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ON THE ASYMPTOTIC BEHAVIOR OF MINIMIZERS OF THE GINZBURG-LANDAU MODEL IN 2 DIMENSIONS

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Dedicated to the memory of Peter Hess

Abstract. Minimizers u_ϵ of the Ginzburg-Landau energy E_ϵ defined in (1) below on an arbitrary domain $\Omega \subset\subset \mathbf{R}^2$ with smooth boundary and boundary data $g: \partial\Omega \to S^1$ as $\epsilon \to 0$ are shown to subconverge weakly in $H^{1,p}$ for $p < 2$ and locally in $H^{1,2}$ away from finitely many points $x_1, \ldots, x_J$ to a smooth harmonic map $u: \Omega \setminus \{x_1, \ldots, x_J\} \to S^1$. The proof is based on simple comparison arguments. The result simplifies and extends previous work of Bethuel-Brezis-Hélein for the same problem on a star-shaped domain.

1. Introduction. Let Ω be a bounded domain in $\mathbf{R}^2$ with smooth boundary $\partial\Omega = \Gamma_1 \cup \cdots \cup \Gamma_K$, where $\Gamma_k \cong S^1$ for $1 \le k \le K$, and let $g = (g_1, \ldots, g_K)$ be smooth functions $g_k: \Gamma_k \to S^1 \subset \mathbf{C} \cong \mathbf{R}^2$, $1 \le k \le K$. Through the identification $\Gamma_k \cong S^1$ we may associate with each g_k a topological degree d_k.

Also let

$$H^1_g(\Omega) = \left\{u \in H^{1,2}(\Omega; \mathbf{R}^2) : u_{|\Gamma_k} = g_k,\ 1 \le k \le K\right\}.$$

It is well-known that $H^1_g(\Omega)$ is non-void. Moreover, for $\epsilon > 0$, $u \in H^1_g(\Omega)$ we define the Ginzburg-Landau energy

$$E_\epsilon(u) = E_\epsilon(u; \Omega) = \frac{1}{2}\int_\Omega \left\{|\nabla u|^2 + \frac{1}{2\epsilon^2}\left(1 - |u|^2\right)^2\right\} dx. \tag{1}$$

It is easy to see that for each $\epsilon > 0$ the infimum

$$\nu(\epsilon) = \inf_{u \in H^1_g} E_\epsilon(u)$$

is attained at a minimizer $u_\epsilon \in H^1_g$.

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Theorem 1.1. *Let $d = \sum_k d_k$ be the total degree of the boundary data g, and for $\epsilon_n \searrow 0$ let (u_n) be a sequence of minimizers for E_{ϵ_n}. Then there exists a subsequence $(u_{n'})$, a finite collection of points $x_1, \ldots, x_J \in \bar{\Omega}$, and a smooth harmonic map $u: \Omega \setminus \{x_1, \ldots, x_J\} \to S^1$ such that*

$$u_{n'} \rightharpoonup u \text{ in } H^{1,2}_{\mathrm{loc}}\left(\Omega \setminus \{x_1, \ldots, x_J\}; \mathbf{R}^2\right)$$

and weakly in $H^{1,p}$ for all $p < 2$.

Remark 1.2. If Ω is simply connected and star-shaped, this result — with $J = |d|$ — is due to Bethuel-Brezis-Hélein [1-4], using the a-priori estimate

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \int_\Omega \left(1 - |u|^2\right)^2$$

derived from the Pohožaev-identity in combination with the results of Brezis-Merle-Rivière [5].

For a sequence $\epsilon_n \searrow 0$ the latter a-priori estimate for minimizers was obtained by this author [7] by a different technique allowing us to extend the afore-mentioned result of [1-4] to domains of arbitrary connectivity for this sequence.

Theorem 1.1 extends this result to any sequence of minimizers. Moreover, we give a simple, direct proof.

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2. Preliminaries. By constructing suitable comparison functions one obtains the following bound for $\nu(\epsilon)$.

Lemma 2.1. *There is a constant $C_1 = C_1(\Omega, g)$ such that for $0 < \epsilon \le 1$ there holds*

$$\nu(\epsilon) \le \pi |d| \, |\ln \epsilon| + C_1. \tag{2}$$

Remark. For simply connected domains this result may be found in [1], Lemma 1.

Proof. (i) If all numbers $d_k = 0$, $k = 1, \ldots, K$, as in [1], one may use the harmonic map $u_0 = e^{i\varphi_0}: \Omega \to S^1$ as comparison function, obtained by solving the Dirichlet problem

$$\begin{aligned} -\Delta \varphi_0 &= 0 \quad &&\text{in } \Omega, \\ \varphi_0 &= \psi_k \quad &&\text{on } \Gamma_k. \end{aligned}$$

Here $\psi_k: \Gamma_k \to \mathbf{R}$ is a smooth map such that $g_k = e^{i\psi_k}$ on Γ_k, $k = 1, \ldots, K$. As in [1], formula (2), one obtains

$$\nu(\epsilon) \le E_\epsilon(u_0) = \frac{1}{2} \int_\Omega |\nabla u_0|^2 \, dx =: C_1.$$

(ii) If, say, $d_1 \neq 0$ we choose smooth curves $\gamma_k\colon [0, 1] \to \bar{\Omega}$ with disjoint images $\gamma_k(]0, 1[) \subset \Omega$ such that

$$\gamma_k(0) \in \Gamma_k, \quad \gamma_k(1) \in \Gamma_1$$

and

$$g_k\big(\gamma_k(0)\big) = g_1\big(\gamma_k(1)\big) =: c_k \in S^1,$$

for $k = 2, \ldots, K$. We then let $\tilde{\Omega} = \Omega \setminus \bigcup_{k=2}^{K} \gamma_k\big(]0, 1[\big)$ and extend g to a continuous function $\tilde{g}\colon \partial\tilde{\Omega} \to S^1$ by letting

$$\tilde{g}\,(\gamma(t)) = c_k, \quad k = 2, \ldots, K, \quad 0 \leq t \leq 1.$$

In this way we obtain a simply connected domain $\tilde{\Omega}$ with piecewise smooth boundary and piecewise smooth boundary data $\tilde{g}$ of degree d.

In particular, if $d = 0$ the desired estimate now follows from step (i).

(iii) If $d \neq 0$, we may in fact assume $d > 0$ or else perform a reflection. Upon deleting d balls $B_{\rho_0}(x_i)$, $i = 1, \ldots, d$, from Ω, where $x_i \in \Omega$ and

$$\rho_0 < \min_{i,j}\Big\{\frac{1}{2}\operatorname{dist}(x_i, \partial\Omega), \frac{1}{4}|x_i - x_j|\Big\},$$

and introducing further Dirichlet boundary conditions

$$g_i(x) = \frac{x - x_i}{|x - x_i|} \quad \text{on } \partial B_{\rho_0}(x_i),$$

we obtain a new domain $\tilde{\Omega} = \Omega \setminus \bigcup_{i=1}^{d} B_{\rho_0}(x_i)$ and boundary data $\tilde{g} \to S^1$ of total degree $\tilde{d} = 0$, whence by steps (i) and (ii) above

$$\tilde{\nu}(\epsilon) = \inf_{u \in H^1_{\tilde{g}}(\tilde{\Omega})} E_\epsilon(u; \tilde{\Omega}) \leq C_1,$$

and it remains to obtain the desired estimate on a ball.

In this case it is very easy to obtain a suitable comparison function. In fact, after scaling it suffices to consider the case $\Omega = B = B_1(0)$, $g(x) = x$.

With the co-rotational ansatz

$$u(re^{i\vartheta}) = e^{i\vartheta} f(r),$$

where

$$f_\epsilon(r) \cong \tanh\Big(\frac{r}{\sqrt{2}\epsilon}\Big),$$

we have

$$|\nabla u_\epsilon|^2 = \frac{1}{2}\frac{(1 - f_\epsilon^2)^2}{\epsilon^2} + \frac{f_\epsilon^2}{r^2}$$

and

$$E_\epsilon(u) = \pi \int_0^1 \frac{f^2}{r}\, dr + \frac{\pi}{\epsilon^2}\int_0^1 (1-f^2)^2 r\, dr.$$

Finally, after a change of variables $s = r/(\sqrt{2}\epsilon)$ we find

$$\frac{\pi}{\epsilon^2}\int_0^1 (1-f^2)^2 r\, dr \le 2\pi \int_0^\infty \left(1-\tanh^2(s)\right)^2 s\, ds < \infty,$$

$$\begin{aligned}\pi \int_0^1 \frac{f^2}{r}\, dr &= \pi \int_1^{\frac{1}{\sqrt{2}\epsilon}} \frac{\tanh^2(s)}{s}\, ds + \pi \int_0^1 \frac{\tanh^2(s)}{s}\, ds \\ &\le \pi \int_1^{\frac{1}{\sqrt{2}\epsilon}} \frac{ds}{s} + \pi \int_0^1 s\, ds \le \pi |\ln(\epsilon)| + \pi \frac{1-\ln 2}{2}.\end{aligned}$$

From [1-4] we also have:

Lemma 2.2. *Any critical point $u \in H^1_g(\Omega)$ of E_ϵ satisfies the estimates $|u| \le 1$ and $|\nabla u| \le C_2 \epsilon^{-1}$ with a uniform constant $C_2 = C_2(g, \Omega)$.*

Again we present the short proof for completeness.

Proof. A critical point u of E_ϵ is a smooth solution of the Euler equation

$$-\Delta u = \frac{1}{\epsilon^2} u\left(1-|u|^2\right) \quad \text{in} \quad \Omega. \tag{3}$$

Multiplying by u, we obtain

$$-\Delta\left(\frac{|u|^2}{2}\right) + |\nabla u|^2 = \frac{1}{\epsilon^2}|u|^2\left(1-|u|^2\right) \quad \text{in } \Omega. \tag{4}$$

Applying the maximum principle to (4) on the region $\Omega_+ = \{x;\, |u(x)| > 1\}$, we find that $\Omega_+ = \emptyset$, whence $|u| \le 1$, as claimed. Moreover, denoting $\tilde{u}(x) = u(\epsilon x)$, by (3) we have

$$-\Delta \tilde{u} = \tilde{u}\left(1-|\tilde{u}|^2\right) \quad \text{in} \quad \Omega_\epsilon = \Omega/\epsilon$$

with uniformly smooth boundary data

$$\tilde{u} = g(\epsilon x) \quad \text{on} \quad \partial\Omega_\epsilon.$$

Elliptic regularity hence guarantees that

$$\|\nabla \tilde{u}\|_{L^\infty} \le C$$

with $C = C(\Omega, g)$ for $\epsilon \le 1$. Scaling back, we obtain

$$\|\nabla u\|_{L^\infty} \le C\epsilon^{-1}, \tag{5}$$

as desired.

Our main lemma deals with the behavior of minimizers $u = u_\epsilon$ on balls $B_\rho(x_0)\cap\Omega$, where $x_0 \in \bar{\Omega}$ and $\rho > 0$.

For $0 < \rho$ let

$$f(\rho) = f(\rho, x_0, \epsilon, u) = \rho \int_{\partial B_\rho(x_0)\cap\Omega} \left\{ |\nabla u|^2 + \frac{1}{2\epsilon^2}\left(1 - |u|^2\right)^2 \right\} do,$$

with do denoting arc-length measure. The following is related to the "Courant-Lebesgue lemma"; see [8, p. 103].

Lemma 2.3. (i) *For* $0 < \epsilon \le e^{-1}$ *we have*

$$\inf_{\epsilon^{1/2}\le\rho\le\epsilon^{1/4}} f(\rho) \le 8\frac{E\left(u_\epsilon;\, \Omega \cap B_{\epsilon^{1/4}}(x_0)\right)}{|\ln \epsilon|} \le 8\pi|d| + 8C_1 =: C_3,$$

and

$$\inf_{5\epsilon^{1/4}\le\rho\le 5\epsilon^{1/8}} f(\rho) \le 2C_3.$$

(ii) *There are constants* γ, $\epsilon_0 = \epsilon_0(\Omega, g) > 0$ *such that for* $0 < \epsilon < \epsilon_0$ *there holds*

$$\inf_{B_\rho(x_0)} |u_\epsilon| \ge \frac{1}{2}$$

whenever $\epsilon^{\frac{1}{2}} \le \rho \le \epsilon^{\frac{1}{4}}$ *and* $f(\rho) \le \gamma$. *(We may assume* $\epsilon_0 < e^{-1}$*).*

Proof. (i) Compute, using Fubini's theorem

$$\begin{aligned} E_\epsilon(u_\epsilon) &\ge E_\epsilon\left(u_\epsilon;\, \Omega \cap B_{\epsilon^{1/4}}(x_0)\right) \ge \frac{1}{2}\int_{\epsilon^{1/2}}^{\epsilon^{1/4}} f(\rho)\,\frac{d\rho}{\rho} \\ &\ge \frac{1}{2}\ln\left(\frac{\epsilon^{1/4}}{\epsilon^{1/2}}\right) \inf_{\epsilon^{1/2}\le\rho\le\epsilon^{1/4}} f(\rho) = \frac{1}{8}|\ln\epsilon| \inf_{\epsilon^{1/2}\le\rho\le\epsilon^{1/4}} f(\rho), \end{aligned}$$

and using Lemma 2.1. Similarly we obtain the second inequality.

(ii) Choose $\epsilon_1 = \epsilon_1(\Omega) > 0$ such that for $0 < \rho < 5\epsilon_1^{\frac{1}{8}}$ the domain $D = \Omega\cap B_\rho(x_0)$ is strongly star-shaped, say, with respect to the origin (but not necessarily with respect to x_0) in the sense that

$$n \cdot x \ge \frac{1}{4}\rho \quad \text{for} \quad x \in \partial D,$$

where n denotes the outer unit normal. Also let τ denote a piecewise smooth unit tangent vector field along ∂D. The Pohožaev identity for (3), dropping ϵ, then yields

$$\begin{aligned} 0 &= \int_D \left(\Delta u + \frac{u}{\epsilon^2}(1 - |u|^2)\right)x \cdot \nabla u\, dx \\ &= \int_{\partial D} \left\{ \partial_n u(x \cdot \nabla u) - n \cdot x\left(\frac{|\nabla u|^2}{2} + \frac{\left(1 - |u|^2\right)^2}{4\epsilon^2}\right)\right\} do + \frac{1}{2\epsilon^2}\int_D \left(1 - |u|^2\right) dx. \end{aligned}$$

Since

$$\partial_n u(x\cdot\nabla u) \geq n\cdot x|\partial_n u|^2 - |\partial_n u|\rho|\partial_\tau u| \geq \frac{n\cdot x}{2}|\partial_n u|^2 - C\rho|\partial_\tau u|^2$$

we obtain

$$\begin{aligned}\frac{1}{\epsilon^2}\int_D \left(1-|u|^2\right)^2 &\leq C\rho\int_{\partial D}\left[\frac{|\partial_\tau u|^2}{2} + \frac{\left(1-|u|^2\right)}{4\epsilon^2}\right]do \\ &\leq Cf(\rho) + C\rho\int_{\partial\Omega\cap B_\rho(x_0)}|\nabla g|^2\,do \leq C_4\gamma + C_5(g)\rho^2.\end{aligned}$$

On the other hand, if $|u(x_1)| < \frac{1}{2}$ for some $x_1 \in D$, by Lemma 2.2 we have

$$|u(y)| \leq \frac{3}{4} \quad \text{for} \quad |x_1 - y| < \frac{\epsilon}{4C_2}$$

and hence for $\rho \geq C^{-1}\epsilon$, in particular, for $\rho \geq \epsilon^{\frac{1}{2}}$ there holds

$$\int_D \frac{\left(1-|u|^2\right)^2}{\epsilon^2}\,dx \geq C_6(\Omega, g) > 0. \tag{6}$$

Hence, if we choose $\epsilon_0 < \epsilon_1$, $\gamma = \epsilon_0^{\frac{1}{2}}$ such that $\left(C_4 + C_5(g)\right)\epsilon_0^{\frac{1}{2}} < C_6(\Omega, g)$ we obtain the assertion of the lemma.

3. Proof of the theorem. For $0 < \epsilon < \epsilon_0$ and minimizers u_ϵ of E_ϵ consider the set

$$S_\epsilon = \left\{x \in \Omega;\ |u_\epsilon(x)| < \tfrac{1}{2}\right\}$$

and its cover $\left(B_{\epsilon^{\frac{1}{4}}}(x)\right)_{x\in S_\epsilon}$. For $x \in S_\epsilon$, let $\epsilon^{\frac{1}{2}} < \rho(x) < \epsilon^{\frac{1}{4}}$ be determined as in Lemma 2.3 such that

$$\frac{8E_\epsilon\left(u_\epsilon;\ \Omega\cap B_{\epsilon^{\frac{1}{4}}}(x)\right)}{|\ln\epsilon|} \geq f(\rho(x), x, \epsilon, u_\epsilon) \geq \gamma. \tag{7}$$

By Vitali's covering lemma there is a (necessarily finite) collection of disjoint balls $B_i = B_{\epsilon^{\frac{1}{4}}}(x_i)$, $x_i \in S_\epsilon$, $1 \leq i \leq I = I(u_\epsilon)$, such that

$$\left(\Omega \cap \bigcup_{x\in S_\epsilon} B_{\epsilon^{\frac{1}{4}}}(x)\right) \subset \bigcup_i B_{5\epsilon^{\frac{1}{4}}}(x_i).$$

Moreover, by (7) and since the balls B_i are disjoint, we obtain the uniform bound

$$I \leq \sum_i \frac{8E_\epsilon(u_\epsilon, \Omega\cap B_i)}{\gamma|\ln\epsilon|} \leq \frac{8E_\epsilon(u_\epsilon)}{\gamma|\ln\epsilon|} \leq C_3\gamma^{-1} =: I_0 \tag{8}$$

on the number of "bad" balls B_i.

We now refine this initial choice further. For $1 \leq i \leq I$ let $\rho_i \in [5\epsilon^{\frac{1}{4}}, 5\epsilon^{\frac{1}{8}}]$ such that

$$f(\rho_i, x_i, \epsilon, u_\epsilon) < 2C_3$$

and let $D_i = \Omega \cap B_{\rho_i}(x_i)$ for brevity. From Pohožaev's identity as in the proof of Lemma 2.3 (ii) we infer:

Lemma 3.1. *There exists a constant* $C_7 = C_7(\Omega, g) > 0$ *such that*

$$\frac{1}{\epsilon^2}\int_{D_i}\left(1-|u_\epsilon|^2\right)^2 dx \le C_7$$

uniformly in $0 < \epsilon < \epsilon_0$, $1 \le i \le I$.

Combining Lemma 3.1 with Lemma 2.2 we conclude:

Lemma 3.2. *There exists a number* $J_0 = J_0(\Omega, g) \in \mathbf{N}$ *such that for any disjoint collection of balls* $B_{\frac{\epsilon}{5}}(x_j)$, $x_j \in \Omega$, $1 \le j \le J$, *with* $|u_\epsilon(x_j)| < \frac{1}{2}$ *there holds* $J \le J_0$.

Proof. By definition of S_ϵ we have $x_j \in S_\epsilon$ for each j and any collection of balls as in the statement above. Hence

$$\Big(\Omega \cap \bigcup_j B_{\frac{\epsilon}{5}}(x_j)\Big) \subset \bigcup_i D_i$$

and by (6) above

$$JC_6(\Omega, g) \le \sum_j \int_{\Omega\cap B_{\frac{\epsilon}{5}}(x_j)} \frac{\left(1-|u_\epsilon|^2\right)^2}{\epsilon^2}\,dx \le \sum_i \int_{D_i} \frac{\left(1-|u_\epsilon|^2\right)^2}{\epsilon^2}\,dx \le I_0 C_7.$$

Now consider the cover $\left(B_{\frac{\epsilon}{5}}(x)\right)_{x\in S_\epsilon}$ of S_ϵ. Again, by Vitali's covering lemma we can find a disjoint collection of balls $B_{\frac{\epsilon}{5}}(x_j)$, $x_j \in S_\epsilon$, $1 \le j \le J$, such that

$$S_\epsilon \subset \bigcup_j B_\epsilon(x_j).$$

Moreover, by Lemma 3.2 we have $J \le J_0$ independent of ϵ. For each $\epsilon > 0$ and any corresponding minimizer u_ϵ we now fix this choice of (x_j). Given $\sigma > 0$ then we denote

$$\Omega^\sigma = \Omega^\sigma_\epsilon = \Omega \setminus \bigcup_j B_\sigma(x_j).$$

Proposition 3.3. *There exists a constant* $C_8 = C_8(\Omega, g) > 0$ *such that for any* $\sigma > 0$ *there holds*

$$E_\epsilon(u_\epsilon; \Omega^\sigma_\epsilon) \le \pi|d|\,|\ln\sigma| + C_8,$$

uniformly in $0 < \epsilon < \epsilon_0$.

Observe that the Theorem easily follows from Proposition 3.3. Indeed, consider any sequence of minimizers $u_k = u_{\epsilon_k}$, where $\epsilon_k \to 0$ as $k \to \infty$. Let $(x_{j,k})$, $1 \le j \le J_k$ denote the corresponding centers of "bad" balls. Note that $J_k \le J_0$ uniformly. Passing to a sub-sequence, if necessary, we may assume that $J_k = J$ is independent of k and $x_{j,k} \to x_j$ as $k \to \infty$ for each $j = 1, \ldots, J$.

For $\sigma > 0$ let

$$\Omega_0^\sigma = \Omega \setminus \bigcup_j B_\sigma(x_j)$$

and let $\Omega_k^\sigma = \Omega_{\epsilon_k}^\sigma$. For any $\sigma > 0$ and $k \geq k_0(\sigma)$ by Proposition 3.3 we then have

$$\frac{1}{2}\int_{\Omega_0^\sigma} |\nabla u_k|^2\, dx \leq E_{\epsilon_k}\left(u_k; \Omega_0^\sigma\right) \leq E_{\epsilon_k}\left(u_k; \Omega_k^{\frac{\sigma}{2}}\right) \leq \pi|d||\ln\sigma| + C.$$

Choosing $\sigma = \sigma_k \to 0$ and passing to a further sub-sequence, if necessary, hence we see that $u_k \rightharpoonup u$ weakly locally in $H^{1,2}_{\mathrm{loc}}(\Omega \setminus \{x_1, \ldots, x_J\}; \mathbf{R}^2)$.

Moreover, $|u| = 1$ on $\Omega \setminus \{x_1, \ldots, x_J\}$. Taking the exterior product of (3) with u_k and passing to the limit as $k \to \infty$ we also obtain

$$-\operatorname{div}(\nabla u \wedge u) = -\lim_{k\to\infty} \operatorname{div}(\nabla u_k \wedge u_k) = 0.$$

This, together with the fact that $|u| = 1$ implies that u is (weakly) harmonic. (The same argument in a different context has been used by Y.M. Chen, Keller-Rubinstein-Sternberg, and Shatah, respectively.) By the regularity result of Hélein [6], u is smooth away from $\{x_1, \ldots, x_J\}$ and hence in fact a classical harmonic map from $\Omega \setminus \{x_1, \ldots, x_J\} \to S^1$.

Finally, for any $p < 2$ and some fixed $\sigma > 0$ we have

$$\begin{aligned}
\int_\Omega |\nabla u_k|^p\, dx &\leq \int_{\Omega_k^\sigma} |\nabla u_k|^p\, dx + \sum_{l=1}^{\infty} \int_{\Omega_k^{2^{-l}\sigma} \setminus \Omega_k^{2^{-l+1}\sigma}} |\nabla u_k|^p\, dx \\
&\leq C(\sigma, \Omega, g) + \sum_l \left(\mu\left(\Omega_k^{2^{-l}\sigma} \setminus \Omega_k^{(2^{-l+1}\sigma)}\right)\right)^{1-\frac{p}{2}} \cdot \left(\int_{\Omega_k^{2^{-l}\sigma}} |\nabla u_k|^2\, dx\right)^{\frac{p}{2}} \\
&\leq C(\sigma, \Omega, g) + C\sum_l 2^{-(1-\frac{p}{2})l}\left(2\pi|d|\left|\ln\left(2^{-l}\sigma\right)\right| + C_8\right)^{\frac{p}{2}} \\
&\leq C(\sigma, \Omega, g)
\end{aligned}$$

and therefore (u_k) remains bounded in $H^{1,p}(\Omega)$ as $\epsilon_k \to 0$. □

The proof of Proposition 3.3 requires the following variant of a result of Brezis-Merle-Rivière [5], Theorem 3.

Choose $R_1 = R_1(\Omega) > 0$ so that for $R < R_1$, $x_0 \in \Omega$ the domain $\Omega \cap B_R(x_0)$ is simply connected.

Proposition 3.4. *Let $\epsilon \leq R_0 < R \leq R_1$. Suppose $\hat{u} \in H^1_g$ satisfies $|\hat{u}| \leq 1$ in Ω, $|\hat{u}(x)| \geq \frac{1}{2}$ in $A_{R,R_0}(x_0) = \Omega \cap B_R \setminus B_{R_0}(x_0)$, and*

$$\frac{1}{\epsilon^2}\int_{\Omega\cap B_R(x_0)} \left(1 - |\hat{u}|^2\right)^2 dx \leq K.$$

Then

$$\int_{A_{R,R_0}} |\nabla \hat{u}|^2\,dx \geq 2\pi \hat{d}^2 \ln\left(\frac{R}{R_0}\right) - C\hat{d}^2 - C,$$

where $C = C(\Omega, g, K)$ *and where* $\hat{d}$ *is the topological degree of* $\hat{u}$, *restricted to* $\partial\big(\Omega \cap B_R(x_0)\big) \cong S^1$.

For completeness the proof of Proposition 3.4 is given in the Appendix.

Proof of Proposition 3.3. Fix any point x_j, $j \in \{1, \dots, J\}$. We may suppose $x_j = 0$. For any $R < R_1$ such that $|u| \geq \frac{1}{2}$ on $\partial\big(\Omega \cap B_R(0)\big)$ denote $d_{j,R}$ the topological degree of the map

$$\frac{u}{|u|} : \partial\big(\Omega \cap B_R(0)\big) \cong S^1 \to S^1.$$

Let $\mathcal{R}_\epsilon^\sigma$ denote the set of all numbers $R \in [\epsilon, \sigma]$ such that $\partial B_R(x_j) \cap B_\epsilon(x_{j'}) = \emptyset$ for all $j \neq j'$ and such that for some collection $J_R \subset \{1, \dots, J\}$, satisfying $J_R \subset J_{R'}$, if $R' \leq R$, the family $\big(B_R(x_j)\big)_{j \in J_R}$ is disjoint and

$$\bigcup_{j \in J} B_\epsilon(x_j) \subset \bigcup_{j \in J_R} B_R(x_j).$$

Note that $\mathcal{R}_\epsilon^\sigma$ is the union of closed intervals $[R_0^{(l)}, R^{(l)}]$, $1 \leq l \leq L$, whose right endpoints correspond to numbers R such that

$$\partial B_R(x_j) \cap \overline{B_R(x_{j'})} \neq \emptyset$$

for some pair $j \neq j' \in J_R$ and whose left endpoints correspond to numbers R_0 such that

$$\overline{B_\epsilon(x_{j'})} \setminus \bigcup_{j \in J_{R_0}} B_{R_0}(x_j) \neq \emptyset$$

for some $j' \notin J_{R_0}$. $J_R = J^{(l)}$ is constant for $R \in [R_0^{(l)}, R^{(l)}]$ and $J^{(l+1)} \subset J^{(l)}$, $J^{(l+1)} \neq J^{(l)}$. Thus $L \leq J \leq J_0 = L_0(\Omega, g)$, independently of ϵ.

Moreover, there exists a constant $M = M(\Omega, g) > 0$ such that

$$R_0^{(1)} \leq M\epsilon, \quad R^{(L)} \geq \frac{\sigma}{M} \quad \text{and} \quad R_0^{(l+1)} \leq M R^{(l)}$$

for all $l = 1, \dots, L-1$. Finally, observe that for all $R \in \mathcal{R}_\epsilon^\sigma$ and $j \in J_R$ the degree $d_{j,R}$ is defined and

$$|d| = \Big|\sum_{j \in J_R} d_{j,R}\Big| \leq \sum_{j \in J_R} |d_{j,R}|^2. \tag{9}$$

Applying Proposition 3.4 and (9), thus we have

$$\begin{aligned}\int_{\Omega_\epsilon^\sigma} |\nabla u_\epsilon|^2\,dx &\le 2E_\epsilon(u_\epsilon) - \sum_{l=1}^{L}\sum_{j\in J^{(l)}}\int_{A_{R^{(l)},R_0^{(l)}}(x_j)} |\nabla u_\epsilon|^2\,dx \\ &\le 2\pi|d||\ln\epsilon| - \sum_l\sum_j 2\pi|d_{j,R^{(l)}}|^2\Big[\ln(R^{(l)}/R_0^{(l)}) - C\Big] + C \\ &\le 2\pi|d|\Big[|\ln\epsilon| - \sum_l\big(\ln R^{(l)} - \ln R_0^{(l)}\big)\Big] + C \le 2\pi|d||\ln\sigma| + C,\end{aligned}$$

as claimed.

4. Concluding remarks. One can combine Theorem 1.1 with the results of [4] to show that the number of singularities $\{x_1, \dots, x_J\}$ of the limiting harmonic map u is exactly d, each of degree $+1$ if $d > 0$, and to obtain further information about the asymptotic behavior of (u_ϵ).

Appendix A. Here we present the proof of Proposition 3.4. In order to reduce the result to Theorem 3 in [5] let U_0 be a tubular neighborhood of $\partial\Omega$ of width $\rho_0 > 0$ such that any point $x \in U_0$ has a unique nearest neighbor $\pi_0(x) \in \partial\Omega$ and such that $\pi_0\colon U_0 \to \partial\Omega$ is smooth. By means of π_0 we extend the data g to a smooth function $\bar g = g\circ\pi_o\colon U_0 \to S^1$. We may assume $R_1 < \rho_0$. Given $\epsilon \le R_0 < R \le R_1$, $\hat u \in H_g^1(A_{R;R_0})$ we extend $\hat u$ to the full annulus

$$\bar A_{R,R_0} = B_R \setminus B_{R_0}(0)$$

by letting

$$\bar u(x) = \begin{cases} \hat u(x), & x \in A_{R,R_0} \\ \bar g(x), & x \in \bar A_{R,R_0}\setminus A_{R,R_0} \subset U_0. \end{cases}$$

Note that

$$\int_{A_{R,R_0}} |\nabla\hat u|^2\,dx \ge \int_{\bar A_{R,R_0}} |\nabla\bar u|^2\,dx - C(\Omega, g).$$

and the degree of $\bar u_{|\partial B_R(0)}$ agrees with the degree $\hat d$ of $\hat u|_{\partial(\Omega\cap B_R(0))}$.

To estimate the Dirichlet integral of $\bar u$ we quote the proof of Theorem 3 in [5] for completeness. Introduce coordinates $x = re^{i\vartheta}$ on $\bar A_{R,R_0}$ and write

$$\bar u(x) = \rho(x)e^{i\varphi(x)},$$

where $\rho = |\bar u|$ and $\varphi = \hat d\vartheta + \psi$, with ψ smooth and single-valued. Note that

$$\nabla\varphi = \frac{\hat d}{r}V + \nabla\psi,$$

where

$$V(re^{i\vartheta}) = ie^{i\vartheta}.$$

Thus

$$|\nabla\varphi|^2 = \frac{\hat{d}^2}{r^2} + \frac{2\hat{d}}{r^2}\frac{\partial\psi}{\partial\vartheta} + |\nabla\psi|^2.$$

Hence

$$\int_{\bar{A}_{R,R_0}} |\nabla\bar{u}|^2\,dx \geq \int_{\bar{A}_{R,R_0}} \rho^2|\nabla\varphi|^2\,dx$$
$$= \int_{\bar{A}_{R,R_0}} \frac{\rho^2\hat{d}^2}{r^2}\,dx + 2\int_{\bar{A}_{R,R_0}} \frac{\rho^2\hat{d}}{r^2}\frac{\partial\psi}{\partial\vartheta}\,dx + \int_{\bar{A}_{R,R_0}} \rho^2|\nabla\psi|^2\,dx = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.$$

Treating each integral separately, we have

$$\mathrm{I}_1 = \int_{\bar{A}_{R,R_0}} \frac{\hat{d}^2}{r^2}\,dx - \int_{\bar{A}_{R,R_0}} (1-\rho^2)\frac{\hat{d}^2}{r^2}\,dx = 2\pi\hat{d}^2\ln\left(\frac{R}{R_0}\right) - \int_{\bar{A}_{R,R_0}} (1-\rho^2)\frac{\hat{d}^2}{r^2}\,dx.$$

Moreover, by Cauchy-Schwarz

$$\int_{\bar{A}_{R,R_0}} (1-\rho^2)\frac{\hat{d}^2}{r^2}\,dx \leq \hat{d}^2(\pi K)^{\frac{1}{2}}.$$

Next, since

$$\int_{\partial B_r(0)} \frac{\partial\psi}{\partial\vartheta}\,d\vartheta = 0, \quad \text{for all} \quad r\in[R_0, R],$$

we have

$$|\mathrm{I}_2| = 2\left|\int_{\bar{A}_{R,R_0}} (\rho^2-1)\frac{\hat{d}}{r^2}\frac{\partial\psi}{\partial\vartheta}\,dx\right| \leq \frac{2\hat{d}}{R_0}\int_{\bar{A}_{R,R_0}} |1-\rho^2||\nabla\psi|\,dx$$
$$\leq 2\hat{d}K^{\frac{1}{2}}\left(\int_{\bar{A}_{R,R_0}} |\nabla\psi|^2\,dx\right)^{\frac{1}{2}} \leq 4\hat{d}^2K + \frac{1}{4}\int_{\bar{A}_{R,R_0}} |\nabla\psi|^2\,dx.$$

Since, finally,

$$\mathrm{I}_3 = \int_{\bar{A}_{R,R_0}} \rho^2|\nabla\psi|^2\,dx \geq \frac{1}{4}\int_{\bar{A}_{R,R_0}} |\nabla\psi|^2\,dx,$$

we obtain the desired conclusion

$$\int_{A_{R,R_0}} |\nabla\hat{u}|^2\,dx \geq 2\pi\hat{d}^2\ln\left(\frac{R}{R_0}\right) - \hat{d}^2\left[(\pi K)^{\frac{1}{2}} + 4K\right] - C(\Omega, g).$$

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