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# A UNIQUENESS RESULT FOR MEASURE-VALUED SOLUTIONS OF NONLINEAR HYPERBOLIC EQUATIONS

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**Abstract.** We will study the solutions of the nonlinear hyperbolic equation $u_t + \mathrm{div}(\mathbf{v}f(u)) = 0$ in $\mathbb{R}^N \times [0,T]$, with given initial condition $u(\cdot,0) = u_0(\cdot)$ in $\mathbb{R}^N$, where $\mathbf{v}$ is a given function from $\mathbb{R}^N \times [0,T]$ to $\mathbb{R}^N$ and $f$ is a given function from $\mathbb{R}$ to $\mathbb{R}$. We will prove that if $\nu$ is a measure-valued solution satisfying some entropy condition, which we define, then $\nu_{x,t} = \delta_{u(x,t)}$ for almost every $(x,t) \in \mathbb{R}^N \times [0,T]$, where $u$ is the unique entropy weak solution to the equation.

We consider the following nonlinear hyperbolic equation, with the initial condition

$$\begin{cases} u_t(x,t) + \mathrm{div}(\mathbf{v}f(u(x,t))) = 0, & x \in \mathbb{R}^N, \ t \in [0,T], \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1}$$

where $T > 0$, $u_t$ denotes the derivative of $u$ with respect to $t$, $\mathrm{div}\,\mathbf{v} = \sum_{i=1}^N \partial_{x_i} v_i$, where $x = (x_1, \dots, x_N)$, $\mathbf{v} = (v_1, \dots, v_N)$, $\mathbf{v} \in C^3(\mathbb{R}^N \times [0,T], \mathbb{R}^N)$. It is assumed that there exists $V > 0$ such that $\sup\{|\mathbf{v}(x,t)|,\ x \in \mathbb{R}^N,\ t \in [0,T]\} \le V$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$, $f$ is a given function of class $C^3$ from $\mathbb{R}$ to $\mathbb{R}$, and $u_0 \in L^\infty(\mathbb{R}^N)$ is a given function. It is also assumed that $\sup\{-\mathrm{div}(\mathbf{v}(x,t))f'(u),\ x \in \mathbb{R}^N,\ t \in [0,T],\ u \in \mathbb{R}\} < \infty$. This is true, for instance, if $\mathrm{div}\,\mathbf{v} = 0$. Under these assumptions, Kruzkov [4] gives results of existence (and uniqueness) of the entropy weak solution $u$; i.e., a function $u \in L^\infty(\mathbb{R}^N \times ]0,T[)$ which satisfies:

$$\begin{aligned} &\int_{\mathbb{R}^N} \int_0^T \eta(u(x,t))\varphi_t(x,t) + \Phi(u(x,t))\mathbf{v}(x,t) \cdot \mathrm{grad}\,\varphi(x,t))\,dt\,dx \\ &+ \int_{\mathbb{R}^N} \int_0^T \mathrm{div}\,\mathbf{v}(x,t)\varphi(x,t)\Big(\Phi(u(x,t)) - \eta'(u(x,t)f(u(x,t))\Big)dt\,dx \\ &+ \int_{\mathbb{R}^N} \eta(u_0(x))\varphi(x,0)\,dx \ge 0 \quad \forall \varphi \in C_c^1(\mathbb{R}^N \times [0,T[, \mathbb{R}_+), \end{aligned} \tag{2}$$

for any $C^1$ convex function $\eta$ from $\mathbb{R}$ to $\mathbb{R}$, and $\Phi$ such that $\Phi' = f'\eta'$,

where $C_c^1(E,F)$ is the set of $C^1$ functions from $E$ to $F$ with compact support in $E$.

Other results of existence and uniqueness of a weak entropy solution to problem (1) are given in [8,] [6], and [5].

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**Remark 1.** In fact, Kruzkov shows the existence (and uniqueness) of a solution $u$ in the following sense:

$$u \in L^\infty(\mathbb{R}^N \times ]0,T[)$$

$$\begin{aligned}\int_{\mathbb{R}^N}\int_0^T \Big[&|u(x,t)-k|\varphi_t(x,t) + \operatorname{sgn}(u(x,t)-k)(f(u)-f(k))\mathbf{v}(x,t)\cdot \operatorname{grad}\varphi(x,t) \\ &- \operatorname{sgn}(u(x,t)-k)\operatorname{div}\mathbf{v}(x,t)f(k)\varphi(x,t)\Big]\,dt\,dx \geq 0, \\ &\forall \varphi \in C_c^1(\mathbb{R}^N \times ]0,T[,\mathbb{R}_+),\ \forall k \in \mathbb{R}.\end{aligned} \tag{3}$$

$$\begin{aligned}&\text{There exists a set } \mathcal{E} \subset [0,T] \text{ of zero measure such that for } t \in [0,T]\setminus\mathcal{E}, \\ &\text{the function } x \mapsto u(x,t) \text{ is defined a.e. on } \mathbb{R}^N, \text{ and for any ball} \\ &B_r = \{x \in \mathbb{R}^N : |x| \leq r\},\quad \lim_{\substack{t\to 0 \\ t\in[0,T]\setminus\mathcal{E}}} \int_{B_r} |u(x,t)-u_0(x)|\,dx = 0.\end{aligned} \tag{4}$$

This result yields the existence of $u \in L^\infty(\mathbb{R}^N \times ]0,T[)$ satisfying (2). This can be proved by approximating $\eta \in C^1(\mathbb{R},\mathbb{R})$ by functions of the form: $\eta_n(s) = \sum_{i=1}^n \alpha_i^{(n)}|s-k_i^{(n)}|$.

Conversely, if $u \in L^\infty(\mathbb{R}^N \times ]0,T[)$ satisfies inequality (2), then $u$ satisfies (3) (by taking regularizations of functions of the type $|\cdot - k|$), and

$$\lim_{\tau\to 0} \frac{1}{\tau}\int_0^\tau \int_{B_r} |u(x,t)-u_0(x)|\,dx\,dt = 0. \tag{5}$$

Note that (4) yields (5), but (5) is weaker than (4). In the following, we shall only use the fact that there exists $u \in L^\infty(\mathbb{R}^N \times ]0,T[)$ satisfying (3) and (5).

**Remark 2.** In previous works, an entropy weak solution is generally defined as a function $u \in L^\infty(\mathbb{R}^N \times ]0,T[) \cap C([0,T[, L^1_{loc}(\mathbb{R}^N))$ satisfying the equation, including the initial condition, in a weak sense; i.e.,

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_0^T u(x,t)\varphi_t(x,t) + f(u(x,t))\mathbf{v}(x,t)\cdot\operatorname{grad}\varphi(x,t))\,dx\,dt \\ &+ \int_{\mathbb{R}^N} u_0(x)\varphi(x,0)\,dx = 0 \quad \forall\varphi \in C_c^1(\mathbb{R}^N \times [0,T[,\mathbb{R}),\end{aligned} \tag{6}$$

and the entropy inequalities, without the initial condition; i.e.,

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_0^T \eta(u(x,t))\varphi_t(x,t) + \Big[\Phi(u(x,t))\mathbf{v}(x,t)\cdot\operatorname{grad}\varphi(x,t)) \\ &+ \operatorname{div}\mathbf{v}(x,t)\Big(\Phi(u(x,t) - f(u(x,t)\eta'(u(x,t))\Big)\varphi(x,t)\Big]\,dt\,dx \geq 0, \\ &\qquad \forall\varphi \in C_c^1(\mathbb{R}^N \times ]0,T[,\mathbb{R}_+), \\ &\quad \text{for any } C^1 \text{ convex function } \eta \text{ from } \mathbb{R} \text{ to } \mathbb{R}, \text{ and } \Phi \text{ such that } \Phi' = f'\eta'.\end{aligned} \tag{7}$$

Such a solution satisfies (2). We introduced formulation (2) in order to be compatible with what follows on measure-valued solutions. In fact, we shall use (7) (more precisely (3)) as a characterization of an entropy weak solution, and write the initial condition under the form (5), which derives from (2) or from the fact that $u \in L^\infty(\mathbb{R}^N \times ]0,T[) \cap C([0,T[, L^1_{loc}(\mathbb{R}^N))$ and satisfies (6).

**Remark 3.** It is possible to replace in problem (1) $\mathbf{v}(x,t)f(u)$ by a (vector) function $\mathbf{v}(x,t,u)$, and a right-hand-side of the type $g(x,t,u)$, with the following assumptions on $\mathbf{v}$ and $g$ (see [4]):

(1) The function $\mathbf{v}$ is of class $C^3$, and $g$ is of class $C^2$,
(2) $\sup\{|\frac{\partial \mathbf{v}}{\partial u}(x,t,u)|,\ x \in \mathbb{R}^N,\ t \in [0,T],\ u \in [-M,M]\} < \infty$, for all $M \in \mathbb{R}$,
(3) $\sup\{|\operatorname{div}\mathbf{v}(x,t,0) - g(x,t,0)|,\ x \in \mathbb{R}^N,\ t \in [0,T]\} < \infty$,
(4) $\sup\{|\frac{\partial(\operatorname{div}\mathbf{v} - g)}{\partial u}(x,t,u)|,\ x \in \mathbb{R}^N,\ t \in [0,T],\ u \in \mathbb{R}\} < \infty$.

Under these hypotheses, the uniqueness theorem stated below still holds. However, for the sake of clarity, we limit ourselves to the case $\mathbf{v}(x,t)f(u)$.

Entropy weak solutions can be seen as limits of the solutions to the viscous perturbations of (1). Some years ago, the notion of a measure-valued solution was introduced with $\mathbf{v}$ constant, which can appear, for instance, as the limit of the solutions of some other perturbations of (1) (KdV equations, see [2]).

**Definition 1.** Let $r > 0$, $T > 0$ and $\nu : (x,t) \mapsto \nu_{x,t}$ be a measurable application from $\mathbb{R}^N \times ]0,T[$ into the set of probabilities on $\mathbb{R}$ supported in $[-r,r]$. For $g \in C(\mathbb{R},\mathbb{R})$, define $\mu_g \in L^\infty(\mathbb{R}^N \times ]0,T[, \mathbb{R})$ by

$$\mu_g(x,t) = \int_{\mathbb{R}} g(\lambda)\, d\nu_{x,t}(\lambda). \tag{8}$$

Let $f \in C(\mathbb{R},\mathbb{R})$, $u_0 \in L^\infty(\mathbb{R}^N)$, $\mathbf{v} \in C^1(\mathbb{R}^N \times [0,T], \mathbb{R}^N)$, then $\nu$ is a measure valued solution to problem (1) if

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_0^T (\mu_{(\cdot)}(x,t)\varphi_t(x,t) + \mu_f(x,t)\mathbf{v}(x,t)\cdot \operatorname{grad}\varphi(x,t))\,dt\,dx\\ &+ \int_{\mathbb{R}^N} u_0(x)\varphi(x,0)\,dx = 0 \quad \forall \varphi \in C^1_c(\mathbb{R}^N \times [0,T[, \mathbb{R}),\end{aligned} \tag{9}$$

where $(\cdot)$ denotes the application $\lambda \mapsto \lambda$.

Note that in the above definition $\mu_g$ is well defined for any $g \in C(\mathbb{R},\mathbb{R})$, due to the assumption that $\nu_{x,t}$ has a compact support and therefore all functions in $C(\mathbb{R},\mathbb{R})$ are integrable over $\mathbb{R}$ for the measure $\nu_{x,t}$, for all $(x,t) \in \mathbb{R}^N \times ]0,T[$.

Examples of measure-valued solutions which are not entropy weak solutions can be built (see [2]). It can be shown that if $(u_\epsilon)_{\epsilon>0} \subset L^\infty(\mathbb{R}^N \times ]0,T[, \mathbb{R})$ is such that $u_\epsilon \to \nu$ as $\epsilon \to 0$ (in the sense that for all $g \in C(\mathbb{R},\mathbb{R})$, $g(u_\epsilon) \to \mu_g$ in $L^\infty(\mathbb{R}^N \times ]0,T[, \mathbb{R})$ for the weak star topology), and $u_\epsilon \to u$ almost everywhere, then $\nu_{x,t} = \delta_{u(x,t)}$ for almost every $(x,t) \in \mathbb{R}^N \times ]0,T[$, where $\delta_a$ denotes the Dirac measure at $a$.

**Definition 2.** With the notations and assumptions introduced in Definition 1, the application $\nu : (x,t) \mapsto \nu_{x,t}$ is said to be an entropy measure valued solution to problem (1) if

$$\int_{\mathbb{R}^N}\int_0^T \Big[\mu_\eta(x,t)\varphi_t(x,t) + \mu_\Phi(x,t)\mathbf{v}(x,t)\cdot \operatorname{grad}\varphi(x,t)) \\ + \operatorname{div}\mathbf{v}(x,t)\mu_{\Phi-f\eta'}(x,t)\varphi(x,t)\Big]\,dx\,dt + \int_{\mathbb{R}^N}\eta(u_0(x))\varphi(x,0)\,dx \geq 0 \qquad (10)$$

$$\forall \varphi \in C_c^1(\mathbb{R}^N\times[0,T[,\mathbb{R}_+),$$

for any $C^1$ convex function $\eta$ from $\mathbb{R}$ to $\mathbb{R}$, and $\Phi$ such that $\Phi' = f'\eta'$.

If $\nu$ is an entropy measure-valued solution to (1), then $\nu$ satisfies the initial condition in the sense stated in the following lemma. This result is an important step of the proof of the following uniqueness theorem.

**Lemma 1.** *Let $T>0$, $f\in C(\mathbb{R},\mathbb{R})$, $u_0\in L^\infty(\mathbb{R}^N)$, $\mathbf{v}\in C^1(\mathbb{R}^N\times[0,T],\mathbb{R}^N)$, and let $\nu : (x,t)\mapsto \nu_{x,t}$ be an entropy measure-valued solution to problem (1). Then*

$$\frac{1}{\tau}\int_0^\tau\int_{B_a}\mu_{|\cdot-u_0(x)|}(x,t)\,dx\,dt \to 0 \quad as \quad \tau\to 0, \quad \forall \quad a\in\mathbb{R}_+, \qquad (11)$$

*where $B_a = \{x\in\mathbb{R}^N : |x|\leq a\}$.*

**Proof.** Let $\psi\in C_c^1(\mathbb{R}^N,\mathbb{R}_+)$ and $0<\tau<T$. We define $\rho$ by

$$\rho(t) = \begin{cases} \frac{\tau-t}{\tau} & \text{if } 0\leq t\leq\tau \\ 0 & \text{if } t>\tau. \end{cases} \qquad (12)$$

Setting $\varphi(x,t)=\psi(x)\rho(t)$ and $\eta(s)=s^2$ in (10) (this is possible by taking regularizations of the function $\rho$) yields

$$-\frac{1}{\tau}\int_0^\tau\int_{\mathbb{R}^N}\mu_{|\cdot|^2}(x,t)\psi(x)\,dx\,dt + \int_0^\tau\int_{\mathbb{R}^N}\mu_\Phi(x,t)\rho(t)\mathbf{v}(x,t)\cdot\operatorname{grad}\psi(x)\,dx\,dt \\ + \int_{\mathbb{R}^N}u_0^2(x)\psi(x)\,dx + \int_0^\tau\int_{\mathbb{R}^N}\mu_{\Phi-f\eta'}(x,t)\operatorname{div}\mathbf{v}(x,t)\rho(t)\psi(x)\,dx\,dt \geq 0. \qquad (13)$$

Since the application

$$(x,t)\mapsto \Big(\mu_\Phi(x,t)\mathbf{v}(x,t)\operatorname{grad}\psi(x) + \mu_{\Phi-f\eta'}(x,t)\operatorname{div}\mathbf{v}(x,t)\psi(x)\Big)\rho(t)$$

belongs to $L^\infty(\mathbb{R}^N\times]0,T[)$, we obtain

$$\limsup_{\tau\to 0}\frac{1}{\tau}\int_0^\tau\int_{\mathbb{R}^N}\mu_{|\cdot|^2}(x,t)\psi(x)\,dx\,dt \leq \int_{\mathbb{R}^N}u_0^2(x)\psi(x)\,dx. \qquad (14)$$

Now, taking $\eta$ to be a linear function in (10), we get an equality and the test function $\varphi$ may be taken in $C_c^1(\mathbb{R}^N\times[0,T[,\mathbb{R})$. Let $(u_0^{(n)})_n\subset C^1(\mathbb{R}^N,\mathbb{R})$ be a sequence such that $\|u_0^{(n)}\|_\infty \leq \|u_0\|_\infty$, for every $n$, and $u_0^{(n)}\to u_0$ almost everywhere in $\mathbb{R}^N$ as $n\to$

$+\infty$. Taking regularizations of $\rho$, it is clear that we can set $\varphi(x,t) = \psi(x)u_0^{(n)}(x)\rho(t)$ in (10). This gives

$$
\begin{aligned}
&-\frac{1}{\tau}\int_0^\tau \int_{\mathbb{R}^N} \Big(\mu_{(\cdot)}(x,t) - u_0(x)\Big)\psi(x)u_0^{(n)}(x)\,dx\,dt \\
&+\int_0^\tau \int_{\mathbb{R}^N} \mu_f(x,t)\mathbf{v}(x,t)\cdot \text{grad}\,\Big(\psi(x)u_0^{(n)}(x)\Big)\rho(t)dx\,dt = 0.
\end{aligned} \tag{15}
$$

Since the application defined by $(x,t) \mapsto \mu_f(x,t)\mathbf{v}(x,t)\cdot\text{grad}\big(\psi(x)u_0^{(n)}(x)\big)\rho(t)$ belongs to $L^\infty(\mathbb{R}^N\times]0,T[)$ and since $\psi$ has a compact support, passing to the limit as $\tau \to 0$ in (15) yields

$$
\lim_{\tau\to 0}\frac{1}{\tau}\int_0^\tau \int_{\mathbb{R}^N} \Big(\mu_{(\cdot)}(x,t) - u_0(x)\Big)\psi(x)u_0^{(n)}(x)\,dx\,dt = 0. \tag{16}
$$

Since $u_0^{(n)}$ tends to $u_0$ in $L^1_{loc}(\mathbb{R}^N)$ as $n$ tends to infinity, we may assert that

$$
\begin{aligned}
&\frac{1}{\tau}\int_0^\tau \int_{\mathbb{R}^N} \Big(\mu_{(\cdot)}(x,t) - u_0(x)\Big)\psi(x)u_0^{(n)}(x)\,dx\,dt \\
&\to \frac{1}{\tau}\int_0^\tau \int_{\mathbb{R}^N} \Big(\mu_{(\cdot)}(x,t) - u_0(x)\Big)\psi(x)u_0(x)\,dx\,dt
\end{aligned} \tag{17}
$$

as $n \to +\infty$, uniformly with respect to $\tau$. Hence, we deduce from (16) that

$$
\lim_{\tau\to 0}\frac{1}{\tau}\int_0^\tau \int_{\mathbb{R}^N} \Big(\mu_{(\cdot)}(x,t) - u_0(x)\Big)\psi(x)u_0(x)\,dx\,dt = 0. \tag{18}
$$

Therefore, for all $\psi \in C_c^1(\mathbb{R}^N, \mathbb{R}_+)$, (14) and (18) imply

$$
\limsup_{\tau\to 0}\frac{1}{\tau}\int_0^\tau \int_{\mathbb{R}^N} \Big(\mu_{|\cdot|^2}(x,t) - 2\mu_{(\cdot)}(x,t)u_0(x) + u_0(x)^2\Big)\psi(x)\,dx\,dt \le 0. \tag{19}
$$

Let $\epsilon > 0$. Note that $|\lambda - u_0(x)| \le \epsilon + \frac{|\lambda - u_0(x)|^2}{\epsilon}$, we may write

$$
\begin{aligned}
&\int_0^\tau \int_{\mathbb{R}^N} \mu_{|\cdot - u_0(x)|}(x,t)\psi(x)\,dx\,dt \\
&\le \epsilon\int_0^\tau \int_{\mathbb{R}^N} \psi(x)\,dx\,dt + \frac{1}{\epsilon}\int_0^\tau \int_{\mathbb{R}^N} \mu_{|\cdot - u_0(x)|^2}(x,t)\psi(x)\,dx\,dt.
\end{aligned} \tag{20}
$$

Hence, passing to the limit as $\tau \to 0$ and using the fact that $\epsilon$ is arbitrary, we deduce that

$$
\lim_{\tau\to 0}\frac{1}{\tau}\int_0^\tau \int_{\mathbb{R}^N} \mu_{|\cdot - u_0(x)|}(x,t)\psi(x)\,dx\,dt = 0. \tag{21}
$$

Let $a > 0$. Setting $\psi = 1$ on $B_a = \{x \in \mathbb{R}^N : |x| \le a\}$ yields assumption (11), and completes the proof of Lemma.

We can now state the uniqueness result.

**Theorem.** *Let $T > 0$, $f \in C^1(\mathbb{R},\mathbb{R})$, $u_0 \in L^\infty(\mathbb{R}^N)$, $\mathbf{v} \in C^1(\mathbb{R}^N \times [0,T],\mathbb{R}^N)$, and let $\nu : (x,t) \mapsto \nu_{x,t}$ be an entropy measure-valued solution to problem* (1). *We assume that* $\sup\{|\mathbf{v}(x,t)|,\ x \in \mathbb{R}^N,\ t \in [0,T]\} = V < \infty$ *and that there exists u, an entropy weak solution to problem* (1) (*that is, $u \in L^\infty(\mathbb{R}^N \times ]0,T[)$ satisfying* (2)). *Then $\nu_{x,t} = \delta_{u(x,t)}$ for almost every $(x,t) \in \mathbb{R}^N \times \mathbb{R}_+$.*

**Remark 4.** Recall that in order to have the existence of an entropy weak solution to (1) it suffices, for instance, to assume that the functions $\mathbf{v}$ and $f$ are of class $C^3$ (see [4]) and that $\sup\{-\mathrm{div}(\mathbf{v}(x,t))f'(u),\ x \in \mathbb{R}^N,\ t \in [0,T],\ u \in \mathbb{R}\} < \infty$. Furthermore the same uniqueness theorem holds in the more general case described in Remark 3 (with an easy generalization of entropy measure-valued solution).

Note that DiPerna (see [2], [3]) had already proved a uniqueness result on the entropy measure-valued solution to (1) for constant $\mathbf{v}$ and with stronger assumptions on $\nu$, in particular with $\nu$ satisfying

$$\frac{1}{\tau}\int_0^\tau \int_{\mathbb{R}^N} \mu_{|\cdot - u_0(x)|}(x,t)\,dx\,dt \to 0 \quad \text{as} \quad \tau \to 0. \tag{22}$$

From Lemma 1 condition (10), which is assumed in our theorem, yields the weaker condition (11), which we recall

$$\frac{1}{\tau}\int_0^\tau \int_{B_a} \mu_{|\cdot - u_0(x)|}(x,t)\,dx\,dt \to 0 \quad \text{as} \quad \tau \to 0,\quad \forall a \in \mathbb{R}_+.$$

Examples can be built to show that (11) can be satisfied without (22) being satisfied. In fact, when studying the convergence of numerical schemes (see [1]), assumption (22) cannot be proved to be satisfied by the limit (in the sense defined below) of approximate solutions which are given by the numerical scheme (except under an "inverse CFL" condition; i.e., a condition of the type $h \le Ck$, where $h$ is the mesh size and $k$ the time step), whereas assumption (11) has been proven to be satisfied. For a constant $\mathbf{v}$, calling $(u_\epsilon)_{\epsilon \ge 0}$ the sequence of approximate solutions, the convergence result may be obtained in the following way. Since $(u_\epsilon)_{\epsilon\ge 0}$ is bounded in $L^\infty(\mathbb{R}^N \times ]0,T[)$, we may write that, for a convenient subsequence, $u_\epsilon \to \nu$ as $\epsilon \to 0$ in the sense that $g(u_\epsilon) \to \mu_g$ in $L^\infty(\mathbb{R}^N \times ]0,T[,\mathbb{R})$ for the weak star topology, for any function $g \in C(\mathbb{R},\mathbb{R})$. Using the fact that, for all $g \in C(\mathbb{R},\mathbb{R})$ and $w \in L^\infty(\mathbb{R}^N \times ]0,T[)$, $|g(u_\epsilon) - w| \to \mu_{|g(\cdot)-w|}$ in $L^\infty(\mathbb{R}^N \times ]0,T[,\mathbb{R})$ for the weak star topology (where $\mu_{|g(\cdot)-w|}(x,t) = \mu_{|g(\cdot)-w(x,t)|}(x,t)$) and the above uniqueness result, we prove $\mu_{|\cdot - u(x,t)|} = 0$, where $u$ is the entropy weak solution to (1), so that $u_\epsilon \to u$ in $L^p_{loc}(\mathbb{R}^N \times ]0,T[,\mathbb{R})$ for any finite $p$. A. Szepessy also gave (see [7]) a uniqueness theorem (for constant $\mathbf{v}$) with condition (11) instead of condition (22).

**Proof of the Theorem.** The proof of the theorem is decomposed in 3 steps. In step 1 we prove that if $\nu$ is an entropy measure-valued solution to (1) and if $u$ is the entropy weak solution to (1), we have

$$\int_{\mathbb{R}^N \times \mathbb{R}_+} (\mu_{|\cdot - u(x,t)|}(x,t)\varphi_t(x,t) + \mu_{F(\cdot,u(x,t))}(x,t)\mathbf{v}(x,t)\cdot \mathrm{grad}\,\varphi(x,t))\,dx\,dt \ge 0$$
$$\forall \varphi \in C_c^1(\mathbb{R}^N \times ]0,T[,\mathbb{R}_+), \tag{23}$$

where $F(\alpha,\beta) = \operatorname{sgn}(\alpha-\beta)(f(\alpha)-f(\beta))$. This result was proved by DiPerna (see [2]) in the case where $\mathbf{v}$ is constant.

Then, for $a > 0$, $\omega = VM_f$ with $M_f = \sup_{s\in[-b,b]} |f'(s)|$, where $b = \sup(r, U)$, and $U = \|u\|_{L^\infty(\mathbb{R}^N\times]0,T[)}$. Let $B_R = \{x \in \mathbb{R}^N : |x| \le R\}$ and define

$$A(t) = \int_{B_{a-\omega t}} \mu_{|\cdot - u(x,t)|}(x,t)\,dx = \int_{B_{a-\omega t}} \int_{\mathbb{R}} |\lambda - u(x,t)|\,d\nu_{x,t}(\lambda)\,dx, \qquad 0 < t < \inf(T, \frac{a}{\omega}). \tag{24}$$

We prove in Step 2 that $A$ is almost everywhere decreasing; i.e.,

$$A(t_1) \le A(t_2) \quad \text{for a.e. } t_1, t_2 \in [0, \inf(T, \frac{a}{\omega})], \ \ t_1 \ge t_2. \tag{25}$$

By Lemma 1, we have

$$\lim_{\tau\to 0} \frac{1}{\tau} \int_0^\tau \int_{B_a} \mu_{|\cdot - u_0(x)|}(x,t)\,dx\,dt = 0. \tag{26}$$

In step 3, we deduce from (25) and (26) that $\mu_{|\cdot - u(x,t)|}(x,t) = 0$ for almost every $(x,t) \in \mathbb{R}^N\times]0,T[$ and therefore that $\nu_{x,t} = \delta_{u(x,t)}$ for almost every $(x,t) \in \mathbb{R}^N\times]0,T[$.

**Step 1 (Proof of assertion (23)).** Taking regularizations of $\varphi$, we need to prove (23) for $\varphi \in C_c^\infty(\mathbb{R}^N\times]0,T[,\mathbb{R}_+)$ only. Since $u$ is the entropy weak solution to (1), that is, $u \in L^\infty(\mathbb{R}^N\times]0,T[,\mathbb{R})$ and $u$ satisfies (2), taking $\eta(\cdot) = |\cdot - k|$, $k \in \mathbb{R}$ in (2) (such functions being obviously admissible by a regularization procedure), we obtain

$$\int_{\mathbb{R}^N\times\mathbb{R}_+} \Big[|u(y,s)-k|\varphi_t(y,s) + F(u(y,s),k)\mathbf{v}(y,s)\cdot\operatorname{grad}\varphi(y,s) - f(k)\operatorname{sgn}(u(y,s)-k)\operatorname{div}\mathbf{v}(y,s)\varphi(y,s)\Big]dy\,ds \ge 0 \quad \forall\varphi \in C_c^\infty(\mathbb{R}^N\times]0,T[,\mathbb{R}_+). \tag{27}$$

Similarly, since $\nu$ is an entropy measure-valued solution to (1), $\nu$ satisfies

$$\int_{\mathbb{R}^N\times\mathbb{R}_+} \Big[\mu_{|\cdot-k|}(x,t)\varphi_t(x,t) + \mu_{F(\cdot,k)}(x,t)\mathbf{v}(x,t)\cdot\operatorname{grad}\varphi(x,t) - f(k)\mu_{\operatorname{sgn}(\cdot-k)}(x,t)\operatorname{div}\mathbf{v}(x,t)\varphi(x,t)\Big]dx\,dt \ge 0 \quad \forall\varphi \in C_c^\infty(\mathbb{R}^N\times]0,T[,\mathbb{R}_+). \tag{28}$$

Since the application defined by $(k,y,s) \mapsto |u(y,s)-k|\varphi_t(y,s) + F(u(y,s),k)\mathbf{v}(y,s)\cdot\operatorname{grad}\varphi(y,s) - f(k)\operatorname{sgn}(u(y,s)-k)\operatorname{div}\mathbf{v}(y,s)\varphi(y,s)$ is integrable over $\mathbb{R}\times\mathbb{R}^N\times]0,T[$ for the measure $\nu_{x,t}\bigotimes\lambda_N\bigotimes\lambda$ for fixed $(x,t)$ (we denote $\lambda$ and $\lambda_N$ the Lebesgue measures on $\mathbb{R}$ and $\mathbb{R}^N$), we integrate (27) over $\mathbb{R}$ with respect to the measure $\nu_{x,t}$ and obtain

$$\begin{aligned}\int_{\mathbb{R}^N}\int_0^T \Big[\int_{\mathbb{R}} &|u(y,s)-\lambda|d\nu_{x,t}(\lambda)\varphi_t(y,s) \\ &+ \int_{\mathbb{R}} F(u(y,s),\lambda)\,d\nu_{x,t}(\lambda)\mathbf{v}(y,s)\cdot\operatorname{grad}\varphi(y,s) \\ &- \int_{\mathbb{R}} f(\lambda)\operatorname{sgn}(u(y,s)-\lambda)\,d\nu_{x,t}(\lambda)\operatorname{div}\mathbf{v}(y,s)\varphi(y,s)\Big]dy\,ds \ge 0 \\ &\forall\varphi \in C_c^\infty(\mathbb{R}^N\times]0,T[,\mathbb{R}_+).\end{aligned} \tag{29}$$

For $p = 1, N$, let $\rho_p \in C_c^\infty(\mathbb{R}^p, \mathbb{R})$; $\mathrm{supp}(\rho_p) = \overline{\{x \in \mathbb{R}^p, \rho_p(x) \neq 0\}} \subset B_1$, $\rho_p \geq 0$ and $\rho_p$ constant on $B_{\frac{1}{2}}$ and $\rho_p(-x) = \rho_p(x)$; also let $\rho_{p,n}$ be defined by $\rho_{p,n} = n^p \rho_p(nx)$, $\forall n \geq 1$. Let $\psi \in C_c^\infty(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}_+)$. Set $\varphi(y,s) = \psi(\frac{x+y}{2}, \frac{t+s}{2})\rho_{N,n}(x-y)\rho_{1,n}(t-s)$ in (29) and integrate over $\mathbb{R}^N \times ]0,T[$, obtaining

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\int_{\mathbb{R}}\Big[\int_{\mathbb{R}} |u(y,s)-\lambda| d\nu_{x,t}(\lambda)\Big(\frac{1}{2}\psi_t(\frac{x+y}{2},\frac{t+s}{2})\rho_{N,n}(x-y)\rho_{1,n}(t-s)\\
&+\psi(\frac{x+y}{2},\frac{t+s}{2})\rho_{N,n}(x-y)\rho'_{1,n}(t-s)\Big) + \int_{\mathbb{R}} F(u(y,s),\lambda)\, d\nu_{x,t}(\lambda)\\
&\mathbf{v}(y,s)\cdot\Big(\frac{1}{2}\mathrm{grad}\,\psi(\frac{x+y}{2},\frac{t+s}{2})\rho_{N,n}(x-y)\rho_{1,n}(t-s)\\
&-\psi(\frac{x+y}{2},\frac{t+s}{2})\mathrm{grad}\,\rho_{N,n}(x-y)\rho_{1,n}(t-s)\Big) - \int_{\mathbb{R}} f(\lambda)\mathrm{sgn}(u(y,s)-\lambda)\, d\nu_{x,t}(\lambda)\\
&\mathrm{div}\,\mathbf{v}(y,s)\psi(\frac{x+y}{2},\frac{t+s}{2})\rho_{N,n}(x-y)\rho_{1,n}(t-s)\Big]\, dyds\, dx\, dt \geq 0.
\end{aligned} \tag{30}$$

Note that (30) makes sense since the expression between brackets in (30) is integrable over $(\mathbb{R}^N \times ]0,T[)^2$ for the measure $(\lambda_N \otimes \lambda) \otimes (\lambda_N \otimes \lambda)$.

Setting $k = u(y,s)$ and $\varphi(x,t) = \psi(\frac{x+y}{2}, \frac{t+s}{2})\rho_{N,n}(x-y)\rho_{1,n}(t-s)$ in (28), we may again integrate over $\mathbb{R}^N \times ]0,T[$, thus obtaining

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\int_{\mathbb{R}}\Big[\int_{\mathbb{R}} |u(y,s)-\lambda|\, d\nu_{x,t}(\lambda)\Big(\frac{1}{2}\psi_t(\frac{x+y}{2},\frac{t+s}{2})\rho_{N,n}(x-y)\rho_{1,n}(t-s)\\
&-\psi(\frac{x+y}{2},\frac{t+s}{2})\rho_{N,n}(x-y)\rho'_{1,n}(t-s)\Big) + \int_{\mathbb{R}} F(u(y,s),\lambda)\, d\nu_{x,t}(\lambda)\mathbf{v}(x,t)\\
&\cdot\Big(\frac{1}{2}\mathrm{grad}\,\psi(\frac{x+y}{2},\frac{t+s}{2})\rho_{N,n}(x-y)\rho_{1,n}(t-s)\\
&+\psi(\frac{x+y}{2},\frac{t+s}{2})\mathrm{grad}\,\rho_{N,n}(x-y)\rho_{1,n}(t-s)\Big) + \int_{\mathbb{R}} \mathrm{sgn}(u(y,s)-\lambda)\, d\nu_{x,t}(\lambda)\\
&f(u(y,s))\mathrm{div}\,\mathbf{v}(x,t)\psi(\frac{x+y}{2},\frac{t+s}{2})\rho_{N,n}(x-y)\rho_{1,n}(t-s)\Big]\, dx\, dt\, dy\, ds \geq 0.
\end{aligned} \tag{31}$$

Summing (30) and (31), we get

$$T_n + X_n + Y_n + Z_n + W_n \geq 0, \tag{32}$$

where,

$$\begin{aligned}
T_n = &\int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}} |u(y,s)-\lambda|\, d\nu_{x,t}(\lambda)\psi_t(\frac{x+y}{2},\frac{t+s}{2})\\
&\rho_{N,n}(x-y)\rho_{1,n}(t-s)\, dx\, dt\, dy\, ds,
\end{aligned} \tag{33}$$

$$\begin{aligned}
X_n = &\int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}} F(u(y,s),\lambda)\, d\nu_{x,t}(\lambda)\mathbf{v}(x,t)\cdot\mathrm{grad}\,\psi(\frac{x+y}{2},\frac{t+s}{2})\\
&\rho_{N,n}(x-y)\rho_{1,n}(t-s)\, dx\, dt\, dy\, ds,
\end{aligned} \tag{34}$$

$$\begin{aligned}
Y_n = \frac{1}{2}&\int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}} F(u(y,s),\lambda)\, d\nu_{x,t}(\lambda)\Big[\mathbf{v}(y,s)-\mathbf{v}(x,t)\Big]\\
&\cdot\mathrm{grad}\,\psi(\frac{x+y}{2},\frac{t+s}{2})\rho_{N,n}(x-y)\rho_{1,n}(t-s)\, dx\, dt\, dy\, ds,
\end{aligned} \tag{35}$$

$$Z_n = \int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}} F(u(y,s),\lambda)\,d\nu_{x,t}(\lambda)\Big[\mathbf{v}(x,t)-\mathbf{v}(y,s)\Big] \cdot \psi(\frac{x+y}{2},\frac{t+s}{2})\mathrm{grad}\,\rho_{N,n}(x-y)\rho_{1,n}(t-s)\,dx\,dt\,dy\,ds \tag{36}$$

$$W_n = \int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{sgn}(u(y,s)-\lambda)\Big[f(u(y,s))\mathrm{div}\,\mathbf{v}(x,t)-f(\lambda) \,\mathrm{div}\,\mathbf{v}(y,s)\Big]d\nu_{x,t}(\lambda)\psi(\frac{x+y}{2},\frac{t+s}{2})\rho_{N,n}(x-y)\rho_{1,n}(t-s)\,dx\,dt\,dy\,ds. \tag{37}$$

By successive changes of variables, we get

$$T_n = \int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}} |u(x-\frac{y}{n},t-\frac{s}{n})-\lambda|\,d\nu_{x,t}(\lambda)\psi_t(x-\frac{y}{2n},t-\frac{s}{2n}) \rho_N(y)\rho_1(s)\,dx\,dt\,dy\,ds, \tag{38}$$

so that

$$\Big|T_n - \int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}} |u(x,t)-\lambda|\,d\nu_{x,t}(\lambda)\psi_t(x,t)\,dx\,dt\Big| \leq \int_{B_1\times B_1} \|u(\cdot-\frac{y}{n},\cdot-\frac{s}{n})-u\|_{L^1(K_\psi)}\|\psi\|_\infty\rho_N(y)\rho_1(s)\,dy\,ds + \frac{1}{n}C_{\psi,b}, \tag{39}$$

where $K_\psi = \overline{K'_\psi}$, $K'_\psi = \{(x,t)\in\mathbb{R}^N\times]0,T[ : \exists(y,s)\in\mathbb{R}^N\times]0,T[,\ |y-x|\leq 1,\ |t-s|\leq 1 \text{ and } \psi(y,s)\neq 0\}$, and $C_{\psi,b}$ depends on $\psi$ (more precisely on the $L^\infty$ norm on the second derivatives of $\psi$ and on the measure of $K_\psi$) and $b$. Here we used the fact that $\rho_N(y)=0$ if $|y|>1$ and $\rho_1(s)=0$ if $|s|>1$. Therefore by the theorem of continuity in means,

$$T_n \to \int_{\mathbb{R}^N\times\mathbb{R}_+}\int_{\mathbb{R}} |u(x,t)-\lambda|\,d\nu_{x,t}(\lambda)\psi_t(x,t)\,dx\,dt \quad \text{as} \quad n\to+\infty. \tag{40}$$

In a similar way, it is easily seen that

$$\Big|X_n - \int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}} F(u(x,t),\lambda)\,d\nu_{x,t}(\lambda)\mathbf{v}(x,t)\mathrm{grad}\,\psi(x,t)\,dx\,dt\Big| \leq VM_f\int_{B_1\times B_1} \|u(\cdot-\frac{y}{n},\cdot-\frac{s}{n})-u\|_{L^1(K_\psi)}\|\mathrm{grad}\,\psi\|_\infty\rho_N(y)\rho_1(s)\,dy\,ds + \frac{1}{n}C_{v,\psi,b}, \tag{41}$$

where $C_{v,\psi,b}$ only depends on $v,\psi$ and $b$, so that

$$X_n \to \int_{\mathbb{R}^N\times\mathbb{R}_+}\int_{\mathbb{R}} F(u(x,t),\lambda)\,d\nu_{x,t}(\lambda)\mathbf{v}(x,t)\mathrm{grad}\,\psi(x,t)\,dx\,dt \quad \text{as} \quad n\to+\infty. \tag{42}$$

Since $\rho_{N,n}(x-y)=0$ if $|x-y|>\frac{1}{n}$ (resp. $\rho_{1,n}(t-s)=0$ if $|t-s|>\frac{1}{n}$), and since $\mathbf{v}$ is continuously derivable, we get $|Y_n|\leq\frac{C_2}{n}$, where $C_2$ depends on $\mathbf{v}$, $\psi$ and $b$.

Denoting by $v_i$, $i=1,\dots,N$, the components of $\mathbf{v}$, we have

$$\Big|(v_i(x,t)-v_i(y,s))\psi(\frac{x+y}{2},\frac{t+s}{2}) - \Big[\sum_{j=1}^{N}\frac{\partial v_i}{\partial x_j}(x,t)(x_j-y_j)+\frac{\partial v_i}{\partial t}(x,t)(t-s)\Big]\psi(x,t)\Big| \leq \frac{C^{(i)}_{\mathbf{v},\psi}}{n^2}, \tag{43}$$

where $C^{(i)}_{\mathbf{v},\psi}$ only depends on $\mathbf{v}$ and $\psi$. Hence we may write $Z_n$ as

$$Z_n = \sum_{i=1}^{N}\Big(\sum_{j=1}^{N} Z_n^{ij}\Big) + Z_n' + Z_n'', \tag{44}$$

where

$$\begin{aligned} Z_n^{ij} = \int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}} F(u(y,s),\lambda)\,d\nu_{x,t}(\lambda)\frac{\partial v_i}{\partial x_j}(x,t)(x_j-y_j) \\ \psi(x,t)\frac{\partial \rho_{N,n}}{\partial x_i}(x-y)\rho_{1,n}(t-s)\,dx\,dt\,dy\,ds, \end{aligned} \tag{45}$$

$$\begin{aligned} Z_n' = \int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}} F(u(y,s),\lambda)\,d\nu_{x,t}(\lambda)\sum_{i=1}^{N}\frac{\partial v_i}{\partial t}(x,t)(t-s) \\ \psi(x,t)\frac{\partial \rho_{N,n}}{\partial x_i}(x-y)\rho_{1,n}(t-s)\,dx\,dt\,dy\,ds, \end{aligned} \tag{46}$$

and

$$\begin{aligned} |Z_n''| \le \int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}} |F(u(y,s),\lambda)|\,d\nu_{x,t}(\lambda)\sum_{i=1}^{N}\frac{C^{(i)}_{\mathbf{v},\psi}}{n^2} \\ |\frac{\partial \rho_{N,n}}{\partial x_i}(x-y)\rho_{1,n}(t-s)|\,dx\,dt\,dy\,ds. \end{aligned} \tag{47}$$

Noting that $\int_{\mathbb{R}} s\rho_{1,n}(s)\,ds = 0$ and that $\int_{\mathbb{R}^N} y_j\frac{\partial\rho_{N,n}}{\partial x_i}(y)\,dy = 0$ for $i \ne j$, we get by successive changes of variables (and using the Lipschitz continuity of $F$ and the theorem of continuity in means) that $Z_n' \to 0$ and, for $i \ne j$, $Z_n^{ij} \to 0$ as $n \to +\infty$. It is also clear that $Z_n'' \to 0$ as $n \to +\infty$. Remarking that $\int_{\mathbb{R}^N} y_i\frac{\partial\rho_{N,n}}{\partial x_i}(y)\,dy = -1$, and thanks to the Lipschitz continuity of $F$, we obtain

$$Z_n^{ii} \to -\int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}} F(u(x,t),\lambda)\,d\nu_{x,t}(\lambda)\frac{\partial v_i}{\partial x_i}(x,t)\psi(x,t)\,dx\,dt \quad \text{as} \quad n\to+\infty. \tag{48}$$

Therefore,

$$Z_n \to -\int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}} F(u(x,t),\lambda)\,d\nu_{x,t}(\lambda)\operatorname{div}\mathbf{v}(x,t)\psi(x,t)\,dx\,dt \quad \text{as} \quad n\to+\infty. \tag{49}$$

We now turn to the study of $W_n$ which we write under the form $W_n = W_n^1 + W_n^2$, with

$$\begin{aligned} W_n^1 = \int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}} \operatorname{sgn}(u(y,s)-\lambda)\Big[f(u(y,s))-f(\lambda)\Big]\,d\nu_{x,t}(\lambda)\operatorname{div}\mathbf{v}(x,t) \\ \psi(\frac{x+y}{2},\frac{t+s}{2})\rho_{N,n}(x-y)\rho_{1,n}(t-s)\,dx\,dt\,dy\,ds \end{aligned} \tag{50}$$

and

$$\begin{aligned} W_n^2 = \int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\int_{\mathbb{R}}\int_{\mathbb{R}} f(\lambda)\operatorname{sgn}(u(y,s)-\lambda)\,d\nu_{x,t}(\lambda)\Big[\operatorname{div}\mathbf{v}(y,s)-\operatorname{div}\mathbf{v}(x,t)\Big] \\ \psi(\frac{x+y}{2},\frac{t+s}{2})\rho_{N,n}(x-y)\rho_{1,n}(t-s)\,dx\,dt\,dy\,ds. \end{aligned} \tag{51}$$

Since

$$|W_n^2| \le \frac{C_{\mathbf{v},f,\psi,b}}{n} \int_{K_\psi} \int_{\mathbb{R}} \int_{K_\psi} \int_{\mathbb{R}} \rho_{N,n}(x-y)\rho_{1,n}(t-s)\,dx\,dt\,dy\,ds,$$

where $C_{\mathbf{v},f,\psi,b}$ depends only on $f$, $\mathbf{v}$, $\psi$ and $b$, we have $W_n^2 \to 0$ as $n \to +\infty$. By a successive change of variables and thanks to the Lipschitz continuity of $F$, we have

$$W_n^1 \to \int_{\mathbb{R}^N} \int_{\mathbb{R}} \int_{\mathbb{R}} F(u(x,t),\lambda)\,d\nu_{x,t}(\lambda) \operatorname{div} \mathbf{v}(x,t)\psi(x,t)\,dx\,dt \quad \text{as} \quad n \to +\infty.$$

Hence, passing to the limit in (32), we obtain (23).

**Step 2: Proof of assertion (25).** We prove here that $A$ defined in (24) is almost everywhere decreasing. Let $\tau = \min(T, \frac{a}{\omega})$, $0 < t_1 < t_2 < \tau$, $0 < \varepsilon < \min(t_1, \tau - t_2)$, and $\delta > 0$. Let $\psi \in C_c^1(\mathbb{R}_+, [0,1])$ with $\psi = 1$ on $[0,a]$, $\psi = 0$ on $[a+\delta, +\infty[$, and $\psi' \le 0$. Let $r_\varepsilon$ be defined by

$$r_\varepsilon(t) = \begin{cases} 0 & \text{if } 0 \le t \le t_1 - \varepsilon \\ \frac{t-(t_1-\varepsilon)}{\varepsilon} & \text{if } t_1 - \varepsilon \le t \le t_1 \\ 1 & \text{if } t_1 \le t \le t_2 \\ \frac{(t_2+\varepsilon)-t}{\varepsilon} & \text{if } t_2 \le t \le t_2 + \varepsilon \\ 0 & \text{if } t_2 + \epsilon \le t < +\infty. \end{cases} \tag{52}$$

Setting $\varphi(x,t) = \psi(|x| + \omega t) r_\varepsilon(t)$ in (23) (this is possible by taking regularizations of the functions $r_\varepsilon$) leads to

$$\begin{aligned} &\frac{1}{\varepsilon} \int_{t_1-\varepsilon}^{t_1} \int_{\mathbb{R}^2} \mu_{|\cdot - u(x,t)|}(x,t)\psi(|x| + \omega t)\,dx\,dt \\ &-\frac{1}{\varepsilon} \int_{t_2}^{t_2+\varepsilon} \int_{\mathbb{R}^2} \mu_{|\cdot - u(x,t)|}(x,t)\psi(|x| + \omega t)\,dx\,dt \ge E, \end{aligned} \tag{53}$$

with

$$E = -\int_0^T \int_{\mathbb{R}^2} \left[\omega \mu_{|\cdot - u(x,t)|}(x,t) + \mu_{F(\cdot,u(x,t))}(x,t) \frac{\mathbf{v}(x,t)\cdot x}{|x|}\right] \psi'(|x| + \omega t) r_\varepsilon(t)\,dx\,dt. \tag{54}$$

Note that since $\nu_{x,t}$ is defined in $[-b,b]$ and $u(x,t) \in [-b,b]$ for almost every $(x,t) \in \mathbb{R}^N \times [0,T]$, then

$$|\mu_{F(\cdot,u(x,t))}| \le \int_{\mathbb{R}} |f(\lambda) - f(u(x,t))|\,d\nu_{x,t}(\lambda) \le \int_{\mathbb{R}} M_f |\lambda - u(x,t)|\,d\nu_{x,t}. \tag{55}$$

Therefore,

$$|\mu_{F(\cdot,u(x,t))}(x,t) \frac{\mathbf{v}(x,t)\cdot x}{|x|}| \le M_f V \mu_{|\cdot - u(x,t)|}(x,t). \tag{56}$$

Hence since $M_f V = \omega$ and $\psi' \le 0$, the inequality $E \ge 0$ holds. Letting $\delta \to 0$ (noting that the application $(x,t) \mapsto \mu_{|\cdot - u(x,t)|}(x,t)$ belongs to $L^\infty(\mathbb{R}^N \times ]0,T[)$), we deduce from (53) that

$$\frac{1}{\varepsilon} \int_{t_1-\varepsilon}^{t_1} \int_{B_{a-\omega t}} \mu_{|\cdot - u(x,t)|}(x,t)\,dx\,dt - \frac{1}{\varepsilon} \int_{t_2}^{t_2+\varepsilon} \int_{B_{a-\omega t}} \mu_{|\cdot - u(x,t)|}(x,t)\,dx\,dt \ge 0, \tag{57}$$

that is,

$$\frac{1}{\varepsilon}\int_{t_1-\varepsilon}^{t_1} A(t)\,dt - \frac{1}{\varepsilon}\int_{t_2}^{t_2+\varepsilon} A(t)\,dt \geq 0. \tag{58}$$

Since $A \in L^1(]0,T[)$ (in fact, $0 \leq A(t) \leq (r+U)\lambda_N(B_{a-\omega t})$), we deduce that if $t_1$ and $t_2$ are Lebesgue points of $A$, then $0 < t_1 \leq t_2 < T \Rightarrow A(t_1) \geq A(t_2)$).

**Step 3: (conclusion of the proof).** First note that

$$\int_{B_a} \mu_{|\cdot-u(x,t)|}(x,t)\,dx\,dt \leq \int_{B_a} \mu_{|\cdot-u_0(x)|}(x,t)\,dx\,dt + \int_{B_a} |u(x,t)-u_0(x)|\,dx\,dt. \tag{59}$$

By (5)

$$\frac{1}{\tau}\int_0^\tau \int_{B_a} |u(x,t)-u_0(x)|\,dx\,dt \to 0 \quad \text{as} \quad \tau \to 0,$$

thus we obtain from (26) that

$$\lim_{\tau\to 0}\frac{1}{\tau}\int_0^\tau \int_{B_a} \mu_{|\cdot-u(x,t)|}(x,t)\,dx\,dt = 0, \tag{60}$$

and hence

$$\frac{1}{\tau}\int_0^\tau A(t)\,dt \to 0 \quad \text{as} \quad \tau \to 0.$$

Therefore since $A$ is decreasing almost everywhere on $]0,\tau[$, $A(t) = 0$ almost everywhere on $]0,\tau[$. Since $a$ is arbitrary, we conclude that $\mu_{|\cdot-u(x,t)|}(x,t) = 0$ almost everywhere in $\mathbb{R}^N\times]0,T[$, so that $\nu_{x,t} = \delta_{u(x,t)}$ for almost every $(x,t) \in \mathbb{R}^N\times]0,T[$.

**Remark 5.** In the case described in Remark 3, the decreasing property of $A(t)$ almost everywhere is no longer valid because of the term $g(x,t,u)$. However, the following property holds

$$0 < t_1 < t_2 < \tau \Longrightarrow A(t_2) \leq A(t_1) + K\int_{t_1}^{t_2} A(t)\,dt, \tag{61}$$

where $K$ depends on the Lipschitz constant of $g$ with respect to $u$. From this property and from (26), which remains valid, we deduce that $A(t) = 0$ almost everywhere. Therefore $\nu_{x,t} = \delta_{u(x,t)}$ for almost every $(x,t) \in \mathbb{R}^N\times]0,T[$. Hence the uniqueness result still holds.

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