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WEAKLY NONLINEAR LARGE TIME BEHAVIOR IN SCALAR CONVECTION-DIFFUSION EQUATIONS

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Abstract. The large time behavior of scalar convection-diffusion equations

$$u_t - \Delta u = \vec{a} \cdot \nabla(|u|^{q-1}u) \quad \text{in } \mathbb{R}^N \times (0,\infty)$$

with initial data in $L^1(\mathbb{R}^N)$ is studied. The case where $q > 1 + 1/N$ is considered. It is by now well known that, in this range of exponents q, solutions behave as the heat kernel as $t \to \infty$. We make more precise this "weakly nonlinear" large time behavior obtaining the second term in the asymptotic development of solutions. The following three cases are distinguished: (a) $q \in (1 + \frac{1}{N}, 1 + \frac{2}{N})$, (b) $q = 1 + \frac{2}{N}$ and (c) $q > 1 + \frac{2}{N}$. The second term in the asymptotic development is of different nature in each of these three cases. In particular, the momentum $\int_{\mathbb{R}^N} x\varphi(x)\,dx$ of the initial data only appears in this second term if $q > 1 + \frac{2}{N}$. The proofs combine scaling arguments with the use of the similarity variables associated to the heat equation.

1. Introduction. In this paper we study the large time behavior of solutions of the scalar equation

$$u_t - \Delta u = \vec{a} \cdot \nabla(|u|^{q-1}) \quad \text{in } \mathbb{R}^N \times (0,\infty) \tag{1}$$

$$u(x,0) = \varphi(x) \in L^1(\mathbb{R}^N), \tag{2}$$

where $\vec{a} \in \mathbb{R}^N$, $q > 1 + \frac{1}{N}$ and $\cdot$ denotes the scalar product in $\mathbb{R}^N$. This equation represents a very simple model of diffusion and convection.

For every $\varphi \in L^1(\mathbb{R}^N)$, system (1)–(2) admits a unique solution $u \in C([0,\infty); L^1(\mathbb{R}^N))$. This solution is smooth for $t > 0$. Integrating equation (1) in $\mathbb{R}^N$ we deduce that the mass of solutions is conserved, i.e.,

$$\int_{\mathbb{R}^N} u(x,t)\,dx = \int_{\mathbb{R}^N} \varphi(x)\,dx, \quad \forall t > 0. \tag{3}$$

On the other hand, the following L^p-estimates hold (cf. [5]):

$$\begin{cases} \text{(i)} \quad \|u(t)\|_p \le C_p \|\varphi\|_1 t^{-\frac{N}{2}(1-\frac{1}{p})}, & \forall t \ge 0 \\ \text{(ii)} \quad \|\nabla u(t)\|_p \le C_p \|\varphi\|_1 t^{-\frac{N}{2}(1-\frac{1}{p})-\frac{1}{2}}, & \forall t \ge 1 \end{cases} \tag{4}$$

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for all $p \in [1,\infty]$ (by $\|\cdot\|_p$ we denote the norm in $L^p(\mathbb{R}^N)$). Similar decay rates were proved by M.E. Schonbek in [7] and [8] under further restrictions on the initial data φ. Note that the decay rates are those of the heat kernel

$$G(x,t) = (4\pi t)^{-N/2} \exp\Big(\frac{-|x|^2}{4t}\Big). \tag{5}$$

The decay rate (4) does not depend on the power q. Therefore, it is natural to expect a weak contribution of the non-linear term on the large time behavior of solutions for large q.

In [5] it was proved that, in a first approximation, when $q > 1 + \frac{1}{N}$ solutions of (1)–(2) behave as the heat kernel. More precisely, the following result was proved.

Theorem. ([5]) *Assume that $q > 1 + \frac{1}{N}$ and $\int_{\mathbb{R}^N} \varphi(x)\,dx = M$, then*

$$t^{\frac{N}{2}(1-\frac{1}{p})}\|u(t) - MG(t)\|_p \to 0 \text{ as } t \to \infty. \tag{6}$$

Moreover, if $\varphi \in L^1(\mathbb{R}^N; 1+|x|)$ (i.e., $\int_{\mathbb{R}^N} |\varphi(x)|(1+|x|)\,dx < \infty$) it follows that

$$\begin{cases} \text{(i)} & \|u(t) - MG(t)\|_p \le C_p t^{-\frac{N}{2}(1-\frac{1}{p})-\frac{1}{2}} & \text{if } q > 1 + \dfrac{2}{N} \\ \text{(ii)} & \|u(t) - MG(t)\|_p \le C_p t^{-\frac{N}{2}(1-\frac{1}{p})-\frac{1}{2}} \log t & \text{if } q = 1 + \dfrac{2}{N} \\ \text{(iii)} & \|u(t) - MG(t)\|_p \le C_p t^{\frac{1}{2}-\frac{N}{2}(q-\frac{1}{p})} & \text{if } 1 + \dfrac{1}{N} < q < 1 + \dfrac{2}{N}. \end{cases} \tag{7}$$

Note that if u is a solution of the heat equation

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \\ u(0) = \varphi \end{cases} \tag{8}$$

with $\int \varphi(x)\,dx = M$, then (7) (i) holds. Therefore, when $q > 1+\frac{2}{N}$ in (7) we get the same decay rate as for the heat equation. The decay rate decreases as q decreases from $q = 1 + \frac{2}{N}$ to $q = 1 + \frac{1}{N}$.

Note that when $q = 1 + \frac{1}{N}$ and $\vec{a} \not\equiv 0$, (6) does not hold. Indeed, in [5] it was proved that when $q = 1 + \frac{1}{N}$, solutions of (1)–(2) behave as self-similar solutions $w_M(x,t) = t^{-N/2} f_M\big(\frac{x}{\sqrt{t}}\big)$ of the nonlinear equation (1). More precisely, for every $M \in \mathbb{R}$ there exists a unique self-similar solution w_M of (1)–(2) of the form above such that

$$\int_{\mathbb{R}^N} f_M dx = M.$$

Moreover, if

$$\int_{\mathbb{R}^N} \varphi\,dx = M$$

then the solution of (1)–(2) satisfies $t^{\frac{N}{2}(1-\frac{1}{p})}\|u(\cdot,t) - w_M(\cdot,t)\|_p \to 0$ as $t \to \infty$ for every $p \in [1,\infty]$. Observe that when $N = 1$, $q = 1 + \frac{1}{N} = 2$ and this corresponds to the viscous Burgers equation.

Therefore we may say that when $q = 1 + \frac{1}{N}$ the large time behavior is of self-similar form. It is weakly nonlinear for $q > 1 + \frac{1}{N}$.

The goal of this paper is to make more precise the weakly nonlinear behavior stated in the theorem above studying the large time behavior of $u(t) - MG(t)$. In the case of the linear heat equation (8) it is easy to check that (cf. J. Duoandikoetxea and E. Zuazua [2])

$$t^{1/2}\|u(t) - [MG(t) - \vec{m}\cdot\nabla G(t)]\|_1 \to 0 \text{ as } t \to \infty \tag{9}$$

for every $\varphi \in L^1(\mathbb{R}^N; 1 + |x|)$ with

$$M = \int_{\mathbb{R}^N} \varphi(x)\,dx$$

and $\vec{m} = (m_1, \dots, m_N)$ where

$$m_i = \int_{\mathbb{R}^N} x_i\varphi(x)\,dx, \quad i = 1, \dots, N.$$

Convergence (9) provides the second term in the asymptotic development of the solution of the heat equation as $t \to \infty$.

In this paper we prove that the contribution of the non-linear term in system (1)–(2) can not be neglected to get the second term in the asymptotic development for any $q > 1 + \frac{1}{N}$. Indeed, (9) does not hold for any $q > 1 + \frac{1}{N}$.

In order to describe the large time behavior of $u(t) - MG(t)$ we must distinguish the following three cases: (a) $1 + \frac{1}{N} < q < 1 + \frac{2}{N}$, (b) $q = 1 + \frac{2}{N}$ and (c) $q > 1 + \frac{2}{N}$. In each of these three cases the contribution of the nonlinear term is of different nature.

To state the main result for the first case we introduce the solution $z = z(x,t)$ of

$$\begin{cases} z_t - \Delta z = \vec{a}\cdot\nabla(G^q) & \text{in } \mathbb{R}^N \times (0,\infty) \\ z(0) = 0 \end{cases} \tag{10}$$

and the function

$$w(x,t) = t^{-\frac{Nq}{2}+\frac{1}{2}} z\left(\frac{x}{\sqrt{t}}, 1\right). \tag{11}$$

We have the following result:

Theorem 1. *Assume that*

$$1 + \frac{1}{N} < q < 1 + \frac{2}{N} \tag{12}$$

and $\varphi \in L^1(\mathbb{R}^N; 1 + |x|))$ *with* $\int_{\mathbb{R}^N} \varphi(x)\,dx = M$*, then*

$$t^{\frac{N}{2}(q-\frac{1}{p})-\frac{1}{2}}\|u(t) - MG(t) - |M|^{q-1}Mw(t)\|_p \to 0 \text{ as } t \to \infty \tag{13}$$

for every $p \in [1,\infty]$.

Remark 1. It is easy to check that

$$\|w(t)\|_p = t^{-\frac{N}{2}(q-\frac{1}{p})+\frac{1}{2}}\|z(1)\|_p, \quad \forall t > 0. \tag{14}$$

Therefore (13) provides the second term in the asymptotic development of the solution u of (1)–(2).

Remark 2. Solution z of (10) can be calculated in terms of the heat kernel. Indeed

$$z = \vec{a} \cdot \nabla g, \tag{15}$$

where

$$\begin{cases} g_t - \Delta g = G^q & \text{in } \mathbb{R}^N \times (0, \infty) \\ g(0) = 0. \end{cases} \tag{16}$$

Thus

$$g(1) = \int_0^1 G(1-s) * G^q(s)\, ds = q^{-\frac{N}{2}} \int_0^1 (4\pi s)^{\frac{N}{2}(1-q)} G(1 + (\frac{1}{q} - 1)s)\, ds. \tag{17}$$

(By $*$ we denote the convolution in $\mathbb{R}^N$).

In order to state our main results for the two other cases ($q = 1 + \frac{2}{N}$ and $q > 1 + \frac{2}{N}$) we will work with the equation in the similarity variables. Let us consider the following weighted L^2-space that was introduced by M. Escobedo and O. Kavian [3] (see also [4] and [6]);

$$\begin{cases} K(y) = \exp\left(\frac{|y|^2}{4}\right) \\ L^2(K) = \{f \in L^2(\mathbb{R}^N) : \|f\|_K^2 := \int_{\mathbb{R}^N} |f(y)|^2 K(y)\, dy < \infty\}. \end{cases} \tag{18}$$

By $(\cdot,\cdot)_K$ we will denote the scalar product in $L^2(K)$.

Concerning the case $q = 1 + \frac{2}{N}$ we have the following result:

Theorem 2. *Assume that $q = 1 + \frac{2}{N}$ and let $\varphi \in L^2(K) \cap L^\infty(\mathbb{R}^N)$ be such that $M = \int_{\mathbb{R}^N} \varphi(x)\, dx$. Then*

$$\frac{t^{\frac{N}{2}(1-\frac{1}{p})+\frac{1}{2}}}{\log t} \|u(t) - MG(t) + \alpha |M|^{q-1} M (\log t) \vec{a} \cdot \nabla G(t)\|_p \to 0 \text{ as } t \to \infty \tag{19}$$

for every $p \in [1, \infty]$ where

$$\alpha = q^{-N/2} (4\pi)^{\frac{N}{2}(1-q)}. \tag{20}$$

In the case $q > 1 + \frac{2}{N}$ the following holds:

Theorem 3. *Assume that $q > 1 + \frac{2}{N}$ and let $\varphi \in L^2(K) \cap L^\infty(\mathbb{R}^N)$ be such that*

$$M = \int_{\mathbb{R}^N} \varphi(x)\, dx \text{ and } \vec{m} = (m_1, \dots, m_N) \text{ with } m_i = \int_{\mathbb{R}^N} \varphi(x) x_i dx.$$

Then

$$t^{\frac{N}{2}(1-\frac{1}{p})+\frac{1}{2}} \|u(t) - MG(t) + \vec{b} \cdot \nabla G(t)\|_p \to 0 \text{ as } t \to \infty \tag{21}$$

for every $p \in [1, \infty]$ with

$$\vec{b} = \vec{a} \int_0^\infty \int_{\mathbb{R}^N} |u|^{q-1} u(x,t)\, dx\, dt - \vec{m}. \tag{22}$$

Remark 3. Note that the momentum $\vec{m}$ of the initial data appears in the second term of the asymptotic development only when $q > 1 + \frac{2}{N}$ as it is shown in (22).

Remark 4. In view of the decay estimates (4i) it is easy to see that the integral $\int_0^\infty \int_{\mathbb{R}^N} |u|^{q-1}u(x,t)\,dx\,dt$ converges when $q > 1 + \frac{2}{N}$. Multiplying in equation (1) by x_i, integrating over $\mathbb{R}^N \times (0,t)$ and letting $t \to \infty$ it is easy to see that $\vec{b} = (b_1, \dots, b_N)$ with

$$b_i = -\lim_{t\to\infty} \int_{\mathbb{R}^N} u(x,t)\, x_i\, dx.$$

We do not know whether $\vec{b}$ can be calculated explicitly in terms of the initial data φ.

The rest of the paper is organized as follows. Section 2 is devoted to proving Theorem 1. In Section 3 we prove Theorem 2 and finally, in Section 4, Theorem 3 is proved.

By C we will denote a positive constant that may change from one line to another. Sometimes we will use the notation C_p to make explicit the fact that the constant may depend on the exponent p of the Lebesgue space we are working in.

2. The case $1 + \frac{1}{N} < q < 1 + \frac{2}{N}$.

This section is devoted to the proof of Theorem 1.

Solution $u = u(x,t)$ of (1)–(2) is given by the integral formula

$$u(t) = G(t) * \varphi + \int_0^t \vec{a} \cdot \nabla G(t-s) * |u|^{q-1}u(s)\,ds. \tag{23}$$

Let us introduce the rescaled functions

$$u_\lambda(x,t) = \lambda^N u(\lambda x, \lambda^2 t), \quad \lambda > 0. \tag{24}$$

It is easy to check that u_λ solves

$$\begin{cases} u_{\lambda,t} - \Delta u_\lambda = \lambda^{N(1-q)+1} \vec{a} \cdot \nabla(|u_\lambda|^{q-1}u_\lambda) & \text{in } \mathbb{R}^N \times (0,\infty) \\ u_\lambda(0) = \varphi_\lambda \end{cases} \tag{25}$$

with $\varphi_\lambda(x) = \lambda^N \varphi(\lambda x)$ and u_λ is given by the integral equation,

$$u_\lambda(t) = G(t) * \varphi_\lambda + \lambda^{N(1-q)+1} \int_0^t \vec{a} \cdot \nabla G(t-s) * |u_\lambda|^{q-1}u_\lambda(s)\,ds. \tag{26}$$

Since $\varphi \in L^1(\mathbb{R}^N; 1+|x|)$, we have (cf. Lemma 3 in [5])

$$\lambda^s[G(1) * \varphi_\lambda - MG(1)] \to 0 \text{ in } L^p(\mathbb{R}^N) \text{ as } \lambda \to \infty \text{ for every } 0 < s < 1. \tag{27}$$

On the other hand, since $1 + \frac{1}{N} < q < 1 + \frac{2}{N}$, it follows that $N(q-1) - 1 < 1$. Therefore, if we have

$$\begin{aligned} &\int_0^1 \vec{a} \cdot \nabla G(t-s) * |u_\lambda|^{q-1}u_\lambda(s)\,ds \\ &\to \int_0^1 \vec{a} \cdot \nabla G(t-s) * |MG|^{q-1}(MG)(s)\,ds = f \text{ in } L^p(\mathbb{R}^N) \text{ as } \lambda \to \infty \end{aligned} \tag{28}$$

then we will have

$$\lambda^{N(q-1)-1}(u_\lambda(1) - MG(1)) \to f \text{ in } L^p(\mathbb{R}^N) \text{ as } \lambda \to \infty. \tag{29}$$

It is easy to check that (29) is equivalent to the desired result (11) since $f = |M|^{q-1}Mz(1)$. Thus, it is enough to prove (28). We have

$$\begin{aligned}
&\left\| \int_0^1 \vec{a} \cdot \nabla G(1-s) * \left\{ |u_\lambda|^{q-1}u_\lambda(s) - |MG(s)|^{q-1}(MG(s)) \right\} dx \right\|_p \\
&\leq |\vec{a}| \int_0^{1/2} \|\nabla G(1-s)\|_p \| \, |u_\lambda|^{q-1}u_\lambda(s) - |MG(s)|^{q-1}(MG(s))\|_1 ds \\
&+ |\vec{a}| \int_{1/2}^1 \|\nabla G(1-s)\|_1 \| \, |u_\lambda|^{q-1}u_\lambda(s) - |MG(s)|^{q-1}(MG(s))\|_p ds \\
&= |\vec{a}|(I_\lambda^1 + I_\lambda^2).
\end{aligned}$$

Let us now prove that $I_\lambda^1 \to 0$ as $\lambda \to \infty$. In view of (4i) using classical estimates on the heat kernel we get

$$\begin{aligned}
&\|\nabla G(1-s)\|_p \, \| |u_\lambda|^{q-1}u_\lambda(s) - |MG(s)|^{q-1}(MG(s))\|_1 \\
&\leq C_p(1-s)^{-\frac{N}{2}(1-\frac{1}{p})-\frac{1}{2}} s^{-\frac{N}{2}(q-1)} \in L^1(0, \frac{1}{2}), \quad \forall \lambda > 0
\end{aligned} \tag{30}$$

since $\frac{N}{2}(q-1) < 1$. On the other hand, (6) implies that

$$u_\lambda(1) \to MG(1) \text{ in } L^p(\mathbb{R}^N) \text{ as } \lambda \to \infty \tag{31}$$

for every $p \in [1, \infty]$. Combining (30)–(31) and applying Lebesgue's Dominated Convergence Theorem we deduce that $I_\lambda^1 \to 0$ as $\lambda \to \infty$. In order to prove $I_\lambda^2 \to 0$ as $\lambda \to \infty$ we proceed in a similar way observing that

$$\begin{aligned}
&\|\nabla G(1-s)\|_1 \, \| |u_\lambda|^{q-1}u_\lambda(s) - |MG(s)|^{q-1}(MG(s))\|_p \\
&\leq C_p(1-s)^{-1/2} s^{-\frac{N}{2}(q-\frac{1}{p})} \in L^1(\frac{1}{2}, 1), \quad \forall \lambda > 0.
\end{aligned}$$

3. The case $q = 1 + \frac{2}{N}$. Let u be the solution of (1)–(2) and let us define the function,

$$v(y, s) = e^{sN/2} u(e^{s/2}y, \, e^s - 1). \tag{32}$$

It is easy to see that $v = v(y, s)$ satisfies the following parabolic equation in the similarity variables $(y, s) = (\frac{x}{\sqrt{t+1}}, \log(t+1)$:

$$\begin{cases} v_s + Lv - \dfrac{N}{2}v = e^{-s/2}\vec{a} \cdot \nabla(|v|^{q-1}v) & \text{in } \mathbb{R}^N \times (0, \infty) \\ v(y, 0) = \varphi(y), \end{cases} \tag{33}$$

where

$$Lv = -\Delta v - \frac{y \cdot \nabla v}{2}. \tag{34}$$

Let us recall some well-known facts about the operator L in the following weighted Sobolev spaces (see [3] and [6] for more details and related questions):

$$H^\ell(K) = \{f \in H^\ell(\mathbb{R}^N) : D^\alpha f \in L^2(K), \quad \forall \alpha \in \mathbb{N}^N : |\alpha| \le \ell\} \text{ for } \ell = 1, 2, \ldots. \tag{35}$$

Endowed with the norm

$$\|\varphi\|^2_{\ell,K} = \sum_{|\alpha|\le\ell} \|D^\alpha \varphi\|^2_K$$

they are Hilbert spaces. Moreover, the embedding $H^{\ell+1}(K) \subset H^\ell(K)$ is compact for every $\ell \in \mathbb{N}$. L is an unbounded self-adjoint operator in $L^2(K)$ with domain $D(L) = H^2(K)$. The eigenvalues of L are the following:

$$\lambda_k = \frac{N+k-1}{2}, \quad k = 1, 2, \ldots. \tag{36}$$

Let us denote by E_k the eigenspace associated to λ_k.

The first eigenvalue $\lambda_1 = \frac{N}{2}$ is simple and its eigenspace E_1 is spanned by $\theta_1(y) = (4\pi)^{-N/2} \exp(-\frac{|y|^2}{4})$ and $E_k = \text{span}\,\{D^\alpha\theta_1;\ \alpha \in \mathbb{N}^N, |\alpha| = k\}$. The fact that convergence (6) holds is equivalent to

$$v(s) \to M\theta_1 \quad \text{in } L^p(\mathbb{R}^N) \quad \text{as } s \to \infty. \tag{37}$$

On the other hand, by using the arguments in [5] it is easy to prove that

$$v(s) \to M\theta_1 \quad \text{in } L^2(K) \quad \text{as } s \to \infty. \tag{38}$$

In order to prove Theorem 2 we have to understand the behavior of

$$w(y,s) = e^{s/2}(v(y,s) - M\theta_1(y)) \tag{39}$$

which satisfies

$$\begin{cases} w_s + Lw - \left(\dfrac{N+1}{2}\right) w = \vec{a} \cdot \nabla(|v|^{q-1}v) & \text{in } \mathbb{R}^N \times (0,\infty) \\ w(y,0) = \varphi(y) - M\theta_1(y). \end{cases} \tag{40}$$

In view of (7ii) it is easy to check that $\frac{1}{s}w(s) \in L^\infty(1,\infty; L^p(\mathbb{R}^N))$ for every $p \in [1,\infty]$. Solution w of (40) splits in two parts: $w = w_1 + w_2$ where

$$\begin{cases} w_{1,s} + Lw_1 - \left(\dfrac{N+1}{2}\right) w_1 = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \\ w_1(0) = \varphi - M\theta_1 \end{cases} \tag{41}$$

and

$$\begin{cases} w_{2,s} + Lw_2 - \left(\dfrac{N+1}{2}\right) w_2 = \vec{a} \cdot \nabla(|v|^{q-1}v) & \text{in } \mathbb{R}^N \times (0,\infty) \\ w_2(0) = 0. \end{cases} \tag{42}$$

Expanding w_1 in Fourier series one easily sees that $w_1 \in L^\infty(0,\infty;L^2(K))$ and therefore

$$\frac{1}{s}w_1(s) \to 0 \quad \text{in} \quad L^2(K) \quad \text{as} \quad s \to \infty. \tag{43}$$

Integrating (42) with respect to y in $\mathbb{R}^N$ we deduce that

$$\int_{\mathbb{R}^N} w_2(y,s)\,dy = 0, \quad \forall s > 0.$$

This means that $w_2(s) \in E_1^\perp = \{\text{orthogonal of } E_1 \text{ in } L^2(K)\}$ for every $s > 0$.

Multiplying equation (42) by y_j for any $j = 1,\dots,N$ and integrating over $\mathbb{R}^N$ we see that the Fourier coefficients of w_2 corresponding to the second eigenvalue

$$c_j(s) = \int_{\mathbb{R}^N} w_2(y,s)y_j\,dy, \quad j = 1,\dots,N$$

satisfy

$$\begin{cases} c_j'(s) = -a_j \displaystyle\int_{\mathbb{R}^N} |v|^{q-1}v(y,s)\,dy \\ c_j(0) = 0, \end{cases}$$

where $\vec{a} = (a_1,\dots,a_N)$. Thus

$$c_j(s) = -a_j \int_0^s \int_{\mathbb{R}^N} |v|^{q-1}v(y,\sigma)\,dy\,d\sigma, \quad j = 1,\dots,N.$$

Thus

$$w_2(y,s) = -\frac{1}{2}\left(\int_0^s \int_{\mathbb{R}^N} |v|^{q-1}v(y,\sigma)\,dy\,d\sigma\right)(\vec{a}\cdot y)\theta_1(y) + w_3(y,s) \tag{44}$$

with $w_3(s) \in (E_1 \oplus E_2)^\perp$ for every $s > 0$.

Using the fact that $v \in L^\infty(0,\infty;L^2(K)\cap L^\infty(\mathbb{R}^N))$ and taking into account that $L - (\frac{N+1}{2})I$ is coercive on $H^1(K) \cap (E_1 \oplus E_2)^\perp$ we deduce that

$$w_3 \in L^\infty(0,\infty;L^2(K)). \tag{45}$$

On the other hand, in view of (6) we have

$$\lim_{s\to\infty} \frac{1}{s}\int_0^s \int_{\mathbb{R}^N} |v|^{q-1}v(y,\sigma)\,dy\,d\sigma = |M|^{q-1}M \int_{\mathbb{R}^N} \theta_1^q(y)\,dy. \tag{46}$$

Combining (43)–(46) we conclude that

$$\frac{1}{s}w(s) = \frac{e^{s/2}}{2}(v(s) - M\theta_1) \to \alpha|M|^{q-1}M\vec{a}\cdot\nabla\theta_1(y) \text{ in } L^2(K) \text{ as } s \to \infty \tag{47}$$

with α as in (20) since $\int_{\mathbb{R}^N} \theta_1^q(y)\,dy = (4\pi)^{\frac{N}{2}(1-q)}q^{-N/2}$.

Taking into account that $L^2(K)$ is continuously embedded into $L^p(\mathbb{R}^N)$ for $1 \le p \le 2$, we deduce that convergence (47) holds in $L^p(\mathbb{R}^N)$ for $1 \le p \le 2$. In order to see that (47) holds in $L^p(\mathbb{R}^N)$ for every $1 \le p \le \infty$, by interpolation, it is sufficient to check that $\frac{1}{s}w(s) \in L^\infty(1,\infty; W^{1,p}(\mathbb{R}^N))$ for every $p \in [1,\infty]$. Since $\frac{1}{s}w(s) \in L^\infty(1,\infty; L^p(\mathbb{R}^N))$, it is sufficient to prove that $\frac{1}{s}\nabla w(s) \in L^\infty(1,\infty; L^p(\mathbb{R}^N))$. This is equivalent to

$$\|\nabla u(t) - M\nabla G(t)\|_p \le C_p t^{-\frac{N}{2}(1-\frac{1}{p})-1}\log t, \quad \forall t > 0. \tag{48}$$

In order to prove (48) we consider the integral equation satisfied by the solution u of (1)–(2):

$$u(t) = G(t) * \varphi + \int_0^t \vec{a}\cdot\nabla G(t-s) * |u|^{q-1}u(s)\,ds.$$

Thus

$$\nabla[u(t) - MG(t)] = \nabla G(t) * (\varphi - M\delta) + \int_0^t \vec{a}\cdot\nabla G(t-s) * \nabla(|u|^{q-1}u(s))\,ds,$$

where δ denotes the Dirac mass at the origin. Proceeding as in Lemma 2 of [5] we deduce that

$$\|\nabla G(t) * (\varphi - M\delta)\|_p \le C_p\|\varphi\|_{L^1(\mathbb{R}^N;|x|)} t^{-\frac{N}{2}(1-\frac{1}{p})-1}, \quad \forall t > 0$$

for every $p \in [1,\infty]$. Therefore, it is sufficient to check that

$$\left\| \int_0^t \vec{a}\cdot\nabla G(t-s) * \nabla(|u|^{q-1}u(s))\,ds \right\|_p \le C_p t^{-\frac{N}{2}(1-\frac{1}{p})-1}\log t. \tag{49}$$

We have

$$\begin{aligned}
&\int_0^t \vec{a}\cdot\nabla G(t-s) * \nabla(|u|^{q-1}u(s))\,ds \\
=&\int_0^{t/2} \vec{a}\cdot\nabla(\nabla G(t-s)) * |u|^{q-1}u(s)\,ds \\
&+\int_{t/2}^t \vec{a}\cdot\nabla G(t-s) * \nabla(|u|^{q-1}u(s))\,ds \\
=&I_1 + I_2.
\end{aligned}$$

Taking into account that

$$\begin{cases} \|\nabla G(t)\|_p \le C_p t^{-\frac{N}{2}(1-\frac{1}{p})-\frac{1}{2}}; & \|D^2G(t)\|_p \le C_p t^{-\frac{N}{2}(1-\frac{1}{p})-1} \\ \|u(t)\|_p \le C_p (t+1)^{-\frac{N}{2}(1-\frac{1}{p})}; & \|\nabla u(t)\|_p \le C_p t^{-\frac{N}{2}(1-\frac{1}{p})-\frac{1}{2}} \end{cases}$$

we get

$$\begin{aligned}
\|I_1\|_p &\le |\vec{a}| \int_0^{t/2} \|D^2G(t-s)\|_p \, \||u|^{q-1}u(s)\|_1\,ds \\
&\le |\vec{a}|C_p \int_0^{t/2} (t-s)^{-\frac{N}{2}(1-\frac{1}{p})-1}(s+1)^{-1}\,ds \\
&\le |\vec{a}|C_p \left(\frac{t}{2}\right)^{-\frac{N}{2}(1-\frac{1}{p})-1} \int_0^{t/2} (s+1)^{-1}\,ds \\
&= C_p t^{-\frac{N}{2}(1-\frac{1}{p})-1}\log(1+t/2)
\end{aligned} \tag{50}$$

and

$$\begin{aligned}\|I_2\| &\le |\vec{a}|C_p \int_{t/2}^{t} \|\nabla G(t-s)\|_1 \|\nabla(|u|^{q-1}u(s))\|_p \, ds \\ &\le C_p|\vec{a}| \int_{t/2}^{t} (t-s)^{-\frac{1}{2}}(s+1)^{-1}s^{-\frac{N}{2}(1-\frac{1}{p})-\frac{1}{2}} \, ds \\ &\le C_p\left(\frac{t}{2}\right)^{-\frac{N}{2}(1-\frac{1}{p})-\frac{3}{2}} \int_{t/2}^{t} (t-s)^{-1/2} ds \\ &= C_p t^{-\frac{N}{2}(1-\frac{1}{p})-1}, \quad \text{as } t \to \infty.\end{aligned} \tag{51}$$

Combining (50) and (51), (49) follows. This concludes the proof of (48) and therefore convergence (47) holds in $L^p(\mathbb{R}^N)$ for every $p \in [1, \infty]$. Finally, it is easy to check that (47) is equivalent to the statement of Theorem 2.

4. The case $q > 1 + \frac{2}{N}$. Let u be the solution of (1)–(2) and define $v = v(y,s)$ as in (32). In this case, v satisfies

$$\begin{cases} v_s + Lv - \dfrac{N}{2}v = e^{-\frac{s}{2}(N(q-1)-1)}\vec{a}\cdot\nabla(|v|^{q-1}v) & \text{in } \mathbb{R}^N \times (0,\infty) \\ v(y,0) = \varphi(y). \end{cases} \tag{52}$$

As in Section 3 we know that

$$v(s) \to M\theta_1 \text{ in } L^2(K) \cap L^p(\mathbb{R}^N) \text{ as } s \to \infty \tag{53}$$

for every $1 \le p \le \infty$.

In order to prove Theorem 3 we must understand the behavior of

$$w(y,s) = e^{s/2}(v(y,s) - M\theta_1(y)) \tag{54}$$

which satisfies

$$\begin{cases} w_s + Lw - \dfrac{(N+1)}{2}w = e^{-\frac{s}{2}(N(q-1)-2)}\vec{a}\cdot\nabla(|v|^{q-1}v) & \text{in } \mathbb{R}^N \times (0,\infty) \\ w(y,0) = \varphi(y) - M\theta_1(y). \end{cases} \tag{55}$$

Note that, since $q > 1 + \frac{2}{N}$, $N(q-1) - 2 > 0$ and therefore the right hand side of (55) decreases exponentially as $s \to \infty$.

In view of (7iii) we have $w \in L^\infty(0,\infty; L^p(\mathbb{R}^N))$ for every $p \in [1,\infty]$. We also have $w \in L^\infty(0,\infty; L^2(K))$. We write $w = w_1 + w_2$ where w_1 solves

$$\begin{cases} w_{1,s} + Lw_1 - \dfrac{(N+1)}{2}w_1 = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \\ w_1(y,0) = \varphi(y) - M\theta_1(y) \end{cases} \tag{56}$$

and w_2 satisfies

$$\begin{cases} w_{2,s} + Lw_2 - \dfrac{(N+1)}{2}w_2 = e^{-\frac{s}{2}[N(q-1)-2]}\vec{a}\cdot\nabla(|v|^{q-1}v) & \text{in } \mathbb{R}^N \times (0,\infty) \\ w_2(y,0) = 0. \end{cases} \tag{57}$$

Expanding w_1 in Fourier series we get that

$$w_1(\cdot,s) \to \frac{1}{2}\left(\int_{\mathbb{R}^N}[\varphi(y) - M\theta_1(y)]y\,dy\right)\cdot y\theta_1(y) = -\vec{m}\cdot\nabla\theta_1(y) \tag{58}$$

in $L^2(K)$ as $s \to \infty$. On the other hand, proceeding as in Section 3, we deduce that

$$w_2(\cdot,s) \to \left(\int_0^\infty e^{-\frac{\sigma}{2}(N(q-1)-2}\int_{\mathbb{R}^N}|v|^{q-1}v(y,\sigma)\,dy\,d\sigma\right)\vec{a}\cdot\nabla\theta_1(y) \tag{59}$$

in $L^2(K)$ as $s \to \infty$. Combining (58) and (59) we obtain

$$w(y,s) \to f(y) \quad \text{in } L^2(K) \text{ as } s \to \infty \tag{60}$$

with

$$f(y) = \left(\vec{a}\int_0^\infty e^{-\frac{\sigma}{2}[N(q-1)-2]}\int_{\mathbb{R}^N}|v|^{q-1}v(y,\sigma)\,dy\,d\sigma - \vec{m}\right)\cdot\nabla\theta_1(y). \tag{61}$$

As in Section 3, convergence (60) holds in $L^p(\mathbb{R}^N)$ for every $p \in [1,\infty]$ and this is equivalent to (21)–(22).

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