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ON SOME CONDITIONS FOR EXISTENCE OF FORCED PERIODIC OSCILLATIONS*

A.M. Krasnosel'skii and M.A. Krasnosel'skii
Institute for Information Transmission Problems, Moscow, USSR

J. Mawhin
Institut de Mathématique, Université de Louvain, B-1348 Louvain-la-Neuve, Belgium

Abstract. This paper is concerned with some sufficient conditions for the existence of forced periodic oscillations for systems of ordinary differential equations. The method introduced here generalizes some ideas of the method of guiding functions for the study of periodic problems under conditions which allow some blow-up of the solutions.

1. Introduction. Consider the system of ordinary differential equations

$$x' = f(t,x), \quad x \in \mathbb{R}^n, \tag{1}$$

with the right-hand side T-periodic in t,

$$f(t+T,x) = f(t,x), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n,$$

and satisfying the Caratheodory conditions. No conditions will be made about the uniqueness of solutions of the Cauchy problem for (1) or about the non-local continuability of such solutions. The Euclidian norm and inner product in $\mathbb{R}^n$ will be denoted by $|\cdot|$ and $(\cdot,\cdot)$, respectively. We shall assume the existence of a real function V on $\mathbb{R}^n$ satisfying the following conditions.

Assumption (V). There exists a positive function V of class C^1 on $\mathbb{R}^n$, a number $r_0 > 0$ and a nonnegative Caratheodory function $a : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{|x|\to\infty} V(x) = \infty \tag{2}$$

and

$$(V'(x), f(t,x)) \le a(t, V(x)), \quad |x| \ge r_0. \tag{3}$$

Such functions are analogues of the guiding functions we considered in [3] in the case where $a(t,v) = a(t)$. The reader can consult [1] and [3] for more references on

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the method of guiding functions and its generalizations. Without loss of generality, we can assume that the function a is T-periodic with respect to t. Let

$$v_0 = \max_{|x|\le r_0} V(x) > 0. \tag{4}$$

Assumption (A). For each $t_1 \in [0,T)$, all the solutions $v(t;t_1,v_0)$ of the Cauchy problem

$$v' = a(t,v), \quad v(t_1) = v_0, \tag{5}$$

are defined for $t \in [t_1, t_1+T]$ and are such that

$$v(t;t_1,v_0) \le v(t_1+T;t_1,v_0) \le v_1 < +\infty, \quad t_1 \le t \le t_1+T, \tag{6}$$

for some $v_1 > 0$.

Example 1. The first example is the function

$$a(t,v) = b(t) + c(t)v,$$

where b and c are nonnegative locally Lebesgue integrable T-periodic functions. Then

$$v(t;t_1,v_0) = \Big[v_0 + \int_{t_1}^{t} \exp\Big(\int_s^{t_1} c(\tau)\,d\tau\Big) b(s)\,ds\Big] \exp\Big(\int_{t_1}^{t} c(s)\,ds\Big),$$

so that (6) is valid with

$$v_1 = \Big[v_0 + \int_{t_1}^{t_1+T} \exp\Big(\int_s^{t_1} c(\tau)\,d\tau\Big) b(s)\,ds\Big] \exp\Big(\int_{t_1}^{t_1+T} c(s)\,ds\Big).$$

In this example, all the solutions of the problem (5) for all values t_1 and v_0 are defined for all $t \ge t_1$. In the following example, the continuability property only holds for some of the solutions of (5) and assumption (6) can be considered as a restriction on the period T.

Example 2. Let $a(t,v) = 1 + v^2$. Then (5) has the form

$$v' = 1 + v^2, \quad v(t_1) = v_0,$$

and its unique solution

$$v(t;t_1,v_0) = \frac{v_0 + \tan(t-t_1)}{1 - v_0 \tan(t-t_1)}$$

satisfies (6) if and only if

$$T < \frac{\pi}{2} - \arctan v_0.$$

Here, $v_0 \tan T < 1$ and

$$v_1 = \frac{v_0 + \tan T}{1 - v_0 \tan T}.$$

Notice that for every nonnegative Caratheodory function a, there always exists a (possibly small) value of T such that assumption (6) holds.

Assumption (W). There exist a function $W \in C^1(\mathbb{R}^n \setminus B(0;r_0),\mathbb{R})$ such that

$$(V'(x), W'(x)) > 0, \quad |x| \geq r_0, \tag{7}$$

and such that for each T-periodic function x from the set

$$\{x \in C(\mathbb{R},\mathbb{R}^n) : \min_{t\in\mathbb{R}} |x(t)| \geq r_0\}, \tag{8}$$

the following inequality holds:

$$\int_0^T (W'(x(t)), f(t,x(t)))\,dt \leq 0. \tag{9}$$

In this assumption, the number r_0 is the one introduced in (3). The strict inequality (7) implies that $V'(x) \neq 0$ for $|x| \geq r_0$.

We can now state the main result of this paper.

Theorem 1. *Let the assumptions* **(V), (A)** *and* **(W)** *hold. Then equation* (1) *has at least one* T*-periodic solution.*

2. Proof of Theorem 1. The proof of Theorem 1 is based upon degree arguments and requires the introduction of the auxiliary family of differential equations

$$x' = \lambda f(t,x) - (1-\lambda)V'(x), \quad 0 \leq \lambda < 1. \tag{10}$$

Let us choose r_1 such that

$$V(x) > v_1 + 1 \quad \text{whenever} \quad |x| \geq r_1, \tag{11}$$

where v_1 is taken from (6). We shall use the following lemma.

Lemma 1. *All possible* T*-periodic solutions* x *of any of the equations* (10) *satisfy the a priori estimate*

$$|x(t)| \leq r_1, \quad t \in \mathbb{R}. \tag{12}$$

Proof. Suppose that there exist some $\lambda_* \in [0,1)$ and some T-periodic solution x_* of (10) with $\lambda = \lambda_*$ which does not satisfy (12). We consider the different cases separately. In the first case, we assume that x_* belongs to the set (8), so that

$$|x_*(t)| \geq r_0, \quad t \in \mathbb{R}, \tag{13}$$

and in the second case, $|x_*(t_*)| < r_0$ for some value t_*. In the first case, the function w_* defined by $w_*(t) = W(x_*(t))$, where W comes from assumption (W), is T-periodic, absolutely continuous and satisfies the relations

$$\begin{aligned} w_*'(t) &= (W'(x_*(t)), x_*'(t)) \\ &= \lambda_*(W'(x_*(t)), f(t,x_*(t)) - (1-\lambda_*)(W'(x_*(t)), V'(x_*(t))). \end{aligned}$$

Therefore, because of assumption (W), we have

$$\int_0^T w_*(t)\,dt = \lambda_* \int_0^T (W'(x_*(t)), f(t, x_*(t))\,dt$$
$$- (1-\lambda_*) \int_0^T (W'(x_*(t)), V'(x_*(t)))\,dt < 0,$$

which is impossible for an absolutely continuous T-periodic function. The second case is more complicated. Let us consider the T-periodic absolutely continuous function v_* defined by $v_*(t) = V(x_*(t))$. Since the function x_* does not satisfy (12), one has $|x_*(t)| > r_1$ for some values of t. Therefore, $v_*(t) > v_1 + 1$ for those values of t (see (11)). Consequently, there exist values t_1 and t_2 with

$$0 \le t_1 < T, \quad t_1 < t_2 \le t_1 + T$$

such that

$$|x_*(t_1)| = r_0, \quad |x_*(t)| > r_0, \quad t_1 < t \le t_2, \tag{14}$$

and

$$v_*(t_2) > v_1 + 1. \tag{15}$$

Consider the function v_* on the interval $[t_1, t_2]$. Since we have

$$v_*'(t) = \lambda_*(V'(x_*(t)), f(t, x_*(t)) - (1-\lambda_*)|V'(x_*(t))|^2$$

and (because of $\lambda_* < 1$)

$$(1-\lambda_*)|V'(x_*(t))|^2 > 0, \quad t_1 \le t \le t_2,$$

we get

$$v_*'(t) < \lambda_*(V'(x_*(t)), f(t, x_*(t))).$$

Therefore, by assumption (A), this gives

$$v_*'(t) < a(t, v_*(t)), \quad t_1 \le t \le t_2.$$

The relations (4) and (14) imply

$$v_*(t_1) = V(x_*(t_1)) \le v_0.$$

By the fundamental theorem on differential inequalities (see, e.g., [2]), the following relation holds:

$$v_*(t) \le v_{\text{upper}}(t; t_1, v_0), \quad t_1 \le t \le t_2, \tag{16}$$

where $v_{\text{upper}}(t; t_1, v_0)$ is the upper solution of the Cauchy problem (5). The estimates (16) and (6) imply the inequality $v_*(t_2) \le v_1$, which contradicts (15). □

Proof of Theorem 1. Theorem 1 can be deduced from Lemma 1 in different ways. For example, we can rely upon the constructions from [3], Chapter VI, which were used there to prove Theorem VI.2. We complete the proof here by using some ideas of relationship theorems given in [1], Chapter IV.

We consider the family of integral operators A_λ defined by

$$A_\lambda x = x(T) + \int_0^T [\lambda f(s, x(s)) - (1-\lambda)V'(x(s))]\, ds, \quad 0 \le \lambda \le 1. \tag{17}$$

Each operator A_λ acts in the space $C = C([0,T], \mathbb{R}^n)$ of continuous functions $x : [0,T] \to \mathbb{R}^n$ and is completely continuous. Every fixed point x_λ of the operator A_λ, for each $\lambda \in [0,1]$, takes equal values at the bounds of the interval $[0,T]$; i.e., $x_\lambda(0) = x_\lambda(T)$. Therefore, for every $\lambda \in [0,1]$, the set of fixed points of A_λ coincides with the set of T-periodic solutions of (10). By Lemma 1, all possible fixed points x_λ of A_λ satisfy the a priori estimate

$$\|x\|_C \le r_1, \quad 0 \le \lambda < 1. \tag{18}$$

Hence, all the completely continuous vector fields Φ_λ defined by

$$\Phi_\lambda x = x - A_\lambda x, \quad 0 \le \lambda < 1, \tag{19}$$

are non-zero and homotopic on the sphere

$$S = \{c \in C : \|x\|_C = r_1 + 1\}.$$

That means that the rotation $\gamma(\lambda)$ of the field (19) on S does not depend upon λ and takes the constant value γ_0. The computation of this value can be made for $\lambda = 0$. In this case, the field (19) has the form

$$(\Phi_0 x)(t) = x(t) - x(T) + \int_0^T V'(x(s))\, ds.$$

Since the zeros of this field coincide with the T-periodic solutions of the dissipative equation $x' = -V'(x)$, we have $\gamma_0 = 1$ (see [1]). Now, if A_1 has a fixed point on S, our theorem is proved. If not, then $\gamma(1) = \gamma_0 = 1 \ne 0$, and this implies the existence of a fixed point x of A_1 with $\|x\|_C < r_1 + 1$.

Remark. If, in Assumption (V), a is of the form $a(t,v) = b(t)c(v)$ with b T-periodic and locally Lebesgue integrable and c positive, continuous and such that

$$\int_1^{+\infty} \frac{dr}{c(r)} = +\infty,$$

then the condition (3) is equivalent to the condition

$$(Z'(x), f(t,x)) \le b(t),$$

for the function Z defined by

$$Z(x) = \int_1^{V(x)} \frac{dr}{c(r)}.$$

Indeed, we have

$$Z'(x) = \frac{V'(x)}{c(V(x))}$$

whenever $V(x) \geq 1$, which we can always assume without loss of generality. Moreover, $Z(x) \to \infty$ when $|x| \to \infty$ and Assumption (W) holds for Z when it holds for V. Thus, in this case, Theorem 1 is equivalent to Theorem VI.2 of [3]. This remark applies to Example 1 of Section 1 in which the comparison function can be replaced by $a(t,v) = e(t)(1+v)$ with $e(t) = \max(b(t), c(t))$.

3. Periodic solutions of some perturbed positively homogeneous systems. As an application, let us consider differential systems of the form

$$x' = h(t,x) + g(t), \tag{20}$$

where we assume, for simplicity, that $h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R} \to \mathbb{R}^n$ are T-periodic with respect to t and continuous and where h is positively homogeneous in x of order $d > 1$; i.e.,

$$h(t, sx) = s^d h(t,x) \tag{21}$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ and $s \geq 0$. The case where $d \in]0,1]$ has already been considered in [3], Section VI.4. It follows from our assumptions that $h(t,0) = 0$ for all $t \in \mathbb{R}$ and that, for $x \neq 0$, one has

$$|h(t,x)| = |x|^d |h(t, \frac{x}{|x|})| \leq H|x|^d,$$

where $H = \max_{t \in \mathbb{R}, |y|=1} |h(t,y)|$. If we set

$$G = \max_{t \in \mathbb{R}} |g(t)|, \quad V(x) = |x|^2,$$

then we have

$$(V'(x), h(t,x) + g(t)) \leq 2H|x|^{d+1} + 2G|x| = 2H(V(x))^{\frac{d+1}{2}} + 2G(V(x))^{\frac{1}{2}}$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$. Therefore, we have to study the Cauchy problem

$$v' = 2Gv^{\frac{d+1}{2}} + 2Gv^{\frac{1}{2}}, \quad v(t_1) = v_0 > 0. \tag{22}$$

If we define the real function Φ by

$$\Phi(v) = \int_0^{v^{1/2}} \frac{ds}{Hs^d + G}\, ds, \quad v > 0,$$

then Φ is increasing and

$$\lim_{v \to +\infty} \Phi(v) =: \Phi(+\infty) = \int_0^{\infty} \frac{ds}{Hs^d + G} = \frac{\frac{\pi}{d}}{G^{1-\frac{1}{d}} H^{\frac{1}{d}} \sin(\frac{\pi}{d})}.$$

Moreover, the solutions of the Cauchy problem (22) satisfy the relation

$$\Phi(v(t)) - \Phi(v_0) = t - t_1, \quad t \geq t_1.$$

Therefore, assumption (A) will be satisfied if and only if

$$T < \Phi(+\infty) - \Phi(v_0) = \frac{\frac{\pi}{d}}{G^{1-\frac{1}{d}} H^{\frac{1}{d}} \sin(\frac{\pi}{d})} - \Phi(v_0). \tag{23}$$

Assume , moreover, that there exist some locally Lebesgue integrable function b such that

$$\bar{b} =: \int_0^T b(t)\, dt < 0 \tag{24}$$

and

$$(y, h(t,y)) \leq b(t) \tag{25}$$

for all $t \in \mathbb{R}$, all $y \in \mathbb{R}^n$ with $|y| = 1$. Then we have

$$(\frac{x}{|x|^{d+1}}, h(t,x) + g(t)) \leq b(t) + \frac{|g(t)|}{|x|^d}$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n \setminus \{0\}$. Consequently, if $x : \mathbb{R} \to \mathbb{R}^n$ is continuous, T-periodic and such that $\min_{t \in \mathbb{R}} |x(t)| \geq r_0$, we have

$$\frac{1}{T} \int_0^T (\frac{x(t)}{|x(t)|^{d+1}}, h(t, x(t)) + g(t))\, dt \leq \frac{1}{T} \int_0^T [b(t) + \frac{|g(t)|}{r_0^d}]\, dt \leq 0$$

if we choose

$$r_0 = (\frac{\overline{|g|}}{|\bar{b}|})^{\frac{1}{d}}. \tag{26}$$

This suggests to take for W the function defined on $\mathbb{R}^n \setminus \{0\}$ by

$$W(x) = \frac{x}{|x|^{d+1}},$$

which clearly satisfies Assumption (W). The application of Theorem 1 provides therefore the following result.

Theorem 2. *Assume that the system* (20) *satisfies the conditions* (21), (24), (25). *Then the system* (20) *has at least one T-periodic solution for each $T > 0$ satisfying condition* (23), *with $v_0 = r_0^2$ and r_0 given by* (26).

In order words, perturbed positive homogeneous systems of degree $d > 1$ satisfying conditions (24) and (25) will have periodic solutions of sufficient small period T for sufficiently small perturbations g. Of course, similar results hold if the opposite sign holds in conditions (24) and (25).

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