\{U_{\alpha}\}_{\alpha > R}$ of compact subsets of $\mathbb{R}^m$ which have the following properties: (i) $\lim_{\alpha \to \infty} U_{\alpha k} = U_k^*$, and (ii) for any neighborhood $O$ of $U$ there is an $M > R$ such that $U_{\alpha} \subset O$ for all $\alpha \geq M$.

(F.4) For each $\epsilon > 0$, there is a $\delta > 0$ such that for any $q \in \mathbb{R}^N$ satisfying $|q| \leq \delta$, any compact subsets $U, V$ of $\mathbb{R}^m$ satisfying $d(U, V) \geq \epsilon$, and any $z \in \Omega$ there is a $j \in A(U, V)$ for which either of the following (i) or (ii) holds:

(i) $j \in A^+(U, V)$ and if $R \geq 1$, $\{U_{\alpha}\}_{\alpha > R} \in C^{+}_{j, R}(U)$, $\{V_{\alpha}\}_{\alpha > R} \in C^{+}_{j, R}(V)$, $\{x_{\alpha}\}_{\alpha > R}$ and $\{y_{\alpha}\}_{\alpha > R}$ are collections of points in $\Omega$ satisfying (5.2) and

$$x_{\alpha}, y_{\alpha} \to z \quad \text{as} \quad \alpha \to \infty,$$

and $\{X_{\alpha}\}_{\alpha > R}$ and $\{Y_{\alpha}\}_{\alpha > R}$ are collections of matrices in $\mathbb{S}^N$ satisfying (5.3), then there is an $\alpha > R$ such that

$$\min \{F_j(x_{\alpha}, r, q + \alpha(x_{\alpha} - y_{\alpha}), X_{\alpha}) : r \in U_{\alpha}, r_j = U_{\alpha j}^*\} > 0$$

or

$$\max \{F_j^*(y_{\alpha}, r, \alpha(x_{\alpha} - y_{\alpha}), -Y_{\alpha}) : r \in V_{\alpha}, r_j = V_{\alpha j}^*\} < 0.$$

(ii) $j \in A^-(U, V)$ and if $R \geq 1$, $\{U_{\alpha}\}_{\alpha > R} \in C^{-}_{j, R}(U)$, $\{V_{\alpha}\}_{\alpha > R} \in C^{-}_{j, R}(V)$, $\{x_{\alpha}\}_{\alpha > R}$ and $\{y_{\alpha}\}_{\alpha > R}$ are collections of points in $\Omega$ satisfying (5.2) and (5.8), and $\{X_{\alpha}\}_{\alpha > R}$ and $\{Y_{\alpha}\}_{\alpha > R}$ are collections of matrices in $\mathbb{S}^N$ satisfying (5.3), then there is an $\alpha > R$ such that

$$\min \{F_j(y_{\alpha}, r, q + \alpha(y_{\alpha} - x_{\alpha}), Y_{\alpha}) : r \in V_{\alpha}, r_j = V_{\alpha j}^*\} > 0$$

or

$$\max \{F_j^*(x_{\alpha}, r, \alpha(y_{\alpha} - x_{\alpha}), -X_{\alpha}) : r \in U_{\alpha}, r_j = U_{\alpha j}^*\} < 0.$$

Theorem 5.2. Let $\Omega$ be bounded and let (F.4) hold. Let $u, v$ be multi-valued solutions of (2.1). Assume that $d(\bar{u}(x), \bar{v}(x)) = 0$ for $x \in \partial \Omega$. Then $d(\bar{u}(x), \bar{v}(x)) = 0$ for $x \in \Omega$.

We need the following well-known lemma for the proof of Theorem 5.2.

Lemma 5.3. Let $f$ be a real-valued upper semicontinuous function on a compact subset $K$ of $\mathbb{R}^N$. Then there is a dense subset $P$ of $\mathbb{R}^N$ such that for each $p \in P$ the function $x \to f(x) - \langle p, x \rangle$ attains a strict maximum over $K$.

For a proof of this lemma, see, e.g., [9].
Proof of Theorem 5.2: We argue by contradiction, and thus suppose that 
\[ \sup_{x \in \bar{\Omega}} d(\bar{u}(x), \bar{v}(x)) > 0. \]

Noting that the function \((x, y) \to d(\bar{u}(x), \bar{v}(y))\) is bounded and upper semicontinuous on \(\bar{\Omega} \times \bar{\Omega}\), we see that the function \(x \to d(\bar{u}(x), \bar{v}(x))\) is also bounded and upper semicontinuous on \(\bar{\Omega}\), and, in view of Lemma 5.3, we can choose a sequence \(\{p^n\} \subset \mathbb{R}^N\) converging to 0 such that the functions \(x \to d(\bar{u}(x), \bar{v}(x)) - \langle p^n, x \rangle\) have a strict maximum over \(\bar{\Omega}\). Fix \(\epsilon > 0\) so that \(\max_{x \in \bar{\Omega}} d(\bar{u}(x), \bar{v}(x)) > \epsilon\), and then choose a \(\delta > 0\) as in (F.4). We select \(q = p^n\) so that \(|q| \leq \delta\). Let \(z \in \bar{\Omega}\) be the unique maximum point of the function \(x \to d(\bar{u}(x), \bar{v}(x)) - \langle q, x \rangle\). We may assume by choosing \(|q|\) small enough that \(z \in \Omega\) and \(d(\bar{u}(z), \bar{v}(z)) \geq \epsilon\). By (F.4), there is a \(j \in A(\bar{u}(z), \bar{v}(z))\) having either property (i) or (ii) of (F.4) with \(\bar{u}(z)\) and \(\bar{v}(z)\) replacing \(U\) and \(V\), respectively.

Assume that \(j\) has property (i). For \(\alpha > 1\), we consider the function

\[ \Psi_\alpha(x, y) = u_j^*(x) - v_{j^*}(y) - \langle q, x \rangle - \frac{\alpha}{2} |x - y|^2 \]
on \(\bar{\Omega} \times \bar{\Omega}\) and let \((x_\alpha, y_\alpha)\) be a maximum point of \(\Psi_\alpha\). We claim that as \(\alpha \to \infty\)

\[ x_\alpha \to z, y_\alpha \to z \quad \text{and} \quad \alpha|x_\alpha - y_\alpha|^2 \to 0. \]  

(5.11)

To see this, we first observe from the inequality

\[ \Psi_\alpha(z, z) \leq \Psi_\alpha(x_\alpha, y_\alpha) \]

(5.12)

that \(\alpha|x_\alpha - y_\alpha|^2\) is bounded and, hence, that \(x_\alpha - y_\alpha \to 0\) as \(\alpha \to \infty\). Next, let \(\{\alpha_n\} \subset (1, \infty)\) be a sequence such that \(\alpha_n \to \infty\) and \(x_{\alpha_n} \to \bar{z}\) for some \(\bar{z} \in \bar{\Omega}\) as \(n \to \infty\). Then we have

\[ \lim_{n \to \infty} y_{\alpha_n} = \bar{z}. \]

Once again, from (5.12), we have

\[ \limsup_{n \to \infty} \frac{\alpha_n}{2} |x_{\alpha_n} - y_{\alpha_n}|^2 \leq u_j^*(\bar{z}) - v_{j^*}(\bar{z}) - \langle q, \bar{z} \rangle - u_j^*(z) + v_{j^*}(z) + \langle q, z \rangle \leq 0. \]

Noting that \(z\) is the unique maximum point of the function \(x \to u_j^*(x) - v_{j^*}(x) - \langle q, x \rangle\), we see that this implies (5.11). Also, from (5.12), we have

\[ u_j^*(z) - v_{j^*}(z) \leq \liminf_{\alpha \to \infty} \{u_j^*(x_\alpha) - v_{j^*}(y_\alpha)\}. \]

From this and the semicontinuitities of \(u_j^*\) and \(v_{j^*}\), we have

\[ \liminf_{\alpha \to \infty} u_j^*(x_\alpha) \geq \liminf_{\alpha \to \infty} \{u_j^*(x_\alpha) - v_{j^*}(y_\alpha)\} + \liminf_{\alpha \to \infty} v_{j^*}(y_\alpha) \geq u_j^*(z) \]

and

\[ \limsup_{\alpha \to \infty} v_{j^*}(y_\alpha) \leq - \liminf_{\alpha \to \infty} \{u_j^*(x_\alpha) - v_{j^*}(y_\alpha)\} + \limsup_{\alpha \to \infty} u_j^*(x_\alpha) \leq v_{j^*}(z). \]
Thus, we see that as $\alpha \to \infty$
\[ u_j^*(x_\alpha) \to u_j^*(z) \quad \text{and} \quad v_j*(y_\alpha) \to v_j*(z). \]  
(5.13)

We now choose an $R \geq 1$ so that $x_\alpha, y_\alpha \in \Omega$ for all $\alpha > R$. In view of Lemma 3.3, for $\alpha > R$, we may choose $X_\alpha, Y_\alpha \in \mathbb{S}^N$ satisfying (5.3) such that
\[ (q + \alpha(x_\alpha - y_\alpha), X_\alpha) \in \mathbb{J}^{2,+}_\alpha u_j^*(x_\alpha) \quad \text{and} \quad (\alpha(x_\alpha - y_\alpha), -Y_\alpha) \in \mathbb{J}^{2,-}_\alpha v_j*(y_\alpha). \]

It follows that, for all $\alpha > R$,
\[ \min\{F_j*(x_\alpha, r, q + \alpha(x_\alpha - y_\alpha), X_\alpha) : r \in \bar{u}(x_\alpha), r_j = u_j^*(x_\alpha)\} \leq 0 \]  
(5.14)
and
\[ \max\{F_j*(y_\alpha, r, \alpha(x_\alpha - y_\alpha), -Y_\alpha) : r \in \bar{v}(y_\alpha), r_j = v_j*(y_\alpha)\} \geq 0. \]  
(5.15)

We set
\[ U_\alpha = \bar{u}(x_\alpha) \quad \text{and} \quad V_\alpha = \bar{v}(y_\alpha). \]

Noting that $\bar{u}$ and $\bar{v}$ are closed, we see from (5.13) that
\[ \{U_\alpha\}_{\alpha > R} \in C_+^{t}(\bar{u}(z)) \quad \text{and} \quad \{V_\alpha\}_{\alpha > R} \in C_-^{t}(\bar{v}(z)). \]

Now, property (i) guarantees that for some $\alpha > R$ we have either
\[ \min\{F_j*(x_\alpha, r, q + \alpha(x_\alpha - y_\alpha), X_\alpha) : r \in U_\alpha, r_j = U_j^*\} > 0 \]
or
\[ \max\{F_j*(y_\alpha, r, \alpha(x_\alpha - y_\alpha), -Y_\alpha) : r \in V_\alpha, r_j = V_j^*\} < 0. \]

This together with (5.14) and (5.15) yields a contradiction.

An argument parallel to the above yields also a contradiction in the case when $j$ has property (ii).

6. Examples. In this section, we illustrate when conditions (F.1) and (F.2) or its variant (F.4) are satisfied.

We begin with the case where the coupling in the unknowns is linear (see [18, 14, 6, 9]), i.e., the case where
\[ F_k(x, r, p, X) = G_k(x, r_k, p, X) + \sum_{j=1}^{m} c_{kj}(x)r_j. \]  
(6.1)

Here we assume for simplicity that, for all $k, j \in A$,
\[ G_k \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N) \quad \text{and} \quad c_{kj} \in C(\Omega). \]  
(6.2)

We also assume that there is a function $\omega \in C([0, \infty))$ satisfying $\omega(0) = 0$ such that if $\alpha > 1$, $X, Y \in \mathbb{S}^N$ and
\[ -3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \]  
then
\[ G_k(y, t, \alpha(x - y), -Y) \leq G_k(x, t, \alpha(x - y), X) + \omega(\alpha|x - y|^2 + 1/\alpha) \]
for all $k \in A, x, y \in \Omega$ and $t \in \mathbb{R}$.
Proposition 6.1. Assume (6.1) and (6.2). Assume, moreover, that for each $k \in A$ and $(x, p, X) \in \Omega \times \mathbb{R}^N \times S^N$ the function $t \rightarrow G_k(x, t, p, X)$ is nondecreasing on $\mathbb{R}$ and that

$$c_{kk}(x) \geq \lambda + \sum_{j \neq k} |c_{kj}(x)|$$

(6.4)

for all $k \in A, x \in \Omega$ and for some $\lambda > 0$. Then (F.1) holds with this $\lambda$. In addition, assume (6.3). Then (F.2) holds.

Proof: Let $U, V$ be compact subsets of $\mathbb{R}^m$. Assume that $d(U, V) > 0$. Fix any $k \in A(U, V)$. Consider the case when $k \in A^+(U, V)$. Let $s \in U$ and $t \in V$ satisfy $s_k = U_k^*$ and $t_k = V_k^*$. Fix $(x, p, X) \in \Omega \times \mathbb{R}^N \times S^N$. We compute that

$$F_k(x, s, p, X) \geq F_k(x, t, p, X) + c_{kk}(x)(s_k - t_k) + \sum_{j \neq k} c_{kj}(x)(s_j - t_j)$$

$$\geq F_k(x, t, p, X) + \lambda (s_k - t_k) + \sum_{j \neq k} |c_{kj}(x)|(s_k - t_k) - \sum_{j \neq k} |c_{kj}(x)|d(U, V)$$

$$= F_k(x, t, p, X) + \lambda (U_k^* - V_k^*).$$

Thus, we have

$$\min \{F_k(x, r, p, X) : r \in U, r_k = U_k^* \}$$

$$\geq \max \{F_k(x, r, p, X) : r \in V, r_k = V_k^* \} + \lambda (U_k^* - V_k^*).$$

Similarly, if $j \in A^-(U, V)$, then we have

$$\min \{F_k(x, r, p, X) : r \in V, r_k = V_k^* \}$$

$$\geq \max \{F_k(x, r, p, X) : r \in U, r_k = U_k^* \} + \lambda (V_k^* - U_k^*).$$

These observations guarantee that (F.1) holds. Also, (F.2) is a direct consequence of (6.3).

The above observation has an obvious generalization as indicated below. Let

$$F_k(x, r, p, X) = G_k(x, r_k, p, X) + \sum_{j=1}^m c_{kj}(x, r, p)r_j,$$

(6.5)

where for all $k, j \in A$,

$$G_k \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N \times S^N) \text{ and } c_{kj} \in C(\Omega \times \mathbb{R}^m \times \mathbb{R}^N).$$

(6.6)

Proposition 6.2. Assume (6.5) and (6.6). Suppose that the functions $t \rightarrow G_k(x, t, p, X)$ are nondecreasing on $\mathbb{R}$ and that

$$c_{kk}(x, r, p) \geq \lambda + \sum_{j \neq k} |c_{kj}(x, r, p)|$$

(6.7)

for all $k \in A, (x, r, p) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^N$ and for some $\lambda > 0$. Then (F.1) holds. Moreover, if (6.3) is also satisfied, then (F.2) holds.

The proof of this proposition parallels the proof of Proposition 6.1, and we omit presenting it here.
Now, let us treat systems arising from optimal control with switching cost (see [1, 19, 15, 9]). The functions \( F_k \) have the form

\[
F_k(x, r, p, X) = \max \{ G_k(x, r_k, p, X), r_k - M_k(x, r) \},
\]

where \( G_k \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N) \), and \( M_k \in C(\Omega \times \mathbb{R}^m) \) are given by

\[
M_k(x, r) = \min \{ r_j + g_{kj}(x) : j \neq k \}, \quad \text{with } g_{kj} \in C(\Omega).
\]

We assume that

\[
g_{kj}(x) > 0 \quad \text{for all } k, j \in A, \quad \text{with } k \neq j, \quad \text{and } x \in \Omega.
\]

We need to assume that there is a function \( \omega \in C([0, \infty)) \) satisfying \( \omega(0) = 0 \) such that

\[
|G_k(x, t, p, X) - G_k(x, t, q, X)| \leq \omega(|p - q|)
\]

for all \( k \in A, \quad x \in \Omega, \quad t \in \mathbb{R}, \quad p, q \in \mathbb{R}^N \) and \( X \in \mathbb{S}^N \).

**Proposition 6.3.** Let the \( F_k \) be represented as (6.8). Assume (6.9), (6.10), (6.3) and that for some constant \( \lambda > 0 \) the functions \( t \to G_k(x, t, p, X) - \lambda t \) are nondecreasing on \( \mathbb{R} \). Then (F.4) holds.

**Proof:** Let \( \omega \) be a function satisfying both of the conditions of (6.3) and (6.10). Fix \( \epsilon > 0 \). We choose a \( \delta > 0 \) so that \( \omega(\delta) < \lambda \epsilon / 2 \). Let \( U, V \) be compact subsets of \( \mathbb{R}^n \) such that \( d(U, V) \geq \epsilon \). Let \( z \in \Omega \).

We first consider the case when \( A^+(U, V) \neq \emptyset \). Our first claim is that there is a \( k \in A^+(U, V) \) for which

\[
\min \{ r_k - M_k(z, r) : r \in U, r_k = U^*_k \} > 0 \quad (6.11)
\]

or

\[
\max \{ r_k - M_k(z, r) : r \in V, r_k = V^*_k \} < 0. \quad (6.12)
\]

To see this, we suppose to the contrary that, for all \( k \in A^+(U, V) \),

\[
\min \{ r_k - M_k(z, r) : r \in U, r_k = U^*_k \} \leq 0 \quad (6.13)
\]

and

\[
\max \{ r_k - M_k(z, r) : r \in V, r_k = V^*_k \} \geq 0. \quad (6.14)
\]

Fix any \( k \in A^+(U, V) \). From (6.14), there is a \( t \in V \) such that

\[
t_k = V^*_k \quad \text{and} \quad t_k \geq M_k(z, t).
\]

Hence, we can choose a \( j \in A, \) with \( j \neq k \), so that

\[
t_k \geq t_j + g_{kj}(z). \quad (6.15)
\]

From (6.13), there is an \( s \in U \) such that

\[
s_k = U^*_k \quad \text{and} \quad s_k \leq M_k(z, s).
\]
Therefore,

\[ s_k \leq s_j + g_{kj}(z). \]

This and (6.15) together yield

\[ s_k - t_k \leq s_j - t_j, \]

which implies that \( s_j - t_j = d(U, V) \). Thus, we see that \( j \in A^+(U, V) \) and \( s_j = U_j^* \), \( t_j = V_j^* \). Thus, we have

\[
U_k^* = U_j^* + g_{kj}(z) \quad \text{and} \quad V_k^* = V_j^* + g_{kj}(z).
\]

Repeating the above argument, we can choose a finite sequence \( \{k^1, \ldots, k^l\} \subset A^+(U, V) \) having the properties:

\[
U_{k^j} = U_{k^{j+1}} + g_{k^j,k^{j+1}}(z) \quad \text{for} \quad j = 1, \ldots, l - 1; \tag{6.16}
\]

\[
V_{k^j} = V_{k^{j+1}} + g_{k^j,k^{j+1}}(z) \quad \text{for} \quad j = 1, \ldots, l - 1; \tag{6.17}
\]

\[
k^1, \ldots, k^{l-1} \text{ are distinct and } k^l = k^1. \tag{6.18}
\]

Adding the sequence of the equalities in (6.16) or (6.17), we obtain

\[
\sum_{j=1}^{l-1} g_{k^j,k^{j+1}}(z) = 0.
\]

This contradicts (6.9), which proves our claim.

Fix \( k \in A^+(U, V) \) so that (5.11) and (6.12) hold. Let \( R \geq 1 \). Fix \( \{U_\alpha\}_{\alpha > R} \in C_{k,R}^+(U) \) and \( \{V_\alpha\}_{\alpha > R} \in C_{k,R}^-(V) \). Let \( \{x_\alpha\}_{\alpha > R}, \{y_\alpha\}_{\alpha > R} \subset \Omega \) converge to \( z \) and satisfy

\[
\alpha|x_\alpha - y_\alpha|^2 \to 0 \quad \text{as} \quad \alpha \to \infty.
\]

Let \( \{X_\alpha\}_{\alpha > R}, \{Y_\alpha\}_{\alpha > R} \subset S^N \) satisfy

\[
-3\alpha \begin{pmatrix} I & 0 & 0 \\ 0 & I & \end{pmatrix} \leq \begin{pmatrix} X_\alpha & 0 \\ 0 & Y_\alpha \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.
\]

We assert that, for some \( \alpha > R \), we have either

\[
\min\{F_k(x_\alpha, r, q + \alpha(x_\alpha - y_\alpha), X_\alpha) : r \in U_\alpha, r_k = U_{ak} \} > 0 \tag{6.19}
\]

or

\[
\max\{F_k(y_\alpha, r, \alpha(x_\alpha - y_\alpha), -Y_\alpha) : r \in V_\alpha, r_k = V_{ak} \} < 0. \tag{6.20}
\]

To see this, we assume that, for all \( \alpha > R \),

\[
\min\{F_k(x_\alpha, r, q + \alpha(x_\alpha - y_\alpha), X_\alpha) : r \in U_\alpha, r_k = U_{ak} \} \leq 0 \tag{6.21}
\]

and will show that, for some \( \alpha > R \),

\[
\max\{F_k(y_\alpha, r, \alpha(x_\alpha - y_\alpha), -Y_\alpha) : r \in V_\alpha, r_k = V_{ak} \} < 0. \tag{6.22}
\]
By this assumption, we easily get
\[ \min \{ r_k - M_k(z, r) : r \in U, r_k = U_k^* \} \leq 0 \]
by letting \( \alpha \to \infty \). Therefore, we have (6.12). Using (6.12), we have
\[
\limsup_{\alpha \to \infty} \max \{ r_k - M_k(y_{\alpha}, r) : r \in V_{\alpha}, r_k = V_{\alpha k^*} \}
\leq \max \{ r_k - M_k(z, r) : r \in V, r_k = V_k^* \} < 0.
\]
Hence,
\[
\max \{ r_k - M_k(y_{\alpha}, r) : r \in V_{\alpha}, r_k = V_{\alpha k^*} \} < 0 \tag{6.23}
\]
for \( \alpha \) sufficiently large. From (6.21), we have
\[
G_k(x_{\alpha}, U_{\alpha k^*}^*, q + \alpha (x_{\alpha} - y_{\alpha}), X_{\alpha}) \leq 0 \quad \text{for all } \alpha > R.
\]
Using (6.3), (6.10) and the monotonicity assumption on \( G_k \), we calculate that
\[
G_k(y_{\alpha}, V_{\alpha k^*}, q + \alpha (x_{\alpha} - y_{\alpha}), X_{\alpha})
\leq G_k(x_{\alpha}, V_{\alpha k^*}, q + \alpha (x_{\alpha} - y_{\alpha}), X_{\alpha}) + \omega(\alpha|x_{\alpha} - y_{\alpha}|^2 + 1/\alpha)
\leq G_k(x_{\alpha}, U_{\alpha k^*}, q + \alpha (x_{\alpha} - y_{\alpha}), X_{\alpha}) - \lambda(U_{\alpha k^*}^* - V_{\alpha k^*})
\quad + \omega(|q|) + \omega(\alpha|x_{\alpha} - y_{\alpha}|^2 + 1/\alpha)
\leq -\frac{\lambda \rho}{2} + \omega(\alpha|x_{\alpha} - y_{\alpha}|^2 + 1/\alpha).
\]
This and (6.23) guarantee that (6.22) is valid for \( \alpha \) sufficiently large.

We now turn to the case when \( A^-(U, V) \neq \emptyset \). We note that \( A^-(U, V) = A^+(V, U) \). Arguing as above, we see that there is a \( k \in A^-(U, V) \) for which
\[
\min \{ r_k - M_k(z, r) : r \in V, r_k = V_k^* \} > 0
\]
and
\[
\max \{ r_k - M_k(z, r) : r \in U, r_k = U_k^* \} < 0.
\]
Fix such a \( k \in A^-(U, V) \). Let \( R \geq 1 \). Fix \( \{ U_{\alpha} \}_{\alpha > R} \in C_{k,R}^-(U) \) and \( \{ V_{\alpha} \}_{\alpha > R} \in C_{k,R}^+(V) \). Let \( \{ x_{\alpha} \}_{\alpha > R} \), \( \{ y_{\alpha} \}_{\alpha > R} \), \( \{ X_{\alpha} \}_{\alpha > R} \), and \( \{ Y_{\alpha} \}_{\alpha > R} \) be as above. Arguments parallel to the above show that, for some \( \alpha > R \),
\[
\min \{ F_k(y_{\alpha}, r, q + \alpha (y_{\alpha} - x_{\alpha}), Y_{\alpha}) : r \in V_{\alpha}, r_k = V_{\alpha k^*} \} > 0
\]
or
\[
\max \{ F_k(x_{\alpha}, r, \alpha (y_{\alpha} - x_{\alpha}), -X_{\alpha}) : r \in U_{\alpha}, r_k = U_{\alpha k^*} \} < 0.
\]
Thus, we conclude that (F.4) holds.

REFERENCES


