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BOUNDARY VALUE PROBLEMS OF A CLASS OF QUASILINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. We consider a class of quasilinear ordinary differential equations. Using the generalized degree theory, we establish the existence of C^1-solutions for periodic and Neumann boundary value problems.

1. Introduction. In this paper, we establish the existence of solutions to the periodic boundary value problem (BVP)

$$(|u'|^{p-2}u')' + f(t,u,u') = y(t), \quad u(0) = u(1),\ u'(0) = u'(1) \tag{1.1}$$

under various conditions on the function $y : [0,1] \to \mathbb{R}$ and the function $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$.

We also consider the problem of the following form

$$\left.\begin{aligned} &Au - f(t,u) = 0 \quad \text{in } (0,1) \\ &u'(0) = u'(1) = 0, \end{aligned}\right\} \tag{1.2}$$

where $Au = -(a(|u'|^2)u')'$, $a : \mathbb{R} \to \mathbb{R}$ is a continuous mapping such that $h(t^2) = \int_0^{t^2} a(\tau)\,d\tau$ is a strictly convex function on $\mathbb{R}$. That is, the equation (1.2) coincides with the equation $g'(u) = 0$, where

$$g(t) = \frac{1}{2}\int_0^1 h(|u'|^2)\,dt - \int_0^1 \int_0^{u(t)} f(\tau, u(\tau))\,d\tau\,dt. \tag{1.3}$$

By a C^1-solution of problem (1.1) we mean that $u \in C^1([0,1])$, $u(0) = u(1)$, $u'(0) = u'(1)$ and u satisfies

$$|u'(t)|^{p-2}u'(t) - |u'(0)|^{p-2}u'(0) = -\int_0^t [f(s,u(s),\, u'(s)) - y(s)]\,ds.$$

By a C^1-solution of problem (1.2) we mean that $u \in C^1([0,1])$ satisfying $u'(0) = u'(1) = 0$ and

$$-(a|u'|^2)u')' - f(t,u) = 0 \quad \text{a.e. in } [0,1].$$

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Let $Au = -(|u'|^{p-2}u')'$ or $-(a(|u'|^2)u')'$ respectively. Then (1.1) and (1.2) may be written

$$Au - Nu = y. \tag{1.4}$$

When A is a linear operator and is Fredholm of index 0, equation (1.4) has been studied by many authors (cf. [6–7, 10–15]) using various theories of topological degree. If A is invertible, Leray-Schauder degree theory may be used; if A is not invertible, some generalized degree theory, such as coincidence degree theory has been employed. When $Au = -(|u'|^{p-2}u')'$ or $Au = -(a|u'|^2)u')'$ is not a Fredholm map or is not linear, these ideas are not directly applicable.

In a recent paper [16], the problem

$$(|u'|^{p-2}u')' + f(t,u) = 0, \quad u(0) = u(1) = 0, \tag{1.5}$$

was considered. Under homogeneous Dirichlet boundary condition, $G_p = A^{-1}$ is compact from $C^0[0,1]$ to $C^0[0,1]$ and equation (1.5) is equivalent to

$$u - G_p(f(t,u)) = 0, \tag{1.6}$$

so Leray-Schauder degree theory can be used. But under the periodic boundary condition or the Neumann boundary condition, A is not invertible, and the approach of [16] is not applicable. Problems of this kind were also studied in [1, 4, 8, 15, 17]. In these papers, all the authors considered the problems in the Sobolev spaces $W_0^{1,p}$ or $W^{1,p}$ and proved the existence of solutions in these spaces by generalized degree theory (see [4, 8, 15]).

It is clear that the methods do not work with our boundary conditions; we need solutions belonging to $C^1([0,1])$. We note that the special case of problem (1.4) with Dirichlet boundary conditions can be easily dealt with in $H_0^1(0,1)$, but we do not know how to deal with the problem with our boundary conditions. Let $Au = -(b(u)u')'$, for $b \in C^0(\mathbb{R}, \mathbb{R}^+)$, and consider the problem

$$\left.\begin{aligned} &Au - f(t,u) = 0 \quad \text{in } (0,1) \\ &u(0) = u(1) = 0 \end{aligned}\right\} \tag{1.7}$$

for $f \in C^0([0,1] \times \mathbb{R})$. In this case, the equation (1.7) does not have a variational structure. But it can be easily shown that $B : H_0^1(0,1) \to (H_0^1(0,1))^*$, $Bu = Au - f(t,u)$ is of type $(S)_+$, where $(H_0^1(0,1))^*$ is the dual space of $H_0^1(0,1)$. So, using the degree theory of the mapping of type $(S)_+$, we can obtain some existence results for (1.7) by using the same ideas as in Section 5 below. But it is not clear how the existence of solutions of the equation in (1.7) with periodic boundary condition or Neumann boundary condition can be treated.

In Sections 2–4, we study the existence of solutions of problem (1.1) and the problem:

$$-(|u'|^{p-2}u')' + f(t,u,u') = y(t), \quad u(0) = u(1), \ u'(0) = u'(1). \tag{1.8}$$

Our methods are closely related to [15], but we consider the problems in $C^2_{\text{per}}(0,1)$, where $C^2_{\text{per}}(0,1) = \{u \in C^2(0,1) : u(0) = u(1), \quad u'(0) = u'(1)\}$. In this case, we do not know whether $A : C^2_{\text{per}}(0,1) \to C^0[0,1]$ belongs to a class of mappings with a degree theory such as A-proper maps because we cannot prove that A maps $C^2_{\text{per}}(0,1)$ onto a linear subspace of $C^0[0,1]$. In order to overcome this difficulty, we replace A by $A_\varepsilon : C^2_{\text{per}}(0,1) \to C^0[0,1]$ and $A_\varepsilon u = -\varepsilon u'' - (|u'|^{p-2}u')'$ where $\varepsilon > 0$. Our results are related to those of the recent papers of [13–14]. Using the properties of A_ε, we can use coincidence degree theory to discuss the same problem, the details we will leave to the reader. In Section 5, we directly use coincidence degree theory to give an existence result for problem (1.2). The idea of this section comes from a recent paper [5] in which the authors discussed the case $p = 2$; some proofs can directly follow from it.

2. Some properties of A_ε. In this section we show that A_ε possesses properties similar to those of a Fredholm map of index 0, We first introduce the following notations. For $u \in C^2([0,1])$, define

$$\|u\|_0 = \sup_{[0,1]} |u(t)|, \ \|u\|_1 = \max\{\|u\|_0, \|u'\|_0\}, \|u\|_2 = \max\{\|u\|_0, \|u'\|_0, \|u''\|_0\}.$$

For $u \in W^{1,p}(0,1)$, define

$$\|u\|_p = \left(\int_0^1 |u(t)|^p \, dt\right)^{1/p}, \quad \|u\|_{1,p} = (\|u\|_p^p + \|u'\|_p^p)^{1/p}.$$

For convenience, we let $\phi_p(s) = |s|^{p-2}s$, where $p > 2$ is fixed and let $g_\varepsilon(s) = \varepsilon s + \phi_p(s)$.

Lemma 2.1. *For any $h \in C^0[0,1]$ with $\int_0^1 h(s)\,ds = 0$, there exists a unique $u \in C^2(0,1)$ such that u satisfies*

$$-\varepsilon u'' - (|u'|^{p-2}u')' = h, \quad u(0) = u(1) = 0, \quad u'(0) = u'(1). \tag{2.1}$$

Proof. For a given $h \in C^0[0,1]$ with $\int_0^1 h(t)\,dt = 0$, we look for a function $u \in C_0^1([0,1])$, such that $g_\varepsilon(u')$ is absolutely continuous and

$$-\varepsilon u'' - (|u'|^{p-2}u')' = h \quad \text{a .e. on } [0,1]. \tag{2.2}$$

First we find a solution $u \in W_0^{1,p}(0,1)$ of (2.2). It is well-known that searching for $u \in W_0^{1,p}(0,1)$ satisfying (2.2) is equivalent to finding critical points of the functional $\psi_h(w) = \frac{\varepsilon}{2}\int_0^1 |w'|^2 + \frac{1}{p}\int_0^1 |w'|^p - \int_0^1 hw$. We find that ψ_h is a continuous functional such that $\psi_h \to \infty$ as $\|w\|_{1,p} \to \infty$. Hence, (see [6]) it possesses a critical point $u \in W_0^{1,p}(0,1)$ at which it reaches its minimum. So, u satisfies $u(0) = u(1) = 0$ and

$$\int_0^1 [\varepsilon u' + \phi_p(u')]\, v' = \int_0^1 hv \tag{2.3}$$

for all $v \in W_0^{1,p}(0,1)$. Then, it follows that $\varepsilon u' + \phi_p(u')$ belongs to $L^q(0,1)$ and satisfies (2.3) for all $v \in C_0^\infty(0,1)$. Here, $q = p/(p-1)$, so $q < p$ for $p > 2$. Therefore, $\varepsilon u' + \phi_p(u') \in W^{1,q}(0,1)$. From this and Theorem VIII of [2] we can see that $g_\varepsilon(u')$ is an absolutely continuous function which satisfies (2.2). Since g_ε is invertible and $g_\varepsilon^{-1} \in C^1(\mathbb{R})$, using Remark 6 of [2] we find that $u \in C^1$. (2.2) means that $u'' = -h/[\varepsilon + (p-1)|u'|^{p-2}]$ almost everywhere in $[0,1]$ and $u'(0) = u'(1)$. The latter equality follows from the fact that the function $g_\varepsilon(s)$ is strictly increasing, (2.2) and $\int_0^1 h(s)\,ds = 0$. Now we can redefine u'' on a set of measure 0 so that $u'' = -h/[\varepsilon + (p-1)|u'|^{p-2}]$, for all $t \in [0,1]$, then $u \in C_0^2(0,1)$ and $u'(0) = u'(1)$.

Now we prove that u is unique. Suppose that there is $u_1 \in C^2(0,1)$ such that $-\varepsilon u_1'' - (|u_1'|^{p-2}u_1')' = h$ and $u_1(0) = u_1(1) = 0$, $u_1'(0) = u_1'(1)$, then

$$\varepsilon(u'' - u_1'') + [(|u'|^{p-2}u')' - (|u_1'|^{p-2}u_1')'] = 0.$$

Multiplying by $(u - u_1)$ and integrating from 0 to 1, we obtain

$$\varepsilon \int_0^1 (u' - u_1')^2\,dt + \int_0^1 (\phi_p(u') - \phi_p(u_1'))(u' - u_1')\,dt = 0.$$

By the monotonicity of ϕ_p we get $u' = u_1'$ on $[0,1]$ and hence $u \equiv u_1$ on $[0,1]$.

Remark 2.2. If $\varepsilon = 0$, then $g_0(s) = \phi_p(s)$. It follows directly that ϕ_p^{-1} does not belong to $C^1(\mathbb{R})$. So, we cannot prove that $u \in C^2(0,1)$ from (2.2).

Let $C^2_{0,\mathrm{per}}(0,1) = \{u \in C^2_{\mathrm{per}}(0,1) : u(0) = u(1) = 0\}$. Then

$$C^2_{\mathrm{per}}(0,1) = \mathbb{R} \oplus C^2_{0,\mathrm{per}}(0,1).$$

From above we know $u_0 \in C^2_{0,\mathrm{per}}(0,1)$. For $h \in C^0(0,1)$, we write $h = h_0 + h_1$, $h_0 = \int_0^1 h$, and $\int_0^1 h_1 = 0$, then $C^0 = \mathbb{R} \oplus Z$, here $Z = \{h \in C^0(0,1) : \int_0^1 h = 0\}$. Let $A_{\varepsilon,1} : C^2_{0,\mathrm{per}}(0,1) \to Z$, $A_{\varepsilon,1}(\bar{u}) = A_\varepsilon(u)$, where $u = u(0) + \bar{u}$, then $A_{\varepsilon,1}$ is invertible.

Lemma 2.3. *$A_\varepsilon : C^2_{\mathrm{per}}(0,1) \to C^0[0,1]$ is a continuous map.*

Proof. Note that the function $T : (0,1) \times \mathbb{R}^2 \to \mathbb{R}$, $T(t,x,y) = (\varepsilon + |x|^{p-2})y$ is continuous, so the proof of this lemma is then routine.

Lemma 2.4. *$A_{\varepsilon,1}^{-1} : Z \to C^1_{0,\mathrm{per}}$ is compact.*

Proof. For a bounded sequence $h_n \in Z$, $A_{\varepsilon,1}^{-1}(h_n) = u_n$, then $u_n \in C^2_{0,\mathrm{per}}(0,1)$ and

$$\|[\varepsilon u_n' + \phi_p(u_n')]'\|_0 \le \|h_n\|_0 \le M. \tag{2.4}$$

As $u_n(0) = u_n(1) = 0$, we have that for any n, there exists a $t_n \in (0,1)$ such that $u_n'(t_n) = 0$ and $\phi_p(u_n'(t_n)) = 0$. From this and $A_{\varepsilon,1}u_n = h_n$, it clearly follows that $-\varepsilon u_n'(t) - \phi_p(u_n'(t)) = \int_{t_n}^t h_n(s)\,ds$. Therefore,

$$\|\varepsilon u_n' + \phi_p(u_n'(t))\|_0 \le \|h_n\|_0 \le M. \tag{2.5}$$

(2.4)–(2.5) imply $\|\varepsilon u_n' + \phi_p(u_n')\|_1 \le M$. Using the fact that $i : C^1([0,1]) \to C^0[0,1]$ is compact, we obtain that there exists a convergent subsequence (still denoted by $\{\varepsilon u_n' + \phi_p(u_n')\}$) such that $\varepsilon u_n' + \phi_p(u_n') \to v$ in $C^0[0,1]$. As $\varepsilon + \phi_p : \mathbb{R} \to \mathbb{R}$ is strictly increasing and continuous, we have $u_n' \to (\varepsilon + \phi_p)^{-1}(v)$ in $C^0[0,1]$. Let $\bar{u}(t) = \int_0^t (\varepsilon + \phi_p)^{-1}(v(s))\,ds$, then, it follows from $u_n' \to (\varepsilon + \phi_p)^{-1}(v)$ in $C^0[0,1]$ that $\bar{u}(1) = 0$ and $\bar{u}'(0) = \bar{u}'(1)$. Therefore, $\bar{u} \in C^1_{0,\mathrm{per}}([0,1])$ and $u_n(t)$ tends to $\int_0^t (\varepsilon + \phi_p)^{-1}(v(s))\,ds$ in $C^1_{0,\mathrm{per}}([0,1])$.

Lemma 2.5. $A_{\varepsilon,1}^{-1} : Z \to C_{0,\mathrm{per}}^2$ *is continuous.*

Proof. Let $h_n \in Z$ and $u_n \in C_{0,\mathrm{per}}^2$ such that $h_n \to h$ in C^0 and $A_{\varepsilon,1}(u_n) = h_n$, then from Lemma 2.4 we have $u_n \to u$ in $C_{0,\mathrm{per}}^1$. As $u_n'' = -h_n/(\varepsilon + (p-1)|u_n'|^{p-2})$ we also have that $u_n \to u$ in $C^2(0,1)$ and $u'' = -h/(\varepsilon + (p-1)|u'|^{p-2})$ on $(0,1)$.

3. An abstract existence theorem. In this section, we define a new scheme and prove that for some N, $A_\varepsilon - \lambda N$ is A-proper with respect to this scheme. We also give an existence result for problem (1.1).

Let E, Y be real Banach spaces. We recall that if $\{E_n\} \subset E$ and $\{Y_n\} \subset Y$ are sequences of finite dimensional oriented spaces and $Q_n : Y \to Y_n$ is a linear projection for each $n \in Z^+$, then the scheme $\Gamma' = \{E_n, Y_n, Q_n\}$ is said to be *admissible* for maps $E \to Y$ provided that $\dim E_n = \dim Y_n$, $\mathrm{dist}\,(e, E_n) \to 0$ for each e in E, and $Q_n y \to y$ for each y in Y.

Definition [13]. A map $T : X \to Y$ is said to be A-proper with respect to Γ if and only if (i) $T_n = Q_n T : X_n \to Y_n$ is continuous; and (ii) if $\{u_{n_j} : u_{n_j} \in X_{n_j}\}$ is any bounded sequence such that $T_{n_j}(u_{n_j}) \to g$ for some g in Y, then there exists a subsequence $\{u_{n_{j(k)}}\}$ and $u \in X$ such that $u_{n_{j(k)}} \to u$ as $k \to \infty$ and $Tu = g$.

Let $Z_n \subset Z$ (Z is as in Section 2) be sequences of oriented finite dimensional spaces, $Y_n = \mathbb{R} \oplus Z_n$ and $Q_n : Y \to Y_n$ be a linear projection of Y onto Y_n for each $n \geq 1$ such that for any $y \in C^0[0,1]$, $Q_n y \to y$. Let $X_n = \mathbb{R} \oplus (A_{\varepsilon,1}^{-1}(Z_n))$, then X_n are sequences of oriented finite dimensional open sets in $C_{\mathrm{per}}^2(0,1)$ and from the properties of Y_n, we have that for any $u \in C_{\mathrm{per}}^2(0,1)$, $\mathrm{dist}\,(u, X_n) \to 0$ as $n \to \infty$.

Now we define a new scheme $\Gamma \equiv \{X_n, Y_n, Q_n\}$ and have the following lemma.

Lemma 3.1. *The Scheme* $\Gamma \equiv \{X_n, Y_n, Q_n\}$ *is admissible.*

Proof. For any n, $A_{\varepsilon,1} : A_{\varepsilon,1}^{-1}(Z_n) \to Z_n$ is a homeomorphism. By the fact that dimension is a topological invariant (see [9], p. 24), we know that

$$\dim (A_{\varepsilon,1}^{-1}(Z_n)) = \dim(Z_n)$$

and so, $\dim (X_n) = \dim (Y_n)$. Here, $A_{\varepsilon,1}^{-1}(Z_n)$ is an open set in $C_{0,\mathrm{per}}^1(0,1)$ and $\dim (A_{\varepsilon,1}^{-1}(Z_n))$ is defined as in [9].

Theorem 3.2. *Suppose* A_ε *is as in Lemma* 2.1, Γ *is as in Lemma* 3.1. *Then* A_ε *is* A*-proper with respect to* Γ.

Proof. Let $\{u_{n_j} : u_{n_j} \in X_{n_j}\}$ be any sequence, bounded in $C_{\mathrm{per}}^2(0,1)$, such that $g_{n_j} \equiv Q_{n_j} A_\varepsilon(u_{n_j}) \to g$ for some g in C^0. Since $u_{n_j}(t) = u_{n_j}(0) + \bar{u}_{n_j}(t)$, $\bar{u}_{n_j} \in X_{1,n_j}$, here $X_{1,n_j} \subset C_{0,\mathrm{per}}^2$, $A_\varepsilon(u_{n_j}) = A_{\varepsilon,1}(\bar{u}_{n_j}) \subset Z_{n_j}$, then $Q_{n_j} A_\varepsilon(u_{n_j}) = A_\varepsilon(u_{n_j})$. We see that $g_{n_j} \equiv A_\varepsilon(u_{n_j}) = A_{\varepsilon,1}(\bar{u}_{n_j}) \to g$ in C^0. By the continuity of $A_{\varepsilon,1}^{-1}$, we know that $\bar{u}_{n_j} \to A_{\varepsilon,1}^{-1} g$. As $\{u_{n_j}(0)\}$ is bounded, we also have $u_{n_j}(0) \to C$ and so, u_{n_j} converges to $C + A_{\varepsilon,1}^{-1} g$. Let $u = C + A_{\varepsilon,1}^{-1} g$, then $u \in C_{\mathrm{per}}^2(0,1)$ and $A_\varepsilon u = g$. $\square$

Now, using the A-proper property of A_ε and the same ideas as in [3, 13], we can establish the generalized degree theory as in [3, 13] and obtain an existence theorem similar to Corollary 2 of [13], but in [3] and [13] the map A is linear and Fredholm of index 0.

Theorem 3.3. (Existence Theorem) *Let $y \in C^0[0,1]$, A_ε be as in Lemma 2.1, let $G \subset C^2_{\text{per}}$ be an open bounded set with $0 \in G$ and $N : C^2_{\text{per}}(0,1) \to C^0[0,1]$ be a bounded continuous nonlinear map such that*

(a) *$A_\varepsilon - \lambda N$ is A-proper* w.r.t. Γ, $\lambda \in (0,1]$,

(b) *$A_\varepsilon u \neq \lambda N u + \lambda y$ for $u \in \partial G$ and $\lambda \in (0,1]$,*

(c) *$QNu + Qy \neq 0$ for $u \in \mathbb{R} \cap \partial G$, where Q is a linear projection of C^0 onto $\mathbb{R}$ with $C^0 = \mathbb{R} \oplus Z$,*

(d) *either*

  (i) $\int_0^1 (QNu + Qy)\, u(t)\, dt \geq 0$ *or*

  (ii) $\int_0^1 (QNu + Qy)\, u(t)\, dt \leq 0$ *for $u \in \mathbb{R} \cap \partial G$.*

Then there exist $u \in \bar{G}$ such that $A_\varepsilon u - Nu = y$.

Proof. Since $A_\varepsilon - \lambda N$ is A-proper, this theorem can be proved by using the same idea as in the proofs of Corollary 2 of [13] and Remark 1.1 of [14].

4. Existence of solutions of (1.1).

In this section we use Theorem 3.3 to establish the existence of solutions to the problems

$$\varepsilon u'' + (|u'|^{p-2}u')' + f(t,u,u') = y(t), \quad u(0) = u(1), \quad u'(0) = u'(1), \qquad (4.1)_\varepsilon$$

where $p > 2$, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $y \in C^0[0,1]$. Then we shall let ε tend to 0 and obtain the existence of a solution to (1.1). We assume that

(H_1) $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and there are positive constants A, B, C such that $B + C < \pi_p$ and

$$|f(t,q,r)| \leq A + B|q|^{p-1} + C|r|^{p-1} \qquad (4.2)$$

for $t \in [0,1]$ and q, $r \in \mathbb{R}$, where $\pi_p > 0$ is the first eigenvalue of the problem (see [16])

$$(|u'|^{p-2}u')' + \lambda |u|^{p-2}u = 0, \quad u(0) = u(1) = 0.$$

(H_1)$'$ $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and satisfies

  (i) there exists a continuous function $f_1 : [0,1] \times \mathbb{R} \to \mathbb{R}$ and a constant C such that $|f(t,q,r)| \leq f_1(t,q) + C|r|^p$,

  (ii) there are constants α, β, $\gamma \geq 0$ and σ, $\tau < p$ such that

$$q\, f(t,q,r) \geq -\alpha |q|^\sigma - \beta |r|^\tau - \gamma \qquad (4.3)$$

  for $t \in [0,1]$, q, $r \in \mathbb{R}$.

Now we prove the following theorem.

Theorem 4.1. *Suppose that in addition to* (H_1) *we assume that*

(H_2) *To a given $y \in C^0[0,1]$ there exists $M > 0$ (depending on y) such that $\int_0^1 \{f(t,u,u') - y(t)\}\, dt \neq 0$, for $u \in C^2(0,1)$ with $|u| > M$ for $t \in [0,1]$.*

(H_3) *There are $M_1 \geq M$ and a, $b \in \mathbb{R}$ such that either* (i) *or* (ii) *holds, where*

  (i) *$a \geq b$, $x \in \mathbb{R}$ and $x \geq M_1 \Longrightarrow f(t,x,0) \geq a$; $x \leq -M \Longrightarrow f(t,x,0) \leq b$ for $t \in [0,1]$ and $b \leq y_1 \leq a$ with $y_1 = \int_0^1 y\, dt$,*

  (ii) *$a \leq b$, $x \in \mathbb{R}$ and $x \geq M_1 \Longrightarrow f(t,x,0) \leq a$; $x \leq -M \Longrightarrow f(t,x,0) \geq b$ for $t \in [0,1]$ and $a \leq y_1 \leq b$.*

Then the periodic BVP (4.1_ε) *has a solution in* $C^2(0,1)$.

Proof. Let $Nu = f(t,u,u')$, then the map $N : C^2(0,1) \to C^0[0,1]$ is compact since $C^2(0,1)$ is compactly embedded in $C^1([0,1])$ and, by Theorem 3.2, $A_\varepsilon - \lambda N$ is A-proper with respect to Γ for each $\lambda \in (0,1]$. Condition (a) in Theorem 3.3 holds. Next we show that if (H_1) holds, then there exists $r > 0$ such that if we let $G\{x \in C^2(0,1) : \|x\|_2 \le r\}$, then (b) of Theorem 3.3 holds. For that it suffices to show that if $u \in C^2(0,1)$ is a solution of

$$-\varepsilon u'' - (|u'|^{p-2}u')' = \lambda f(t,u,u') - \lambda y, \quad u(0) = u(1), \quad u'(0) = u'(1) \tag{4.4}$$

for some $\lambda \in (0,1]$, then $\|u\|_2 \le M_2$ for some $M_2 > 0$ independent of u and λ.

So, let $u \in C^2(0,1)$ be a solution of (4.4) and integrate from 0 and 1 to obtain

$$\int_0^1 \{f(t,u(t),u'(t)) - y(t)\}\,dt = 0. \tag{4.5}$$

It follows from (4.5) and (H_2) that there exists $t_0 \in [0,1]$ such that $|u(t_0)| \le M$. We write $u(t) = u(t_0) + \int_0^1 u'(s)\,ds$, and so

$$|u(t)| \le M + \|u'\|_p, \quad \text{where} \quad \|v\|_p = \left(\int_0^1 |v|^p\,dt\right)^{1/p}. \tag{4.6}$$

For $u \in C^2_{\text{per}}(0,1)$, we write $u(t) = u(0) + h$ with $h \in C^2_{0,\text{per}}$, then $u' = h'$ and $\|u'\|_p = \|h'\|_p$. Therefore, $\|u(t)\|_p \le M + \|h'\|_p$. From the equality (4.4), we obtain

$$\varepsilon \int_0^1 h'^2\,dt + \int_0^1 |h'(t)|^p\,dt = \lambda \int_0^1 \{f(t,u,u') - y(t)\}h(t)\,dt, \tag{4.7}$$

then,

$$\begin{aligned} \int_0^1 |h'|^p dt &\le \|(|f(t,u,u')| + |y|)\|_q \|h\|_p \\ &\le (A + B\|u\|_p^{p-1} + C\|h'\|_p^{p-1} + \|y\|_0)\|h\|_p. \end{aligned} \tag{4.8}$$

In view of the fact that $h(0) = h(1) = 0$ and $\|h\|_p \le \pi_p^{-1}\|h'\|_p$ (see [16]), one easily derives from (4.6) and (H_1) that there exists $A_1 > 0$ independent of λ and ε such that $\|h'\|_p \le A_1$ and so, $|u| \le M + A_1$ for $t \in [0,1]$ and $\|u\|_p \le M + A_1$.

It follows from (4.4) that

$$\varepsilon h'(t) + |h'(t)|^{p-2}h'(t) = \lambda \int_{t_1}^t \{f(s,u,h') - y(s)\}\,ds, \tag{4.9}$$

where $t_1 \in (0,1)$ is such that $h'(t_1) = 0$. Then, we get from (4.9) that

$$|\varepsilon h'(t) + |h'(t)|^{p-2}h'(t)| \le A_2, \tag{4.10}$$

here $A_2 = A_2(M, A_1, A, B, C, \|y\|_0)$. Using the facts that ϕ_p is strictly increasing and A_2 is independent of ε and (4.10), we easily obtain that there exists A_3, such that $|h'| \le A_3$ for $t \in (0,1)$, and A_3 is also independent of λ and ε. Hence,

$\|u\|_1 \le A_3 + M$. From (4.4) we also obtain $-(\varepsilon + |u'|^{p-2})u'' = \lambda f(t,u,u') - \lambda y$, therefore,

$$|u''| \le \varepsilon^{-1}(|f(t,u,u')| + |y|) \le \varepsilon^{-1}A_4, \tag{4.11}$$

where $A_4 > 0$ is independent of ε, λ. Now let

$$r > \max\{A_3 + M, \varepsilon^{-1}A_4\} \text{ and } G = \{u \in C^2_{\text{per}}(0,1) : \|u\|_2 \le r\},$$

then for $\lambda \in (0,1]$ and $u \in \partial G$,

$$-\varepsilon u'' - (|u'|^{p-2}u')' \ne \lambda f(t,u,u') - \lambda y, \tag{4.12}$$

that is, condition (b) of Theorem 3.3 holds. Note that Theorem 3.3 (c) holds, for

$$QNx - Qy = \int_0^1 \{f(t,x,0) - y(t)\}\,dt \ne 0, \quad x \in \mathbb{R} \cap \partial G.$$

This follows from the fact that if $x \in \mathbb{R} \cap \partial G$, then $|x| = r > M$, using (H_2). Theorem 3.3 (d) follows directly from (H_3) (see [13]). Hence, the conclusion of Theorem 4.1 follows from Theorem 3.3.

Corollary 4.2. *Suppose that* (H_1)–(H_3) *of Theorem* 4.1 *hold,* $y \in C^0[0,1]$. *Then BVP* (1.1) *has at least one solution in* $C^1([0,1])$.

Proof. From the proof of Theorem 4.1, we know that for any $\varepsilon > 0$, there exist at least one $u_\varepsilon \in C^2_{\text{per}}(0,1)$ which satisfies

$$\varepsilon u''_\varepsilon + (|u'_\varepsilon|^{p-2}u'_\varepsilon)' + f(t,u_\varepsilon,u'_\varepsilon) = y(t), \tag{4.13}$$

and $\|u_\varepsilon\|_1 \le A_3 + M$, $A_3 + M$ is independent of ε. Let $u_\varepsilon = u_\varepsilon(0) + \bar{u}_\varepsilon$, then $\bar{u}_\varepsilon$ satisfies $A_{\varepsilon,1}(\bar{u}_\varepsilon) = f(t,u_\varepsilon,u'_\varepsilon) - y(t)$ and $\|f(t,u_\varepsilon,u'_\varepsilon) - y\|_0 \le A_5$ (here $A_5 > 0$ is independent of ε). Lemma 2.4 implies that there exists $\bar{u} \in C^1_{0,\text{per}}([0,1])$ such that $\bar{u}_\varepsilon \to \bar{u}$ in $C^1([0,1])$ as $\varepsilon \to 0$. The boundedness of $\|u_\varepsilon\|_0$ implies that $u_\varepsilon(0) \to C$ as $\varepsilon \to 0$, here $C \in \mathbb{R}$, hence $u_\varepsilon \to u$ in $C^1([0,1])$, $u = C + \bar{u}$. Integrating (4.13) from 0 to t and letting $\varepsilon \to 0$, we obtain that $u \in C^1_{\text{per}}([0,1])$ is a solution of (1.1).

Remark 4.3. If the function f in problem (1.1) is independent of u', then $N(u) = f(t,u)$ is also compact as a map from $C^2_{\text{per}}(0,1)$ to $C^0[0,1]$ and so Corollary 4.2 yields existence results for the periodic BVP

$$(|u'|^{p-2}u')' + f(t,u) = y, \quad u(0) = u(1), \quad u'(0) = u'(1). \tag{4.14}$$

Combining the facts in [14] and Theorem 4.1, we have the following theorem.

Theorem 4.4. *Let* $y_1 = 0$ *and suppose* $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous and satisfies the conditions*

- (H_4) *There is* $M > 0$ *such that* $f(t,u,u')\,u < 0$ *for* $u \in C^2_{\text{per}}(0,1)$ *with* $|u(t)| > M$ *for* $t \in [0,1]$.
- (H_5) *There exists a continuous function* $f_1 : [0,1] \times \mathbb{R} \to \mathbb{R}$ *and a constant* $C \in \mathbb{R}^+$ *with* $C < \pi_p$ *such that* $|f(t,q,r)| \le f_1(t,q) + C|r|^{p-1}$ *for* $t \in [0,1]$ *and* q, $r \in \mathbb{R}$. *Then the periodic BVP* (1.1) *has a solution in* $C^1([0,1])$.

Now we give an existence theorem for a different version of the BVP $(4.1)_\varepsilon$, namely,

$$-\varepsilon u'' - (|u'|^{p-2}u')' + f(t,u,u') = y(t), \quad u(0) = u(1), \quad u'(0) = u'(1). \tag*{$(4.1)'_\varepsilon$}$$

Theorem 4.5. *Suppose that in addition to* $(\mathrm{H}_1)'$ *we assume that* (H_2) *and* (H_3) *in Theorem* 4.1 *hold. Then the periodic BVP* $(4.1)'_\varepsilon$ *has a solution in* $C^2(0,1)$.

Proof. Let $Nu = -f(t,u,u')$, then condition (a) of Theorem 3.3 holds. To prove this theorem, it suffices to show that $u \in C^2(0,1)$ is a solution of

$$-\varepsilon u'' - (|u'|^{p-2}u')' = -\lambda f(t,u,u') + \lambda y, \quad u(0) = u(1), \quad u'(0) = u'(1) \tag{4.15}$$

for some $\lambda \in (0,1]$, then $\|u\|_2 \le M_3$ for some $M_3 > 0$ independent of u and λ.

Following the same ideas as in the proof of Theorem 4.1, we obtain that

$$\|u\|_p \le M + \|u'\|_p$$

and

$$\varepsilon \int_0^1 u'^2\,dt + \int_0^1 |u'|^p\,dt + \lambda \int_0^1 f(t,u,u')u\,dt = \lambda \int_0^1 yu\,dt.$$

In view of (4.3), one easily derives that

$$\int_0^1 f(t,u,u')u\,dt \ge -\alpha \int_0^1 |u|^\sigma - \beta \int_0^1 |u'|^\tau - \gamma \ge -\alpha\|u\|_p^\sigma - \beta\|u'\|_p^\tau - \gamma.$$

Then, $\|u'\|_p^p \le \alpha\|u\|_p^\sigma + \beta\|u'\|_p^\tau + \|y\|_0\|u\|_p + \gamma$. So, there exists M_4 which depends on M, α, β, γ, σ, τ, and $\|y\|_0$, but is independent of λ and ε such that $\|u'\|_p \le M_4$. Therefore, $\|u\|_{1,p} \le M + 2M_4$. Since $W^{1,p}(0,1)$ is embedded in $C^0(0,1)$, $\|u\|_0 \le M_5$, where M_5 is independent of ε and λ. Let

$$M_6 = \sup\{|f(t,q)| : 0 \le t \le 1,\ |q| \le M_5\},$$

then M_6 is also independent of λ, ε. From this, $(\mathrm{H}_1)'$ and (4.15), it easily follows that

$$|\varepsilon u' + (|u'|^{p-2}u')| \le \int_0^1 |f(t,u,u')|\,dt + \|y\|_0 \le M_6 + C(M+2M_4)^p + \|y\|_0. \tag{4.16}$$

Let $M_7 = M_6 + C(M+2M_4)^p + \|y\|_0$, we know that M_7 is independent of λ and ε. Therefore, $|u'|_0 \le M_7^{1/(p-1)}$. From (4.15) we also get that there exists $M_8 > 0$ independent of λ, ε such that $|u''| \le \varepsilon^{-1}M_8$. Now, let

$$r > \max\{M_5,\ M_7^{1(p-1)}, \varepsilon^{-1}M_8\} \text{ and } G = \{u \in C^2_{\mathrm{per}}(0,1) : \|u\|_2 \le r\},$$

then for $\lambda \in (0,1]$ and $u \in \partial G$,

$$-\varepsilon u'' - (|u'|^{p-2}u')' \ne -\lambda f(t,u,u') + \lambda y. \tag{4.17}$$

Condition (b) of Theorem 3.3 holds.

Corollary 4.6. *Suppose that the conditions of Theorem* 4.5 *hold,* $y \in C^0[0,1]$. *Then BVP* (1.8) *has at least one solution in* $C^1([0,1])$.

If the function f in $(4.1)'_\varepsilon$ is independent of u', then we have the following corollary.

Corollary 4.7. *Suppose that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist constants α, $\beta \geq 0$ and $\sigma < p$ such that*

$$qf(t,q) \geq -\alpha|q|^{\sigma} - \beta \tag{4.18}$$

for $t \in [0,1]$ and $q \in \mathbb{R}$. We also assume that

$(\mathrm{H}_2)'$ *To a given $y \in C^0[0,1]$ there exist $M > 0$ (depending on y) such that $\int_0^1 \{f(t,u) - y(t)\}\,dt \neq 0$, for $u \in C^2_{\mathrm{per}}(0,1)$ with $|u| > M$ for $t \in [0,1]$.*

$(\mathrm{H}_3)'$ *There are $M_1 \geq M$ and a, $b \in \mathbb{R}$ such that either* (i) *or* (ii) *holds, where*

(i) *$a \geq b$, $x \in \mathbb{R}$ and $x \geq M_1 \Longrightarrow f(t,x \geq a$; $x \leq -M \Longrightarrow f(t,x) \leq b$ for $t \in [0,1]$ and $b \leq y_1 \leq a$ with $y_1 = \int_0^1 y\,dt$.*

(ii) *$a \leq b$, $x \in \mathbb{R}$ and $x \geq M_1 \Longrightarrow f(t,x) \leq a$; $x \leq -M \Longrightarrow f(t,x) \geq b$ for $t \in [0,1]$ and $a \leq y_1 \leq b$.*

Then the periodic BVP

$$-(|u'|^{p-2}u')' + f(t,u) = y, \quad u(0) = u(1), \quad u'(0) = u'(1)$$

has a solution in $C^1([0,1])$.

Proof. This Corollary follows from Theorem 4.5 and Corollary 4.6.

5. Existence results for problem (1.2).

In this section, we directly use coincidence degree theory [12] to discuss the problem

$$\left.\begin{aligned} &Au - f(t,u) = 0 \quad \text{in } (0,1) \\ &u'(0) = u'(1) = 0, \end{aligned}\right\} \tag{5.1}$$

where $Au = -(a(|u'|^2)u')'$, $a : \mathbb{R} \to \mathbb{R}$ is a continuous mapping that satisfies the following conditions

(a_1) the mapping $h(t^2)$ is strictly convex, where

$$h(t^2) = \int_0^{t^2} a(\tau)\,d\tau;$$

(a_2) there exist $p > 1$, c_0, $c_3 > 0$ and c_1, $c_2 \geq 0$ such that

$$c_0|t|^{p-2} + c_1 \leq a(t^2) \leq c_2 + c_3|t|^{p-2} \tag{5.2}$$

for all $t \in \mathbb{R}$.

Using the same ideas as above, we can obtain some existence results for problem (5.1). But in this section, we use a simpler method.

The condition (a_2) holds if $a(t)$ is a polynomial with $a(t) \geq c_1$. Other examples are $a(t^2) = 1 + (1+t^2)^{-2}$ and for any $p > 1$, $a(t^2) = (t^2)^{(p-2)/2}$. We first show the following lemma.

Lemma 5.1. *Let* $Au = -(a|u'|^2)u')'$, $S(u) = a(|u|^2)$ *and*

$$C^1_*([0,1]) = \{u \in C^1([0,1]) : u'(0) = u'(1) = 0\}.$$

For any $\delta > 0$, *let* $J_\delta(u) = Au + \delta\, S(u)u$. *Suppose the function a satisfies conditions* (a_1) *and* (a_2). *Then,* J_δ *is invertible and*

$$J_\delta^{-1} : L^q(0,1) \to C^1_*([0,1]) \quad \textit{is compact,}$$

where $q = p/(p-1)$, p *is as in* (5.2).

Proof. For a given $x \in L^q(0,1)$ with $q = p/(p-1)$, we look for a function $u \in C^1([0,1])$ satisfying

$$\left.\begin{aligned} &Au + \delta\, S(u)u = x \quad \text{a.e. on } [0,1] \\ &u'(0) = u'(1) = 0 \end{aligned}\right\} \tag{5.3}$$

with $a(|u'|^2)u'$ an absolutely continuous function on $[0,1]$. Clearly, if u is such a solution, then it satisfies

$$\int_0^1 a(|u'|^2)u'v' + \delta \int_0^1 a(|u|^2)uv = \int_0^1 xv, \tag{5.4}$$

for all $v \in W^{1,p}(0,1)$. Conversely, if $u \in W^{1,p}(0,1)$ satisfies (5.4) for all $v \in W^{1,p}(0,1)$, then by condition (a_2), $a(|u'|^2)u' \in L^q(0,1)$, $a(|u|^2)u \in L^q(0,1)$ and satisfies (5.4) for all $v \in C_0^\infty(0,1)$. Hence, $a(|u'|^2)u' \in W_0^{1,p}(0,1)$; $a(|u'|^2)u'$ is an absolutely continuous function on $[0,1]$. The embedding of $W_0^{1,p}(0,1)$ to $C^0[0,1]$ implies that $a(|u'|)u' \in C^0(0,1)$ and $a(|u'(0)|^2)u'(0) = 0 = a(|u'(1)|^2)u'(1)$. Since the function $\phi(t) = a(t^2)t$ is strictly increasing, we conclude that $u \in C^1([0,1])$ and $u'(0) = u'(1) = 0$. This means $u \in C^1_*([0,1])$.

Now, we prove that there exists $u \in W^{1,p}(0,1)$ such that (5.4) holds. Consider

$$I(u) = \frac{1}{2}\int_0^1 h(|u'|^2) + \frac{\delta}{2}\int_0^1 h(|u|^2) - \int_0^1 xu. \tag{5.5}$$

By the properties of $h(t)$, we know that

$$I(u) \geq C\|u\|^p_{1,p}, \quad \text{(with some } C > 0). \tag{5.6}$$

We also find I is a continuous convex functional on $W^{1,p}(0,1)$. Hence, it possesses a critical point $u \in W^{1,p}(0,1)$ at which it reaches its minimum. We also know that at u, (5.4) holds for all $v \in W^{1,p}(0,1)$.

For $x \in L^q(0,1)$, there is only one $u \in W^{1,p}(0,1)$ satisfying (5.4). Suppose not, then,there are u_1, $u_2 \in C^1_*([0,1])$, $u_1 \not\equiv u_2$ such that

$$\int_0^1 a(|u_1'|^2)\, u_1'v' + \delta \int_0^1 a(|u_1|^2)\, u_1 v = \int_0^1 xv, \tag{5.7}$$

$$\int_0^1 a(|u_2'|^2)\, u_2'v' + \delta \int_0^1 a(|u_2|^2)\, u_2 v = \int_0^1 xv, \tag{5.8}$$

for all $v \in W^{1,p}(0,1)$. Then,

$$0 = (J_\delta(u_1) - J_\delta(u_2),\ u_1 - u_2) = \int_0^1 (a(|u_1'|^2)u_1' - a(|u_2'|^2)u_2')(u_1' - u_2')$$
$$+ \delta \int_0^1 (a(|u_1|^2)u_1 - a(|u_2|^2)u_2)(u_1 - u_2)$$
$$\geq \int_0^1 (a(|u_1'|^2)|u_1'| - a(|u_2'|^2)|u_2'|)(|u_1'| - |u_2'|)$$
$$+ \delta \int_0^1 (a(|u_1|^2)|u_1| - a(|u_2|^2)|u_2|)(|u_1| - |u_2|) > 0.$$

This is a contradiction. Therefore, J_δ is invertible.

To end the proof of this lemma, we have to prove that J_δ^{-1} is compact. We first show that $J_\delta^{-1} : L^q(0,1) \to C_*^1([0,1])$ is continuous. Let $\{x_n\}$ be a sequence in $L^q(0,1)$ such that $x_n \to x$ as $n \to \infty$. Suppose that $J_\delta^{-1}(x_n)$ does not converge to $J_\delta^{-1}(x)$ as $n \to \infty$. Hence, there exists an $\varepsilon > 0$ and a subsequence of $\{x_n\}$ which we will call again $\{x_n\}$ such that

$$\|J_\delta^{-1}(x_n) - J_\delta^{-1}(x)\|_1 \geq \varepsilon, \tag{5.9}$$

for all $n \in \mathbb{N}$. Based on the definition of the mappings ϕ, J_δ^{-1} and the fact that for a solution u of problem (5.3), $\phi(u')$ is an absolutely continuous function on $[0,1]$, setting

$$u_n = J_\delta^{-1}(x_n); \tag{5.10}$$

$$u = J_\delta^{-1}(x), \tag{5.11}$$

we find that

$$-(\phi(u_n'))' + \delta\, S(u)u = x_n, \tag{5.12}$$

for each fixed $n \in \mathbb{N}$, $u_n \in C_*^1([0,1])$. Equation (5.12) and the boundedness of $\{x_n\}$ in $L^q(0,1)$ tell us that $\|u_n\|_{1,p}$ is bounded. There exists a subsequence of $\{u_n\}$ (still call it $\{u_n\}$) such that there exists $w \in C^0[0,1]$,

$$u_n \to w \quad \text{in } C^0[0,1]. \tag{5.13}$$

Here we use the compactness of the embedding of $W^{1,p}(0,1)$ in $C^0[0,1]$. From (5.12) we also know that the sequence $\{\phi(u_n')\}$ meets the requirements of Ascoli-Arzela's theorem in $C^0[0,1]$. Therefore, there exists a subsequence of $\{\phi(u_n')\}$ (still call it $\{\phi(u_n')\}$) which is convergent in C^0. This and the fact that the function ϕ has a continuous inverse imply that $\{u_n\}$ contains a convergent subsequence in $C^1([0,1])$ and $u_n \to w$ in $C^1([0,1])$. From the equality

$$\int_0^1 [\phi(u_n')v' + \delta\, S(u_n)u_n v] = \int_0^1 x_n v, \tag{5.14}$$

for all $v \in W^{1,p}(0,1)$ and $n \in \mathbb{N}$, and recalling that ϕ is continuous, we can let n go to infinity in (5.14) to obtain

$$\int_0^1 [\phi(w')v' + \delta\, S(w)v] = \int_0^1 xv, \tag{5.15}$$

for all $v \in W^{1,p}$. This and the argument above imply that $w = u$, which is a contradiction in light of (5.9). Following the same ideas as above, we can show that J_δ^{-1} is compact.

Theorem 5.2. *Let $f : (0,1) \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function, i.e., $f(\cdot, u)$ is measurable for every $u \in \mathbb{R}$ and $f(t,\cdot)$ is continuous for almost every $t \in (0,1)$. Moreover, assume that for every $R > 0$, there is a $k_R \in L^1(0,1)$ such that*

$$|f(t,u)| \leq k_R(t),$$

for all $|u| \leq R$ and almost every $t \in (0,1)$. Let $\alpha \in L^1(0,1)$ be such that

(A$_1$) *for any $\varepsilon > 0$, there exist $\beta_\varepsilon \in L^p(0,1)$, $\gamma_\varepsilon \in L^1(0,1)$ such that*

$$f(t,u)u \leq (\alpha(t) + \varepsilon)|u|^p + \beta_\varepsilon(t)|u|^{p-1} + \gamma_\varepsilon(t);$$

(B$_1$) *for any $u \in W^{1,p}(0,1)$, one has $\int_0^1 (|u'|^p - \frac{\alpha}{c_0}|u|^p)\,dt > 0$.*

Then problem (5.1) has a solution in C^1_* where c_0 is as in (5.2).

Remark 5.3. It is easy to show that assumption (A$_1$) is true if for some $\alpha \in L^1(0,1)$,

$$\limsup_{|u|\to\infty} (f(t,u)/|u|^{p-1}) \leq \alpha(t),$$

for almost every $t \in (0,1)$.

Remark 5.4. The condition (B$_1$) is equivalent to

(B$_2$) there exists $\bar{\varepsilon} > 0$ such that for any $u \in W^{1,p}(0,1)$ one has

$$\int_0^1 (|u'|^p - \frac{\alpha}{c_0}|u|^p)\,dt \geq \bar{\varepsilon}\|u\|_{1,p}^p. \tag{5.16}$$

Suppose (B$_2$) is false, we can find a sequence $\{u_n\}$ in $W^{1,p}(0,1)$ such that $\|u_n\|_{1,p} = 1$ and $\int_0^1 (|u_n'|^p - \frac{\alpha}{c_0}|u_n|^p) \to 0$. Taking a subsequence, we can assume $u_n \to u$ weakly in $W^{1,p}(0,1)$. Then $\{u_n\}$ converges to u in $L^p(0,1)$ and the weak semicontinuity of the L^p-norm of u_n' implies $\int_0^1 (|u'|^p - \frac{\alpha}{c_0}|u|^p) \leq 0$. By (B$_1$), $u = 0$ and the above implies that $\{u_n\}$ converges to 0 in $L^p(0,1)$. Since $\int_0^1 (|u_n'|^p - \frac{\alpha}{c_0}|u_n|^p) \to 0$, it follows that $\|u_n\|_{1,p} \to 0$, which is impossible.

Lemma 5.5. *For any $\delta > 0$ and $\alpha \in L^1(0,1)$, (B$_1$) holds. Then the equation*

$$-(a(|u'|^2)u')' + \delta\, S(u)u - \alpha(t)|u|^{p-2}u = 0 \tag{5.17}$$

has only the trivial solution in $C^1_*([0,1])$.

Proof. Suppose $u \in C^1_*([0,1])$ is a solution of (5.17), then

$$\int_0^1 [a(|u'|^2)|u'|^2 + \delta\, S(u)u^2 - \alpha(t)|u|^p]\, dt = 0. \tag{5.18}$$

So,

$$0 \geq c_0 \int_0^1 \left[|u'|^p + \delta|u|^p - \frac{\alpha(t)}{c_0}|u|^p\right] dt > 0.$$

This is a contradiction.

Proof of Theorem 5.2. From Lemma 5.1 we know that for any $\delta > 0$, problem (5.1) is equivalent to

$$u - J_\delta^{-1}(\delta\, S(u)u + f(t,u)) = 0 \tag{5.19}$$

and $J_\delta^{-1} : L^q(0,1) \to C^1_*([0,1])$ is compact. Now we establish *a priori* bounds necessary for application of coincidence degree.

We consider the family of equations

$$u - J_\delta^{-1}(\lambda\delta\, S(u)u + \lambda f(t,u) - (1-\lambda)\alpha(t)|u|^{p-2}u) = 0, \tag{5.20}$$

for $\lambda \in [0,1]$. Let $u \in C^1_*([0,1])$ be a solution of (5.20), then u satisfies

$$Au + \delta\, S(u)u - \lambda(\delta\, S(u)u + f(t,u)) - (1-\lambda)\alpha(t)|u|^{p-2}u = 0 \text{ a.e. on } [0,1],$$

$$\int_0^1 [a(|u'|^2)|u'|^2 + \delta\, S(u)u^2 - \lambda(\delta\, S(u)u^2 + f(t,u)u) - (1-\lambda)\alpha(t)|u|^p]\, dt = 0.$$

Fix $\varepsilon < c_0\bar{\varepsilon}$. Condition (A_1) and Remark 2.4 imply that

$$\begin{aligned}0 &\geq \int_0^1 \{a(|u'|^2)|u'|^2 - \lambda[(a(t)+\varepsilon)|u|^p + \beta_\varepsilon(t)|u|^{p-1} + \gamma_\varepsilon(t)] - (1-\lambda)\alpha(t)|u|^p\} \\ &\geq c_0 \int_0^1 [|u'|^p - \frac{\alpha}{c_0}|u|^p) - \varepsilon|u|^p - \lambda\beta_\varepsilon(t)|u|^{p-1} - \gamma_\varepsilon(t)]\, dt \\ &\geq (c_0\bar{\varepsilon} - \varepsilon)\|u\|^p_{1,p} - \int_0^1 (\beta_\varepsilon(t)|u|^{p-1} + \gamma_\varepsilon(t))\, dt.\end{aligned}$$

From this we obtain that there exists $M_9 > 0$ independent of λ and u such that

$$\|u\|_{1,p} \leq M_9.$$

The embedding of $W^{1,p}(0,1)$ to $C^0[0,1]$ imply that there exists $M_{10} > 0$ independent of λ and u such that

$$\|u\|_0 \leq M_{10}.$$

Directly integrating in (5.21), we obtain that $\|u\|_1$ is bounded.

By Lemma 5.5 and the arguments above, using the coincidence degree theory, we easily prove this theorem.

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