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ON ASYMPTOTIC STABILITY FOR LINEAR VISCOELASTIC FLUIDS

MAURO FABRIZIO AND BARBARA LAZZARI
Dipartimento di Matematica, Piazza di Porta S. Donato, 5, 40127 Bologna, Italy

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Abstract. The constitutive equation of the classical theory of infinitesimal viscometric fluids is considered, and some restrictions for this equation are found as a direct consequence of thermodynamics principles. These restrictions are proved to be sufficient to prove existence, uniqueness and stability theorems for the boundary-initial history value problem. Subsequently it is shown that the asymptotic stability of the rest state fails when these conditions are not satisfied.

Introduction. This paper presents an evolution problem in a bounded domain Ω for viscoelastic fluids of the kind discussed in [2], [15], [21] and [22][1]. We confine our attention to the classical theory of infinitesimal visco-elasticity and consider isotropic, homogeneous, incompressible fluids. For these materials the linearized constitutive theory states that the *symmetric stress tensor* $\mathbf{T}$ is determined by the *infinitesimal strain history* $\mathbf{E}^t(x,s) = \mathbf{E}(x,t-s)$ through the hereditary law[2]

$$\mathbf{T}(x,t) = -p(x,t)\mathbf{I} + 2\int_0^\infty \mu'(s)[\mathbf{E}^t(x,s) - \mathbf{E}(x,t)]\,ds, \tag{0.1}$$

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[1]A viscoelastic fluid (see Truesdell-Noll [22]) "may remember everything that ever happened to it, yet it cannot recall any one configuration as being physically different from any other except in regard to its mass density", and (see Truesdell [21]) "[a] fluid may have definite memory of all its past experience, [yet] it reacts to those experiences only by comparing them with its present configuration". In other words the stress in a fluid should be unchanged by a change of the reference configuration. Therefore the present configuration is used as reference.

[2]This hereditary law which postulates a linear relationship between the stress and the history is obtained as linearization of the frame-independent constitutive equation of Boltzmann's type

$$\mathbf{T}(x,t) = -p(x,t)\mathbf{I} + 2\int_0^\infty \mu'(s)[\mathbf{C}_t^{-1}(x,t-s) - \mathbf{I}]\,ds,$$

where $\mathbf{C}_t^{-1}$ is a relative strain tensor defined as inverse of the right relative Cauchy-Green tensor (see [15], page 21). Equation (0.1) is not frame indifferent because $\mathbf{E}$ is not properly invariant under changes of frame, thus this theory cannot possibly apply to any material in general finite deformation. However (see [22], page 117), "It is possible... that these [infinitesimal] theories describe the behavior of *some* material for arbitrary finite deformations, even though only in limiting case can they be expected to apply to *all* simple materials with fading memory".

where p is a scalar function called *reaction pressure* which, due to the constraint of incompressibility, is constitutively indeterminate, while μ' is a material function such that

$$\mu(s) = -\int_s^\infty \mu'(r)\,dr, \text{ called } \textit{shear relaxation function}, \text{ belongs to } L^1(\mathbb{R}^+).$$

The stability question for a fluid obeying constitutive equation (0.1) was considered under a variety of conditions in several papers [14], [15], [17], [18].[3] In particular Slemrod proved [17] that if $\mu \in C^2(\mathbb{R}^+)$, $\mu(s) \to 0$ as $s \to \infty$, $\mu(s) > 0$, $\mu'(s) < 0$ and $\mu''(s) \geq 0$, then the rest state of the fluid is stable in an appropriate "fading memory" norm and the solution to the linearized boundary-initial history value problem converges to the rest state weakly in this norm as $t \to \infty$. Next he proved that the additional assumption, $-\int_0^\infty s^2\mu'(s)\,ds < \infty$, yields asymptotic stability.

The aim of this paper is to point out the strict connection between thermodynamic requirements on the relaxation function μ and existence, uniqueness and stability theorems for the boundary-initial history value problem for a viscoelastic fluid obeying the constitutive equation (0.1).

In Section 1, taking the point of view explained in [9] and [5], we shall derive the restrictions that the Second Law of Thermodynamics places on the constitutive equation (0.1). Specifically, we will show the following thermodynamic requirement:

$$\hat{\mu}_c(\omega) = \int_0^\infty \mu(s)\cos\omega s\,ds > 0, \quad \omega \in \mathbb{R}. \tag{0.2}$$

Next, in Section 3, the condition (0.2) is shown to be sufficient to prove existence, uniqueness and asymptotic stability of the solution to the linearized initial-boundary value problem.

Finally, in Section 4 we prove that asymptotic stability of the rest state fails when condition (0.2) is replaced by the weaker condition $\hat{\mu}_c(\omega) \geq 0$, $\omega \in \mathbb{R}$. Indeed we exhibit a particular family of relaxation functions leading to solutions which are not asymptotically stable.

1. Viscoelastic fluids and thermodynamics. A linear isotropic, homogeneous, incompressible viscoelastic fluid characterized by the constitutive equation (0.1) is a simple system in the sense of the definition given in [3] and [7]. According to the formalism of [7], we introduce the notion of *deformation process* as a mapping $P : [0, d_P) \to Lin$[4] piecewise continuous and defined for any $t \in [0, d_P)$ by

$$P(t) = \mathbf{L}(t), \tag{1.1}$$

where $\mathbf{L}$ is the spatial velocity gradient and $d_P \in \mathbb{R}^{++}$ is called the *duration* of the process.

[3] Several authors have studied the problem of the asymptotic behaviour of the boundary-initial value problem for a viscoelastic solid, see for example [4] and [6].

[4] *Lin* is the usual set of the second order tensors, later on we denote with *Sym* the subset of the second order symmetric tensors.

Let P a deformation process and t_1, $t_2 \in [0, d_P)$, $t_1 < t_2$. Then the process

$$P_{[t_1,t_2)} = P(t_1 + t), \quad t \in [0, t_2 - t_1) \tag{1.2}$$

is called a *segment* of the process P. The segment $P_{[0,t)}$ of P will be denoted by P_t.

Letting P_1 and P_2 be two processes, we define the process $P_1 * P_2$, the *continuation* of P_1 with P_2, as

$$(P_1 * P_2)(t) = \begin{cases} P_1(t), & t \in [0, d_{P_1}) \\ P_2(t - d_{P_1}), & t \in [d_{P_1}, d_{P_1} + d_{P_2}). \end{cases} \tag{1.3}$$

Definition 1.1. A *simple material element* is a set $(\Pi, \Sigma, \hat{\rho}, \hat{\mathbf{T}})$ with the following properties:

(a) Π is the set of all processes of the system and satisfies the conditions:
  (i) if $P \in \Pi$ so is every segment of P;
  (ii) if P_1, $P_2 \in \Pi$ then $P_1 * P_2 \in \Pi$.

(b) Σ is a topological space, whose elements σ are called *states.*

(c) $\hat{\rho} : \Sigma \times \Pi \to \Sigma$ is a mapping called *evolution function* and to any "initial" state σ^i and process P the function $\hat{\rho}$ assigns the "final" state σ^f, $\sigma^f = \hat{\rho}(\sigma^i, P)$. Moreover, the function $\hat{\rho}$ is such that $\hat{\rho}(\sigma, P_1 * P_2) = \hat{\rho}(\hat{\rho}(\sigma, P_1), P_2)$.

(d) Letting E be a connected, open subset of *Lin*, the response functional $\hat{\mathbf{T}}$: $\Sigma \times E \to$ *Sym* maps the pair $(\sigma(t), P(t))$ into the stress tensor $\mathbf{T}(t)$; i.e., $\mathbf{T}(t) = \hat{\mathbf{T}}(\sigma(t), P(t))$, with $\sigma(t) = \hat{\rho}(\sigma, P_t)$.

Definition 1.2. A pair $(\sigma, P) \in \Sigma \times \Pi$ is called a *cyclic process* if $\hat{\rho}(\sigma, P) = \sigma$.

We denote by w the *power of the stress*

$$w(\sigma(t), P(t)) = \hat{\mathbf{T}}(\sigma(t), P(t)) \cdot \mathbf{L}(t) = \hat{\mathbf{T}}(\sigma(t), P(t)) \cdot \mathbf{D}(t), \tag{1.4}$$

where $\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T)$ is the *strain rate tensor*, and by $W(\sigma, P)$ the work on the path $\mathcal{C}$ performed by going from σ to $\hat{\rho}(\sigma, P)$ via the process P; i.e.,

$$W(\sigma, P) = \int_{\mathcal{C}} \hat{\mathbf{T}}(\sigma(t), P(t)) \cdot \mathbf{D}(t)\, dt. \tag{1.5}$$

The traditional statement of the Second Law involves the notion of a reversible process. A general study of this notion has been given by Serrin [16]; in this paper we use a definition of reversible process that is a restriction to the class of simple materials [10] of the general definition of reversible processes given in [16].

Definition 1.3. A cyclic process (σ, P) is called *reversible* if the new process P^- defined in $[0, d_P)$ as $P^-(\tau) = -\mathbf{L}(d_P - \tau)$, for $\tau \in [0, d_P)$, belongs to Π and, moreover,

(a) $\hat{\rho}(\sigma, P_t) = \hat{\rho}(\sigma, P^-_{d_P - t})$,

(b) $w(\sigma(t), P(t)) = -w(\sigma(t), P^-(d_P - t))$.

Second law of thermodynamics for isothermal processes.[5] On any cyclic process (σ, P) on the state space corresponding to the closed path $\mathcal{C}$, we have

$$W(\sigma, P) = \oint_{\mathcal{C}} \mathbf{T}(t) \cdot \mathbf{D}(t)\, dt \geq 0, \tag{1.6}$$

where the equality holds if and only if the cyclic process is reversible.

We now consider isotropic, homogeneous,[6] incompressible viscoelastic fluids in the linear, isothermal approximation, characterized by the constitutive equation (0.1) in which $\mu \in L^1(\mathbb{R}^+)$ and $\int_0^\infty \mu(s)\, ds \neq 0$; for these models the state is given by the *relative history* up to time t of the infinitesimal strain tensor $\mathbf{E}$ relative to the configuration at time t; i.e.,

$$\sigma(t) = \mathbf{E}_r^t(s) = \mathbf{E}(t-s) - \mathbf{E}(t), \quad s > 0;$$

moreover by virtue of linear approximation we have $\mathbf{D}(t) = \dot{\mathbf{E}}(t)$.

The state space Σ is a normed space and a relative history $\mathbf{E}_r^t$ belongs to Σ if for $\mathbf{E}_r^t$ the constitutive equation (0.1) is well defined.

In order to obtain analytical restrictions for relaxation function $\mu \in L^1(\mathbb{R}^+)$, it is necessary to describe the pairs (σ, P) that are closed cycles. It follows from the definition of state that (σ, P) is a closed cycle if and only if $\mathbf{E}^t$ is a periodic history and P is a process given by one or more periods of the history $\mathbf{E}^t$. Particular closed cycles are the pairs defined in the following manner:

$$\mathbf{E}^t(s) = \frac{1}{\omega}[\mathbf{E}_1 \cos\omega(t-s) + \mathbf{E}_2 \sin\omega(t-s)], \quad \omega \neq 0, \tag{1.7}$$

$$P(\tau) = \mathbf{D}(\tau) = -\mathbf{E}_1 \sin\omega\tau + \mathbf{E}_2 \cos\omega\tau, \quad \tau \in [t, t+d_P), \tag{1.8}$$

where $\mathbf{E}_1$, $\mathbf{E}_2 \in Sym$ and $d_P = \frac{2\pi}{|\omega|}$.

Moreover, for viscoelastic fluids obeying the constitutive equation (0.1), Definition 1.3 assures (see [10]) that only those pairs (σ, P), where σ is the history identically equal to zero and P is the null process, are reversible cycles.

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[5]The statements of the First and the Second Law of Thermodynamics given in [3] can be written in the following form: *On any cyclic process on the state space corresponding to the closed path $\mathcal{C}$, we have*

$$\oint_{\mathcal{C}} [h(t) + \mathbf{T}(t) \cdot \mathbf{D}(t)]\, dt = 0, \qquad \text{First Law}, \tag{$*$}$$

$$\oint_{\mathcal{C}} \Big[\frac{h(t)}{\vartheta(t)} + \frac{\mathbf{q}(t) \cdot \mathbf{g}(t)}{\vartheta^2(t)}\Big]\, dt \leq 0, \qquad \text{Second Law}, \tag{$**$}$$

where h is the rate at which heat is absorbed per unit mass, $\mathbf{q}$ is the heat flux vector, ϑ is the absolute temperature and $\mathbf{g}$ its gradient. When the study of simple material elements is restricted to isothermal cases, i.e., to cases in which $\mathbf{g} = \mathbf{0}$ and $\vartheta(t) = \vartheta_0$, $(*)$ and $(**)$ give (1.6) except for the final statement concerning reversibility.

[6]This study can be at once extended to inhomogeneous materials.

Theorem 1.1. *The constitutive law* (0.1) *for linear viscoelastic fluids is compatible with the Second Law of Thermodynamics if and only if for every relaxation function*

$$\mu \in L^1(\mathbb{R}^+), \text{ so that } \int_0^\infty \mu(s)\,ds \neq 0, \text{ the following inequality holds:}$$

$$\hat{\mu}_c(\omega) = \int_0^\infty \mu(s)\cos\omega s\,ds > 0, \quad \forall\omega \in \mathbb{R}. \tag{0.2}$$

Proof. Take the closed cycle (σ, P) defined in (1.7), (1.8); the mechanical work done on this cycle is

$$\begin{aligned} W(\sigma,P) = &\int_t^{t+d_P} -p(\tau)\mathbf{I}\cdot\mathbf{D}(\tau)\,d\tau + \int_t^{t+d_P}\int_0^\infty 2\omega^{-1}\mu'(s)(\mathbf{E}_1\cos\omega(\tau-s) \\ &+\mathbf{E}_2\sin\omega(\tau-s) - \mathbf{E}_1\cos\omega\tau - \mathbf{E}_2\sin\omega\tau)\cdot(-\mathbf{E}_1\sin\omega\tau + \mathbf{E}_2\cos\omega\tau)ds\,d\tau \\ = &\int_t^{t+d_P} -p(\tau)\mathbf{I}\cdot\mathbf{D}(\tau)\,d\tau + \int_t^{t+d_P}\int_0^\infty 2\omega^{-1}\mu(s)(\mathbf{E}_1\sin\omega(\tau-s) \\ &- \mathbf{E}_2\cos\omega(\tau-s))\cdot(\mathbf{E}_1\sin\omega\tau - \mathbf{E}_2\cos\omega\tau)\,ds\,d\tau. \end{aligned} \tag{1.9}$$

The first integral is equal to zero because the fluid is incompressible; i.e., $\mathbf{I}\cdot\mathbf{D} = \nabla\cdot\mathbf{v} = 0$. Moreover, carrying out the integration in (1.9) with respect to τ and recalling (1.6), we obtain after a short calculation

$$W(\sigma,P) = d_P[\mathbf{E}_1^2 + \mathbf{E}_2^2]\int_0^\infty \mu(s)\cos\omega s\,ds \geq 0. \tag{1.10}$$

If $[\mathbf{E}_1^2 + \mathbf{E}_2^2] \neq 0$, the pair defined in (1.7), (1.8) does not represent a reversible process; thus to comply with the Second Law, the inequality (1.10) must be strong for every $\omega \neq 0$. Finally we observe that (0.2) holds for every $\omega \in \mathbb{R}$, because $\int_0^\infty \mu(s)\,ds \neq 0$. Thus the "only if" part of the theorem is proved.

To show that (0.2) is a sufficient condition for the validity of the Second Law, we recall that a closed cycle (σ, P) is a pair where $\sigma = \mathbf{E}_r^t$, with $\mathbf{E}^t$ periodic function, and the process P coincides with one or more periods of the history $\mathbf{E}^t$. However any periodic history $\mathbf{E}_T^t$ with period T can be expressed through its Fourier series as

$$\mathbf{E}_T^t(s) = \sum_{k=0}^\infty[\mathbf{A}_k\cos k\omega(t-s) + \mathbf{B}_k\sin k\omega(t-s)], \quad \omega > 0, \tag{1.11}$$

where $\mathbf{A}_k$, $\mathbf{B}_k \in Sym$ and $\omega = \frac{2\pi}{T}$.

Let P_T be the process of duration T defined by

$$P_T(\tau) = \mathbf{D}_T^t(t+\tau), \quad \tau\in[0,T); \tag{1.12}$$

then the mechanical work performed along the cycle $(\mathbf{E}_T^t, P_T)$ is given by

$$\begin{aligned} W(\sigma,P) = &\int_t^{t+T}\int_0^\infty 2\mu'(s)\sum_{k=0}^\infty\sum_{h=0}^\infty[\mathbf{A}_k\cos k\omega(\tau-s) + \mathbf{B}_k\sin k\omega(\tau-s) \\ &- \mathbf{A}_k\cos k\omega\tau - \mathbf{B}_k\sin k\omega\tau]\cdot h\omega[-\mathbf{A}_h\sin k\omega\tau + \mathbf{B}_h\cos k\omega\tau]\,ds\,d\tau. \end{aligned} \tag{1.13}$$

Term by term integration of the double series shows that the only non vanishing terms are those with $h = k$. Thus we have

$$W(\sigma, P) = 2\sum_{k=0}^{\infty}\int_t^{t+T}\int_0^{\infty} k^2\omega^2\mu(s)[\mathbf{A}_k \sin k\omega(\tau - s) - \mathbf{B}_k \cos k\omega(\tau - s)] \cdot [\mathbf{A}_k \sin k\omega\tau - \mathbf{B}_k \cos k\omega\tau]\, ds\, d\tau, \tag{1.14}$$

and so

$$W(\sigma, P) = 2\omega\pi \sum_{k=0}^{\infty} k^2[\mathbf{A}_k^2 + \mathbf{B}_k^2] \int_0^{\infty} \mu(s) \cos k\omega s\, ds, \quad \omega > 0. \tag{1.15}$$

Because the relaxation function μ satisfies (0.2), it follows from (1.15) that $W(\sigma, P) > 0$. Hence the second part of the theorem is proved.

2. Formulation of initial-boundary value problem. Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain occupied by a linear viscoelastic incompressible fluid satisfying the constitutive equation (0.1) and let $\mathbf{v}$ and p be the velocity and pressure fields respectively.

The linear approximation of the equations of motion for the initial boundary value problem with Dirichlet boundary conditions are[7]

$$\begin{cases} \nabla \cdot \mathbf{v}(x,t) = 0, \quad x \in \Omega,\ t > 0 \\ \dfrac{\partial}{\partial t}\mathbf{v}(x,t) = \nabla \cdot [-p(x,t)\mathbf{I} + \displaystyle\int_0^{\infty} \mu(s)\nabla\mathbf{v}^t(x,s)\, ds] + f(x,t), \quad x \in \Omega,\ t > 0 \\ \mathbf{v}(x,t) = \mathbf{0}, \quad x \in \partial\Omega,\ t > 0 \\ \mathbf{v}(x,\tau) = \mathbf{v}_0(x,\tau), \quad x \in \Omega,\ \tau \le 0, \end{cases} \tag{2.1}$$

where a constant mass density ρ is understood and not written, $\mathbf{f}$ represents the body force and $\mathbf{v}_0$ the history of the velocity vector up to time $t = 0$.

The following assumptions are made: for the relaxation function μ, which characterizes the behaviour of the viscoelastic fluid, $\mu \in L^1(\mathbb{R}^+)$ and satisfies the thermodynamic inequality

$$\int_0^{\infty} \mu(s) \cos \omega s\, ds > 0, \qquad \omega \in \mathbb{R}. \tag{0.2}$$

For later convenience we introduce the following notation: let $L_s^2(\Omega)$ and $H_{s0}^2(\Omega)$ be the Hilbert spaces obtained by the completion of solenoidal vector fields $\mathbf{v} \in C_0^{\infty}(\Omega)$ in the $L^2(\Omega)$ and in the $H_0^1(\Omega)$ norm respectively. Moreover, let $L_{\pi}^2(\Omega)$ be the Hilbert space obtained by the completion of irrotational vector fields $\mathbf{v} \in C_0^{\infty}(\Omega)$ in the $L^2(\Omega)$ norm. Then we have $L^2(\Omega) = L_s^2(\Omega) \oplus L_{\pi}^2(\Omega)$. The symbol $H_s^{-1}(\Omega)$ denotes the dual of $H_{s0}^1(\Omega)$.

3. Existence, uniqueness and stability. Now we proceed to the statement of the main result, that condition (0.2) alone on the relaxation function μ is sufficient to enforce existence, uniqueness and asymptotic stability of the solution to the problem (2.1).

[7] *For incompressible fluids we have* $\nabla \cdot \mathbf{v}(x,t) = 0$. Consequently, $\nabla \cdot \mathbf{D}(x,t) = \frac{1}{2}\nabla \cdot [\nabla\mathbf{v}(x,t)]$.

Theorem 3.1. *If the relaxation function* $\mu \in L^1(\mathbb{R}^+)$ *and satisfies* (0.2), *the supply* $\mathbf{f} \in L^2(\mathbb{R}^+; H^{-1}(\Omega))$, *the initial history* $\mathbf{v}_0(\cdot, s) \in H^1_{s0}(\Omega)$ *for all* $s \leq 0$, *and the function* $\mathbf{V}_0(x,t) = \nabla \cdot \int_t^\infty \mu(s)\nabla \mathbf{v}_0(x, t-s)\, ds \in L^2(\mathbb{R}^+; H_s^{-1}(\Omega))$, *then problem* (2.1) *has one and only one solution* $\mathbf{v} \in L^2(0, +\infty; H^1_{s0}(\Omega))$.

For study of the system 2.1 the Laplace transform method is used. If $\varphi : [0, \infty) \to \mathbb{R}$ is a smooth function, then $\tilde{\varphi}(z) = \int_0^\infty \exp(-zs)\varphi(s)\, ds$, $z \in \mathbb{C}$, denotes the Laplace transform of φ.

Now we consider the Laplace transform of problem (2.1):

$$\begin{cases} \nabla \cdot \tilde{\mathbf{v}}(x,z) = 0, & x \in \Omega \\ z\tilde{\mathbf{v}}(x,z) = -\nabla \tilde{p}(x,z) + \nabla \cdot [\tilde{\mu}(z)\nabla \tilde{\mathbf{v}}(x,z)] + \mathbf{F}(x,z), & x \in \Omega \\ \tilde{\mathbf{v}}(x,z) = \mathbf{0}, & x \in \partial\Omega, \end{cases} \tag{3.1}$$

where

$$\begin{aligned} \mathbf{F}(x,z) &= \tilde{\mathbf{f}}(x,z) + \mathbf{v}_0(x,0) + \int_0^\infty \exp(-zs)\nabla \cdot) \int_t^\infty \mu(\tau)\nabla \mathbf{v}_0(x, s-\tau)\, d\tau\, ds \\ &= \tilde{\mathbf{f}}(x,z) + \mathbf{v}_0(x,0) + \tilde{\mathbf{V}}_0(x,z). \end{aligned} \tag{3.2}$$

The hypotheses $\mathbf{f} \in L^2(\mathbb{R}^+; H^{-1}(\Omega))$ and $\mathbf{V}_0 \in L^2(\mathbb{R}^+; H_s^{-1}(\Omega))$ make $\mathbf{F}$ well defined for every complex number $z \in \mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Re}\{z\} \geq 0\}$.

Now we consider the variational formulation of the linear differential system (3.1), and recall the definition of a weak solution.

Definition 3.1. A function $\tilde{\mathbf{v}} \in H^1_{s0}(\Omega)$ is called a *weak solution* to (3.1) if[8]

$$\int_\Omega [\tilde{\mu}(z)\nabla \tilde{\mathbf{v}}(x,z) \cdot \nabla \bar{\mathbf{u}}(x) + z\tilde{\mathbf{v}}(x,z) \cdot \bar{\mathbf{u}}(x)]\, dx = \int_\Omega \mathbf{F}(x,z) \cdot \bar{\mathbf{u}}(x)\, dx, \tag{3.3}$$

for every complex vector $\mathbf{u} \in H^1_{s0}(\Omega)$.

Following Temam ([19], Lemma 2.1) we can prove that if $\tilde{\mathbf{v}}$ is a weak solution, then there exists a scalar field $\tilde{p} \in L^2(\Omega)$ such that

$$\begin{cases} \nabla \cdot \tilde{\mathbf{v}}(x,z) = 0, & x \in \Omega \\ z\tilde{\mathbf{v}}(x,z) = -\nabla \tilde{p}(x,z) + \nabla \cdot [\tilde{\mu}(z)\nabla \tilde{\mathbf{v}}(x,z)] + \mathbf{F}(x,z), & x \in \Omega \end{cases}$$

in the sense of distributions, and moreover $\tilde{\mathbf{v}}(x,z) = \mathbf{0}$ on $\partial\Omega$.

Lemma 3.1. *Under the hypotheses of Theorem* 3.1 *the problem* (3.1) *has one and only one weak solution.*

By general theorems on elliptic systems [11], [19], [20], the coerciveness of the bilinear form

$$a(\mathbf{v}, \mathbf{u}; z) = \int_\Omega [\tilde{\mu}(z)\nabla \mathbf{v}(x) \cdot \nabla \bar{\mathbf{u}}(x) + z\mathbf{v}(x) \cdot \bar{\mathbf{u}}(x)]\, dx \tag{3.4}$$

[8] $\bar{\mathbf{u}}$ denotes the conjugate of $\mathbf{u}$.

in $H^1_{s0}(\Omega)$) assures existence and uniqueness of a weak solution to (3.3) for every $\mathbf{F} \in H^{-1}(\Omega)$. Thus, to prove Lemma 3.1 we have to show that there exists $\alpha > 0$, possibly dependent on z, such that

$$|a(\mathbf{v}, \mathbf{v}; z)| \geq \alpha(z) \|\mathbf{v}\|^2_{H^1_{s0}(\Omega)}. \tag{3.5}$$

Our aim is to show that the coerciveness of a is a consequence of the thermodynamic restriction (0.2). To prove that this is so we begin by recalling a property of Fourier integrals.

Proposition. *Let φ_1, $\varphi_2 \in L^1(\mathbb{R})$, with $\varphi_1\varphi_2 \in L^1(\mathbb{R})$. Let $\hat{\varphi}$ denote the Fourier transform of φ. If*

$$\Phi(\omega) = \int_{-\infty}^{+\infty} \hat{\varphi}_1(\tau)\hat{\varphi}_2(\omega - \tau)\, d\tau \tag{3.6}$$

is continuous in ω, then

$$\begin{aligned} &\int_{-\infty}^{+\infty} \exp(-i\omega s)\varphi_1(s)\varphi_2(s)\, ds \\ &\qquad = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{\varphi}_1(\tau)\hat{\varphi}_2(\omega - \tau)\, d\tau \text{ for every } \omega \in \mathbb{R}. \end{aligned} \tag{3.7}$$

The proof follows immediately from Parseval's theorem [1] and so is omitted.

Lemma 3.2. *If the relaxation function $\mu \in L^1(\mathbb{R}^+)$ and satisfies* (0.2), *then*

$$\int_0^\infty \exp(-\sigma s) \cos \omega s \mu(s)\, ds > 0 \quad \text{for } \omega \in \mathbb{R}, \sigma \geq 0. \tag{3.8}$$

Proof. When $\sigma = 0$ the inequality (3.8) coincides with (0.2). Let $\sigma > 0$ and consider the functions [12]

$$\varphi_1(s) = \mu(|s|), \qquad \varphi_2(s) = \begin{cases} 0 & s < 0 \\ \exp(-\sigma s) & s \geq 0. \end{cases}$$

It follows at once that φ_1, φ_2, $\varphi_1\varphi_2 \in L^1(\mathbb{R})$ and

$$\hat{\varphi}_1(\tau) = 2\int_0^\infty \cos \tau s \mu(s)\, ds, \qquad \hat{\varphi}_2(\tau) = \frac{1}{\sigma + i\tau};$$

hence

$$\Phi(\omega) = \int_{-\infty}^{+\infty} \frac{2}{\sigma + i(\omega - \tau)} \int_0^\infty \cos \tau s \mu(s)\, ds\, d\tau$$

is continuous. From (3.7) we obtain

$$\int_0^\infty \exp[-(\sigma + i\omega)s]\mu(s)\, ds = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1}{\sigma + i(\omega - \tau)} \int_0^\infty \cos \tau s \mu(s)\, ds\, d\tau, \tag{3.9}$$

the real part of (3.9) yields

$$\int_0^\infty \exp(-\sigma s)\cos\omega s\mu(s)\,ds = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\sigma}{\sigma^2+(\omega-\tau)^2}\int_0^\infty \cos\tau s\mu(s)\,ds\,d\tau, \tag{3.10}$$

and condition (0.2) provides the desired result (3.8) for $\sigma > 0$. □

We are now in a position to establish the coerciveness of the bilinear form $a(\mathbf{v},\mathbf{u};z)$.

Lemma 3.3. *If the relaxation function $\mu \in L^1(\mathbb{R}^+)$ and satisfies* (0.2), *then the bilinear form a is coercive for every complex number $z \in \mathbb{C}^+$.*

Proof. Since $|a(\mathbf{v},\mathbf{v};z)| \geq \mathrm{Re}\{a(\mathbf{v},\mathbf{v};z)\}$, it is enough to show that

$$\mathrm{Re}\{a(\mathbf{v},\mathbf{v};z)\} \geq \alpha(z)\|\mathbf{v}\|^2_{H^1_{s0}(\Omega)} \tag{3.11}$$

for every $z \in \mathbb{C}^+$. Letting $z = \sigma + i\omega$, $\omega \in \mathbb{R}$, $\sigma \geq 0$ we have

$$\tilde{\mu}(x,z) = \int_0^\infty \exp(-\sigma s)\cos\omega s\mu(s)\,ds - i\int_0^\infty \exp(-\sigma s)\mathrm{sen}\omega s\mu(s)\,ds,$$

whence

$$\mathrm{Re}\,\{a(\mathbf{v},\mathbf{v};z)\} = \int_0^\infty \exp(-\sigma s)\cos\omega s\mu(s)\,ds\cdot \left(\int_\Omega |\nabla\mathbf{v}(x)|^2 dx + \sigma\int_\Omega |\mathbf{v}(x)|^2 dx\right). \tag{3.12}$$

Hence by Korn's inequality we obtain

$$\mathrm{Re}\{a(\mathbf{v},\mathbf{v};z)\} \geq C(\Omega)\int_0^\infty \exp(-\sigma s)\cos\omega s\mu(s)\,ds\|\mathbf{v}\|^2_{H^1_{s0}(\Omega)},$$

where $C(\Omega)$ is a strictly positive constant which depends on the domain Ω. Recourse to Lemma 3.2 and to inequality (3.8) completes the proof. □

Because Lemma 3.1 assures existence and uniqueness of the solution to (3.3), we can investigate the properties of the solution $\tilde{\mathbf{v}}$. To this end we consider a representation in terms of the Green tensor function Γ, defined as a solution of the problem

$$\int_\Omega [\tilde{\mu}(z)\nabla_{x'}\cdot\Gamma(x,x';z)\nabla_{x'}\mathbf{u}(x') + z\Gamma(x,x';z)\mathbf{u}(x')]\,dx' = \int_\Omega \delta(x-x')\mathbf{u}(x')\,dx' \tag{3.13}$$

for every $\mathbf{u} \in H^1_{s0}(\Omega)$, where δ denotes the Dirac delta function.

In terms of Γ, the solution to (3.3) can be written as

$$\tilde{\mathbf{v}}(x,z) = \int_\Omega \Gamma(x,x';z)\mathbf{F}(x',z)\,dx'. \tag{3.14}$$

In the following lemma we prove existence, uniqueness and asymptotic behaviour with respect to the parameter z of solutions Γ of the equation (3.13).

Lemma 3.4. *Under the hypotheses of Theorem* 3.1, *there exists a unique solution* Γ *to the problem* (3.13) *such that*

(i) $\Gamma(x, \cdot; z) \in H^1_{s0}(\Omega)$ *for every* $z \in \mathbb{C}^+$;

(ii) $\Gamma(x, x'; \cdot)$ *is continuous in* $\mathbb{C}^+$;

(iii) *for* $z \in \mathbb{C}^+$,

$$\lim_{z\to\infty} z^{1-\alpha}\Gamma(x, x'; z) = \mathbf{0}, \qquad \alpha > 0, \tag{3.15}$$

$$\lim_{z\to\infty} \int_\Omega z\Gamma(x, x'; z)\mathbf{u}(x')\, dx' = \mathbf{u}(x) \text{ in the sense of distributions.} \tag{3.16}$$

Proof. (i) is a consequence of the coerciveness of the bilinear form $a(\mathbf{v}, \mathbf{v}; z)$ because δ is in $H^{-1}(\Omega)$. To prove (ii) we observe that the continuity of $\Gamma(x, x'; \cdot)$ follows from the continuity of the bilinear form $a(\mathbf{u}, \mathbf{v}; \cdot)$ with respect to the third argument (see [20], Lemma 44.1). As to (iii), by (3.13) we have

$$\int_\Omega z^{1-\alpha}\Gamma(x, x'; z)[\mathbf{u}(x') - z^{-1}\tilde{\mu}(z)\nabla_{x'} \cdot \nabla_{x'}\mathbf{u}(x')]\, dx' = z^{-\alpha}\mathbf{u}(x)$$

for every $\alpha > 0$ and $\mathbf{u} \in C_0^\infty(\Omega)$. Because $\tilde{\mu}$ is a bounded function of z, taking the limit as $z \to \infty$ yields (iii) □

Further recourse to Green's function provides a representation for $\nabla_x\tilde{\mathbf{v}}$. For convenience, we denote by $\nabla_x\Gamma$ the (third order) tensor function such that

$$\begin{aligned}\int_\Omega [\tilde{\mu}(z)\nabla_{x'} \cdot (\nabla_x\Gamma(x, x'; z))\nabla_{x'}\mathbf{u}(x') + z(\nabla_x\Gamma(x, x'; z))\mathbf{u}(x')]\, dx' \\ = \int_\Omega \delta_x(x - x')\mathbf{u}(x')\, dx'\end{aligned} \tag{3.13'}$$

for every $\mathbf{u} \in H^1_{s0}(\Omega)$. In terms of $\nabla_x\Gamma$ we have

$$\nabla_x\tilde{\mathbf{v}}(x, z) = \int_\Omega \nabla_x\Gamma(x, x'; z)\mathbf{F}(x', z)]\, dx'. \tag{3.14'}$$

By paralleling the proof of Lemma 3.4 and replacing Γ with $\nabla_x\Gamma$, we obtain the following result.

Lemma 3.5. *Under the hypotheses of Theorem* 3.1, *there exists a solution* $\nabla_x\Gamma$ *to the problem* (3.13′) *such that*

(i) $\nabla_x\Gamma(x, \cdot; z) \in L^2_s(\Omega)$ *for every* $z \in \mathbb{C}^+$ (*see* [20], *Lemma* 23.2);

(ii) $\nabla_x\Gamma(x, x'; \cdot)$ *is continuous on* $\mathbb{C}^+$;

(iii) *for* $\alpha > 0$, $z \in \mathbb{C}^+$,

$$\lim_{z\to\infty} z^{1-\alpha}\nabla_x\Gamma(x, x'; z) = \mathbf{0} \tag{3.15'}$$

in distributional sense.

These results enable us to state existence, uniqueness and stability of the solution to problem (2.1).

Proof of Theorem 3.1. Since $\mathbf{f} \in L^2(\mathbb{R}^+; H^{-1}(\Omega))$, and

$$\mathbf{V}_0(x,t) = \nabla \cdot \int_t^\infty \mu(s) \nabla \mathbf{v}_0(x, t-s)\, ds$$

belongs to $L^2(\mathbb{R}^+; H_s^{-1}(\Omega))$, we have

$$\begin{aligned} \lim_{z\to\infty} \mathbf{F}(x,z) = \lim_{z\to\infty} [\tilde{\mathbf{f}}(x,z) + \mathbf{v}_0(x,0) \\ + \int_0^\infty \exp(-zs) \int_t^\infty \mu(\tau)\nabla \mathbf{v}_0(x, s-\tau)\, d\tau\, ds] = \mathbf{v}(x,0). \end{aligned} \tag{3.17}$$

Then (3.14) and property (iii) of Γ yield (for $\alpha > 0$)

$$\begin{aligned} \lim_{z\to\infty} z^{1-\alpha}\tilde{\mathbf{v}}(x,z) &= \lim_{z\to\infty} \int_\Omega z^{1-\alpha}\Gamma(x,x';z)\tilde{\mathbf{F}}(x',z)\,dx' \\ &= \lim_{z\to\infty} \int_\Omega z^{1-\alpha}\Gamma(x,x';z)\mathbf{v}(x',0)\,dx' = \mathbf{0}. \end{aligned} \tag{3.18}$$

Let $z = i\omega$ and $0 < \alpha < \frac{1}{2}$. Then (3.18) implies that $\tilde{\mathbf{v}}(x, i\omega)$ is in L^2 with respect to ω; we can view it as the Fourier transform of $\mathring{\mathbf{v}}(x,t) = \begin{cases} 0 & t<0 \\ \mathbf{v}(x,t) & t \geq 0 \end{cases}$.
According to Parseval's theorem,

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_\Omega |\tilde{\mathbf{v}}(x,i\omega)|^2 dx\, d\omega &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_\Omega |\mathring{\mathbf{v}}(x,t)|^2 dx\, dt \\ &= \frac{1}{2\pi} \int_0^\infty \int_\Omega |\mathbf{v}(x,t)|^2 dx\, dt. \end{aligned} \tag{3.19}$$

By the same token we have

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_\Omega |\nabla_x \tilde{\mathbf{v}}(x,i\omega)|^2 dx\, d\omega &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_\Omega |\nabla_x \mathring{\mathbf{v}}(x,t)|^2 dx\, dt \\ &= \frac{1}{2\pi} \int_0^\infty \int_\Omega |\nabla_x \mathbf{v}(x,t)|^2 dx\, dt. \end{aligned} \tag{3.20}$$

It follows from (3.19) and (3.20) that

$$\int_0^\infty \int_\Omega [|\nabla \mathbf{v}(x,t)|^2 + |\mathbf{v}(x,t)|^2]\, dx\, dt < \infty.$$

This proves the required result: $\mathbf{v} \in L^2(\mathbb{R}^+; H^1_{s0}(\Omega))$.

4. Counterexamples to asymptotic stability. In this section, for the sake of simplicity, we restrict our attention to "one dimensional evolution" problems.

Let us assume that an incompressible viscoelastic fluid fills the strip $0 < x < l$, $(x,y) \in \mathbb{R}^2$, between two fixed plates. If the supply $\mathbf{f}$ and the initial history $\mathbf{v}^0$ have the form $\mathbf{f} = f(x,t)\mathbf{k}$, $\mathbf{v}^0 = v^0(x,t)\mathbf{k}$, then the vector $\mathbf{v} = v(x,t)\mathbf{k}$ must be a solution of the (scalar) boundary-initial history value problem

$$\begin{cases} v_t(x,t) = \displaystyle\int_0^\infty \mu(s) v_{xx}(x,t-s)\,ds + f(x,t), & x \in (0,l), \quad t > 0 \\ v(0,t) = v(0,l) = 0, \qquad t > 0 \\ v(x,\tau) = v^0(x,\tau), \qquad x \in (0,l), \quad \tau \le 0. \end{cases} \tag{4.1}$$

Because Theorem 2.1 can be applied to domains which are bounded in some direction (see for example [19], Theorem 2.1) if the relaxation function μ satisfies (0.2), it follows that problem (4.1), for any $f \in L^2(0,+\infty; H^{-1}(0;l))$ and v^0 such that $\int_t^\infty \mu(s) v^0_{xx}(x,t-s)\,ds \in L^2(0,+\infty; H_0^{-1}(0;l))$, has one and only one solution $v \in L^2(0,+\infty; H_0^1(0;l))$.

The aim of this section is to show that there exist relaxation functions which comply with weaker thermodynamic requirements yet do not allow the asymptotic stability of the solution to problem (4.1) under the hypotheses of Theorem 3.1. To this end we consider a non-negative relaxation function μ that satisfies the assumptions

P_1: μ is a positive decreasing function belonging to $H^1(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$\mu(s) < \frac{k}{(1+s)^{(2+\epsilon)}}, \quad k, \epsilon > 0;$$

P_2: μ satisfies the "weak formulation" of Second Law of Thermodynamics for isothermal processes; that is, $\int_0^\infty \mu(s)\cos\omega s\,ds \ge 0$ for every $\omega \in \mathbb{R}$, and at least one circular frequency $\omega^* \ne 0$ exists such that

$$\hat{\mu}(\omega^*) = \int_0^\infty \mu(s)\cos\omega^* s\,ds = 0. \tag{4.2}$$

Theorem 4.1. *Let μ be a relaxation function which agrees with P_1 and P_2. Then there exists a critical length l^* for the strip such that for $l = l^*$ and $f = 0$, and for initial history $v^0(x,\tau) = \sin(\frac{\pi x}{l^*})[c_1\cos\omega^*\tau + c_2\sin\omega^*\tau]$, problem (4.1) has a unique periodic solution (not belonging to $L^2(0,+\infty; H_0^1(0;l^*))$,*

$$v(x,t) = \sin\Big(\frac{\pi x}{l^*}\Big)[c_1\cos\omega^* t + c_2\sin\omega^* t].$$ [9]

We start with the following auxiliary result:

[9] It is possible to show that if P_1 holds, then for any $v^0 \in L^\infty(\mathbb{R}^-, H_0^1(0,l))$, the function

$$V^0(x,t) = \int_t^\infty \mu(s) v^0_{xx}(x,t-s)\,ds \in L^2(0,\infty; H_s^{-1}(0,l)) \quad \text{(see [13])}.$$

Lemma 4.1. *Let μ be a relaxation function which agrees with* P_1 *and* P_2*, let* $\omega^* \neq 0$ *be a circular frequency satisfying* (4.2) *and* $\hat{\mu}$ *the half range Fourier transform of* μ*. Then there exists a critical length* l^* *such that the problem*

$$\begin{cases} i\omega^* v(x,t) - \hat{\mu}(\omega^*) v_{xx}(x) = 0, & x \in (0,l) \\ v(0) = v(l) = 0 \end{cases} \tag{4.3}$$

has infinitely many complex valued solutions for $l = l^*$.

Proof. We substitute (4.2) into (4.3). Then both the real and imaginary parts of the solution v to (4.3) have to satisfy the problem

$$\begin{cases} \omega^* u(x,t) + \hat{\mu}_s(\omega^*) u_{xx}(x) = 0, & x \in (0,l) \\ u(0) = u(l) = 0. \end{cases} \tag{4.4}$$

By partial integration formula, we obtain

$$\frac{1}{\omega^*}\hat{\mu}_s(\omega^*) = \frac{1}{\omega^{*2}}[\mu(0) + \int_0^\infty \mu'(s)\cos\omega^* s\, ds] = \frac{1}{\omega^{*2}}\int_0^\infty \mu'(s)(\cos\omega^* s - 1)\, ds > 0,$$

because hypothesis P_1 implies $\mu'(s)\ (\cos\omega^* s - 1) \geq 0$ for any $s \in \mathbb{R}^+$.[10]

If we choose

$$l = l^* = \sqrt{\frac{\hat{\mu}_s'(\omega^*)}{\omega^*}}\,\pi, \tag{4.5}$$

then $u^*(x) = c\sin(\frac{\pi x}{l^*})$, $c \in \mathbb{R}$, is a solution to problem (4.4), because $\frac{\omega^*}{\hat{\mu}_s'(\omega^*)}$ is an eigenvalue of $-\Delta$. Recalling the connection between problems (4.3) and (4.4), we conclude that any function $v(x,t) = (c_1 + ic_2)\sin(\frac{\pi x}{l^*})$, c_1, $c_2 \in \mathbb{R}$, is a solution to (4.3).

Proof of Theorem 4.1. Let l^* be as in (4.5). Then by virtue of Lemma 4.1 the only solution to the problem

$$\begin{cases} v_t(x,t) = \displaystyle\int_0^\infty \mu(s) v_{xx}(x,t-s)\, ds, & x \in (0,l^*),\ t > 0 \\ v(0,t) = v(l^*,t) = 0, \quad t > 0 \\ v(x,\tau) = \sin\left(\dfrac{\pi x}{l^*}\right) c_1 \cos\omega^*\tau + c_2 \sin\omega^*\tau], & x \in (0,l^*),\ \tau \leq 0 \end{cases}$$

which agrees with the prescribed initial history value is $v(x,t) = \sin(\frac{\pi x}{l^*})\ [c_1\cos\omega^* t + c_2 \sin\omega^* t]$. In fact it is easy to show that $v_t(x,t) = \omega^* \sin(\frac{\pi x}{l^*})\ [-c_1 \sin\omega^* t + c_2\cos\omega^* t]$ and

$$\int_0^\infty \mu(s) v_{xx}(x,t-s)\, ds = \left(\frac{\pi}{l^*}\right)^2 \sin\left(\frac{\pi x}{l^*}\right)\hat{\mu}_s(\omega^*)[-c_1\sin\omega^* t + c_2\cos\omega^* t].$$

Finally we exhibit a family of non-negative relaxation functions which comply with requirements P_1 and P_2. A similar family was exhibited first by Fabrizio and

[10]We note that if $\mu'(s)(\cos\omega^* s - 1) = 0$ for any $s \in \mathbb{R}^+$, then $\mu(s) = 0$ for any $s \in \mathbb{R}^+$.

Morro [8], to show that there exist relaxation functions for linear viscoelastic solids which do not allow the quasistatic problem to have a unique solution in the space of sinusoidal-in-time strain histories. Subsequent as Giorgi and Lazzari [13] have made use of the same family (although with a different choice of parameters) to obtain counterexamples to the asymptotic stability of the rest state for initial-boundary value problems for linear viscoelastic solid materials.

We consider the two parameter family

$$\mu(s) = \left[\frac{s^2}{\beta} - \frac{\alpha - 3}{\beta^2}\, s + \frac{\alpha^2 - 8\alpha + 24}{8\beta^3}\right] \exp(-\beta s)$$

with $\alpha \in (0, 2+\sqrt{2}]$, $\beta > 0$. The function μ satisfies P_1, and in addition

$$\hat{\mu}_c(\omega) = \int_0^\infty \mu(s) \cos \omega s \, ds = \frac{\alpha^2}{8\beta^2(\beta^2+\omega^2)^3}\left(\omega^2 - \frac{8-\alpha}{\alpha}\beta^2\right)^2 \geq 0.$$

Hence for $\omega^* = \beta\sqrt{\frac{8-\alpha}{\alpha}}$, we have $\hat{\mu}_c(\omega^*) = 0$, so that also P_2 holds.

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