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ENERGY DECAY FOR THE WAVE EQUATION WITH A NONLINEAR WEAK DISSIPATION

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Abstract. We derive a precise decay estimate of the energy of the solutions to the initial boundary value problem for the wave equation with a nonlinear dissipation $\rho(u_t)$, where $\rho(v)$ is a function like $v/\sqrt{1+v^2}$. Since our dissipation is weak as $|u_t|$ tends to ∞ we treat strong solutions rather than usual energy finite solutions.

1. Introduction. In this paper we are concerned with the decay property of the solutions to the initial-boundary value problem for the wave equation with a dissipative term,

$$u_{tt} - \Delta u + \rho(u_t) = 0 \quad \text{in} \quad \Omega \times [0, \infty), \tag{1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{and} \quad u\,|_{\partial\Omega} = 0, \tag{2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with (C^2-class) boundary $\partial\Omega$ and $\rho(v)$ is an increasing function in v.

When $\rho(v)$ is a function like $|v|^r v$, $-1 < r \le 8/(N-4)^+$, the decay property of the solutions of (1)–(2) was investigated in detail in our previous papers [6, 10, 11, 12]. For related works see Yamada [17], Harau and Zuazua [2], Zuazua [18], Komornik [3] and Nakao [7–9], where our results for $0 \le r \le 4/(N-2)$ are generalized in various directions. Quite recently, Komornik [4] has extended our result for $4/(N-2) < r \le 8/(N-4)^+$ (see [11]) to a wider class of ρ. See also [14, 15] for further related works.

If $r = -1$ in this example, i.e., $\rho(v) = \mathrm{sgn}(v)$, the problem is a critical case and no result on the decay rate of the solutions is known. (See Haraux [1], where existence of periodic solutions and some asymptotics are considered for such an equation.) Motivated by this critical case we are interested here in the decay property of the solutions of the problem (1)–(2) with $\rho(v)$ such that

$$-\infty < \lim_{v\to-\infty} \rho(v) < \lim_{v\to\infty} \rho(v) < \infty. \tag{3}$$

If $\rho(v)$ satisfies at most (3) the dissipative effect by $\rho(u_t)$ is weak as $|u_t|$ is large and for convenience we call such a term weak dissipation. Let us consider the most typical example $\rho(v) = v/\sqrt{1+v^2}$, which satisfies $\lim_{v\to\pm\infty} \rho(v) = \pm 1$. For this case, once the boundedness of $\|u_t(t)\|_\infty$ was established for a solution $u(t)$ we find $\rho(u_t)u_t \ge const.|u_t|^2$ and hence it is easy to see that the energy $E(t) = \frac{1}{2}\{\|u_t(t)\|^2 + \|\nabla u(t)\|^2\}$ of the solution $u(t)$ decays exponentially as $t \to \infty$. However, such a

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boundedness is known only for the one space dimensional equations. Thus, our problem is delicate even for such a simple example when $N \geq 2$. As far as the smooth and small amplitude solutions are concerned it is easy to show the global existence and exponential decay of them (cf. [13]). But, of course, our interest lies in the solutions with large data.

The object of this paper is to derive a precise decay rate of the energy $E(t)$ of the solutions $u(t)$ of (1)–(2) when $\rho(u_t)$ is a weak dissipation. For generality, however, we consider in fact a class of $\rho(v)$ including the functions $\rho(v) = |v|^r v/(1+v^2)^{\alpha/2}$ with $-1 < r < \alpha + 8/(N-4)^+$ and $0 \leq \alpha \leq r+1$. Thus, the main results concerning the decay property of our previous papers [6, 10, 11, 12] are reproduced here as special cases. Our result can be applied even for the case such as $\rho(v) = k_0 \operatorname{sgn} v$ for $|v| \geq \epsilon_0 > 0$. Since we assume in fact that (u_0, u_1) belongs to $H_2 \cap H_1^0 \times H_1^0$ the solutions we treat are so-called strong or H_2 solutions except for the case $0 \leq \alpha \leq r \leq \alpha + 4/(N-2)$.

2. Statement of result. The function spaces we use are all familiar and the definitions of them are omitted. We denote by $\|\cdot\|_p$ the L^p norm in Ω and set $\|\cdot\| = \|\cdot\|_2$ for convenience. We make the following assumption on $\rho(v)$.

Hypothesis 1. ρ is a continuous function on $\mathbb{R}$ and satisfies the conditions

$$(i) \qquad k_1(|v|^{p+2} + |v|^{q+2}) \geq \rho(v)v \geq k_0|v|^{r+2}/(1+|v|)^\alpha$$

and

$$(ii) \quad \rho \text{ is nondecreasing on } \mathbb{R}, \quad \text{i.e.,} \quad \{\rho(v_2) - \rho(v_1)\}(v_2 - v_1) \geq 0 \text{ for } v_1, v_2 \in \mathbb{R},$$

where k_0, k_1 are positive constants, and α, p, q and r are constants such that

$$-1 < p \leq r < \infty, \quad p \leq 0 \leq q \leq 8/(N-4)^+ \quad \text{and} \quad 0 \leq \alpha \leq r+1. \tag{4}$$

(We use the notation $a^+ = \max\{a, 0\}$ for $a \in \mathbb{R}$.) Note that it follows easily from the first inequality in Hypothesis 1(i) that

$$|\rho(v)| \leq C\left\{(\rho(v)v)^{(p+1)/(p+2)} + (\rho(v)v)^{(q+1)/(q+2)}\right\}.$$

Our result reads as follows.

Theorem 1. *Assume that* $(u_0, u_1) \in H_2 \cap H_1^0 \times H_1^0$. *Then, under Hypothesis* 1 *the problem* (1)–(2) *admits a unique solution* $u(t)$ *such that*

$$u \in W^{1,\infty}([0,\infty); H_1^0) \bigcap L^\infty([(0,\infty); H_2 \cap H_1^0),$$

$$\rho(u_t)u_t \in L^1_{loc}([0,\infty); L^1),$$

and it satisfies the decay property

$$E(t) \leq C(1+t)^{-2\eta} \tag{5}$$

with

$$\eta = \min\Big\{\frac{1}{r^+}, \frac{2}{(N-2)^+(\alpha - r)^+}, \frac{2(q+1)}{((N-4)q-4)^+}, \frac{p+1}{(-p)^+}\Big\}, \tag{6}$$

where C denotes constants which may depend on $\|u_0\|_{H_2}+\|u_1\|_{H_1}$ and other known constants.

Remark 1. If we assume in addition $\rho(u_1)\in L^2$, then $u(t)$ belongs also to $W^{2,\infty}([0,\infty);L^2)$.

Remark 2. When $0\le\alpha\le r$, $p=0$ and $0\le q\le 4/(N-2)$ the constant C depends only on $\|u_0\|_{H_1}+\|u_1\|$ and the estimate holds for finite energy solutions, which is a generalization of the result in [6]. (See also [2, 3, 18] for closely related results.)

Remark 3. When $0\le\alpha\le r+1$ the results for $-1<p\le 0$ and $4/(N-2)\le q\le 8/(N-4)^+$ generalize the main results in [11] and [12], respectively. See Komornik [5], where very similar result is given for the case that $0\le\alpha\le r$, $p=0$, and $4/(N-2)<q\le 8/(N-4)^+$, which is also a generalization of our result [11].

Remark 4. When $r=0$ and $N=2$ the right-hand side of (5) should be replaced by $C(m)(1+t)^{-m}$ for any $m>0$, and when $r=0$ and $N=1$ the right-hand side of (5) can be replaced by $C\exp(-\lambda t)$ for some $\lambda>0$. This is also true when $r=\alpha=0$ and $(N-4)q\le 4$.

For illustration we give a result for a typical case as a corollary.

Corollary 1. *Let $\rho(v)=|v|^r v/(1+v^2)^{\alpha/2}$ and assume that*

$$-1<r\le\alpha+8/(N-4)^+\quad\textit{and}\quad 0\le\alpha\le r+1.$$

Then, the solution $u(t)$ in Theorem 1 satisfies the decay property

$$E(t)\le C(1+t)^{-2\eta}$$

with η such that

$$\eta=\min\Big\{\tfrac{2}{(N-2)(\alpha-r)^+},\ \tfrac{1}{r},\ \tfrac{2((r-\alpha)^++1)}{((N-4)(r-\alpha)^+-4)^+}\Big\}\quad\textit{if}\ \ r\ge 0$$

and

$$\eta=\min\Big\{\tfrac{2}{(N-2)(\alpha-r)^+},\ \tfrac{1+r}{-r}\Big\}\quad\textit{if}\quad -1<r<0.$$

(*When $r=0$ and $N=1$, $E(t)$ decays exponentially.*) *In particular, if $\alpha=r+1$ we have*

$$\eta=\begin{cases}\min\{\frac{2}{N-2},\ \frac{1}{r}\} & \textit{if}\ \ r\ge 0,\\ \min\{\frac{2}{N-2},\ \frac{1+r}{-r}\} & \textit{if}\ \ -1<r<0.\end{cases}$$

Proof. It suffices to take $p=0$ and $q=(r-\alpha)^+$ if $r\ge 0$, and to take $p=r$ and $q=0$ if $r<0$. $\square$

For the proof of Theorem 1 we use the following two lemmas.

Lemma 1 (Gagliardo-Nirenberg)**.** *Let $1\le r<p\le\infty$, $1\le q\le p$ and $m\ge 0$. Then, the inequality*

$$\|v\|_p\le C\|D^m v\|_q^{\theta}\|v\|_r^{1-\theta}\quad\textit{for}\quad v\in W^{m,q}\cap L^r \tag{7}$$

holds with some $C>0$ and

$$\theta=\Big(\frac{1}{r}-\frac{1}{p}\Big)\Big(\frac{m}{N}+\frac{1}{r}-\frac{1}{q}\Big)^{-1} \tag{8}$$

provided that $0<\theta\le 1$ ($0<\theta<1$ if $p=\infty$ and $mq=N$).

Lemma 2. *Let $\phi(t)$ be a nonnegative function on $\mathbb{R}^+ \equiv [0,\infty)$, satisfying*

$$\sup_{t\le s\le t+1} \phi(s)^{1+\gamma} \le k_0(1+t)^\beta\{\phi(t)-\phi(t+1)\} \tag{9}$$

for some $k_0>0$, $\gamma>0$, $\beta<1$. Then, $\phi(t)$ has the decay property

$$\phi(t) \le C_0(1+t)^{-(1-\beta)/\gamma}, \tag{10}$$

where C_0 is a constant depending on $\phi(0)$. (*When $\gamma=0$ the right hand side of* (10) *should be replaced by $C_0\exp(-\lambda t^{1-\beta})$, $\lambda>0$.*)

For a proof of Lemma 2 see Nakao [6, 8].

3. Proof of Theorem 1. The existence and uniqueness part of our theorem is proved by standard monotonicity and compactness arguments based on usual a priori estimates (cf. J.L. Lions [5], W. Strauss [16]), and it suffices for the proof of Theorem 1 to derive the decay estimates (5) and (6). First, we note that

$$\|u_t\|^2+\|\nabla u(t)\|^2 \le E(0)<\infty, \tag{11}$$

$$\|\nabla u_t\|^2+\|\Delta u(t)\|^2 \le \|\nabla u_1\|^2+\|\Delta u_0\|^2<\infty \tag{12}$$

and

$$\|u_{tt}(t)\|^2+\|\nabla u_t(t)\|^2 \le \|u_{tt}(0)\|^2+\|\nabla u_1\|^2 \le C(\|u_0\|_{H_2},\|\rho(u_1)\|_{L^2},\|\nabla u_1\|). \tag{13}$$

The second estimate follows, formally, by multiplying the equation by $-\Delta u_t$ and integrating since

$$\int_\Omega \rho(u_t)(-\Delta u_t)dx = \int_\Omega \rho'(u_t)\|\nabla u_t\|^2dx \ge 0.$$

The third one follows, formally, by differentiating the equation in t, multiplying by u_{tt} and integrating. When $\rho(v)$ is not differentiable we have only to approximate it by appropriate smooth increasing functions $\rho_\epsilon(v)$. (It is easy to see the approximate solutions u_ϵ converge to the desired solution $u(t)$ as $\epsilon\to 0$.) The a priori estimates (11) and (12) are sufficient for our existence and uniqueness result. If we assume in addition $\rho(u_1)\in L^2$ then we can use also (13) to get further $u\in W^{2,\infty}([0,\infty);L^2(\Omega))$. For the argument below the third one is not needed.

Now, multiplying equation by u_t and integrating over $\Omega\times[t,t+1]$, $t>0$, we have

$$k_0\int_t^{t+1}\int_\Omega \frac{|u_t|^{r+2}}{(1+|u_t|)^\alpha}dxds \le \int_t^{t+1}\int_\Omega \rho(u_t)u_tdxds = E(t)-E(t+1)\equiv D(t)^{r+2}, \tag{14}$$

where we have used the main assumption Hypothesis (i).

To derive an estimate for $\int_t^{t+1}\|u_t(s)\|^2ds$ from (14) we see, for $0\le\epsilon\le 1$,

$$\int_t^{t+1}\|u_t(s)\|_{1+\epsilon}^{1+\epsilon}ds = \int_t^{t+1}\int_\Omega\Big\{\frac{|u_t|^{r+2}}{(1+|u_t|)^\alpha}\Big\}^{(1+\epsilon)/(r+2)}(1+|u_t|)^{\alpha(1+\epsilon)/(r+2)}dxds$$
$$\le \Big\{\int_t^{t+1}\int_\Omega\frac{|u_t|^{r+2}}{(1+|u_t|)^\alpha}dxds\Big\}^{\frac{1+\epsilon}{r+2}}\Big\{\int_t^{t+1}\int_\Omega(1+|u_t|)^{(1+\epsilon)\alpha/(r+1-\epsilon)}dxds\Big\}^{\frac{r+1-\epsilon}{r+2}}$$
$$\le CD(t)^{1+\epsilon}\Big\{1+\int_t^{t+1}\|u_t\|_{(1+\epsilon)\alpha/(r+1-\epsilon)}^{(1+\epsilon)\alpha/(r+1-\epsilon)}ds\Big\}^{\frac{r+1-\epsilon}{r+2}} \le CD(t)^{1+\epsilon}, \tag{15}$$

where we assumed $(1+\epsilon)\alpha/(r+1-\epsilon) \le 2N/(N-2)$ at the last stage and used (12) and (14). Let us take ϵ as follows:

$$\epsilon = \min\{\frac{2N(r+l)-\alpha(N-2)}{2N+\alpha(N-2)}, 1\}. \tag{16}$$

(Note that $(1+\epsilon)\alpha/(r+1-\epsilon) = 2N/(N-2)$ if $N > 2$. Also note that if $\alpha \le r$ we can take $\epsilon = 1$ without use of (12) to get (15).)

When $0 \le \epsilon < 1$, i.e., $Nr < \alpha(N-2)$ we have, by (12),

$$\int_t^{t+1} \|u_t(s)\|^2 ds \le C\int_t^{t+1} \|u_t(s)\|_{1+\epsilon}^{2(1-\theta)} \|\nabla u_t(s)\|^{2\theta} ds \le C\int_t^{t+1} \|u_t(s)\|_{1+\epsilon}^{2(1-\theta)} ds \tag{17}$$

with $\theta = N(1-\epsilon)/(N+2-(N-2)\epsilon)$. Here, we note that since $0 \le \epsilon < 1$,

$$2(1-\theta) = \frac{4(1+\epsilon)}{N+2-(N-2)\epsilon} < (1+\epsilon) \tag{18}$$

and hence, from (17) and (15),

$$\int_t^{t+1} \|u_t(s)\|^2 ds \le C\Big(\int_t^{t+1} \|u_t(s)\|_{1+\epsilon}^{1+\epsilon} ds\Big)^{2(1-\theta)/(1+\epsilon)} \le CD(t)^{2(1-\theta)}. \tag{19}$$

When $r \ge \alpha(N-2)/N$ we see $2\alpha/r \le 2N/(N-2)$ and we take $\epsilon = 1$ in (5) to get

$$\int_t^{t+1} \|u_t(s)\|^2 ds \le CD(t)^2. \tag{20}$$

We set for convenience

$$\theta^* = \begin{cases} \theta & \text{if } r < \alpha(N-2)/N, \\ 0 & \text{if } r \ge \alpha(N-2)/N. \end{cases}$$

Then, from (19) or (20) there exist $t_1 \in [t, t+\frac{1}{4}]$, $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|u_t(t_i)\| \le CD(t)^{1-\theta^*}. \tag{21}$$

Thus, multiplying the equation by u and integrating we have

$$\begin{aligned} &\int_{t_1}^{t_2} \|\nabla u(s)\|^2 ds \qquad (22) \\ &= -(u_t(t_2), u(t_2) + (u_t(t_1), u(t_1)) - \int_{t_1}^{t_2}\int_\Omega \rho(u_t) u dx ds + \int_{t_1}^{t_2} \|u_t(s)\|^2 ds \\ &\le CD(t)^{(1-\theta^*)} \sup_{t\le s\le t+1} \|u(s)\| + CD(t)^{2(1-\theta^*)} + \int_{t_1}^{t_2}\int_\Omega |\rho(u_t)u| dx ds. \end{aligned}$$

To treat the last term of (22) set for convenience $r_1 = p$ and $r_2 = q$. Then, we see by Hypothesis (i), (12) and (14) that

$$\int_{t_1}^{t_2}\int_\Omega |\rho(u_t)u|dxds \le C\sum_{i=1}^2 \int_{t_1}^{t_2}\int_\Omega (\rho(u_t)u_t)^{(r_i+1)/(r_i+2)}|u|dxds \tag{23}$$
$$\le C\sum_{i=1}^2 \Big(\int_{t1}^{t_2}\int_\Omega \rho(u_t)u_t dxds\Big)^{(r_i+1)/(r_i+2)}\Big(\int_{t_1}^{t_2}\|u(s)\|_{r_i+2}^{r_i+2}ds\Big)$$
$$\le C\sum_{i=1}^2 D(t)^{(r_i+1)(r_i+2)/(r+2)} \sup_{t\le s\le t+1}\|\nabla u\|^{(1-\nu_i)}\|\Delta u\|^{\nu_i}$$
$$\le C\sum_{i=1}^2 D(t)^{(r_i+1)(r_i+2)/(r+2)} E(t)^{(1-\nu_i)/2},$$

where ν_i is given by

$$\nu_i = \begin{cases} 0 & \text{if} \quad -1 < r_i \le 4/(N-2), \\ \frac{N-2}{2} - \frac{N}{r_i+2} & \text{if} \quad 4/(N-2) < r_i \le 8/(N-4). \end{cases} \tag{24}$$

Thus, it follows from (22) and (23) that

$$\int_{t_1}^{t_2} \|\nabla u(s)\|^2 ds \le C\{D(t)^{1-\theta^*}E(t)^{1/2} + D(t)^{2(1-\theta^*)}$$
$$+ \sum_{i=1}^2 D(t)^{(r_i+1)(r+2)/(r_i+2)}E(t)^{(1-\nu_i)/2}\} \equiv A(t)^2 \tag{25}$$

and hence, by (19) or (20),

$$\int_{t_1}^{t_2} E(s)ds \le A(t)^2 + D(t)^{2(1-\theta^*)}. \tag{26}$$

Since $E(t)$ is nonincreasing we have from above

$$E(t+1) \le 2\{A(t)^2 + D(t)^{2(1-\theta^*)}\}$$

and consequently (see (14)),

$$E(t) = E(t+1) + \int_t^{t+1}\int_\Omega \rho(u_t)u_t dxds \tag{27}$$
$$\le C\{D(t)^{2(1-\theta^*)} + D(t)^{r+2} + CD(t)^{1-\theta^*}E(t)^{1/2}$$
$$+ C\sum_{i=1}^2 D(t)^{(r_i+1)(r+2)/(r_i+2)}E(t)^{(1-\nu_i)/2}\}$$

which implies

$$E(t) \le C\Big\{D(t)^{r+2} + D(t)^{2(1-\theta^*)} + \sum_{i=1}^2 D(t)^{2(r_i+1)(r+2)/(r_i+2)(1+\nu_i)}\Big\}. \tag{28}$$

In order to derive the decay rate of $E(t)$ from (28) we consider two cases: (i) $-1<r<\alpha(N-2)/N$ and (ii) $r\geq\alpha(N-2)^+/N$, separately.

Case (i): $-1<r<\alpha(N-2)/N$. In this case we recall

$$\epsilon=\frac{2N(r+1)-\alpha(N-2)}{2N+\alpha(N-2)}$$

and

$$\theta^*=\theta=\frac{N(1-\epsilon)}{N+2-(N-2)\epsilon}=\frac{\alpha(N-2)-Nr}{\alpha(N-2)-Nr+2(r+2)}\,.$$

Thus, we have from (28) that

$$E(t)\leq C\Big\{D(t)^{r+2}+D(t)^{2(1-\theta)}+\sum_{i=1}^{2}D(t)^{2(q+1)(r+2)/(q+2)(1+\nu_2)}\Big\}\leq CD(t)^{2\kappa_1}$$

or

$$E(t)^{(r+2)/2\kappa_1}\leq CD(t)^{r+2}=C\left(E(t)-E(t+1)\right) \tag{29}$$

with

$$\kappa_1=\min\{1-\theta,(q+1)(r+2)/(q+2)(1+\nu_2),(p+1)(r+2)/(p+2)\}.$$

The inequality (29) yields, by Lemma 2,

$$E(t)\leq C(1+t)^{-2\eta_1} \tag{30}$$

with

$$\begin{aligned}\eta_1&=\{(r+2)/2\kappa_1-1\}^{-1}/2\\&=\min\{(1-\theta)/(r+2\theta),2(q+1)/((N-4)q-4)^+,-(p+1)/p\}\\&=\min\{2/(N-2)(\alpha-r)^+,2(q+1)/((N-4)q-4)^+,-(p+1)/p\}.\end{aligned}$$

Case (ii): $Nr\geq\alpha(N-2)^+$. In this case we recall $\epsilon=1$ and $\theta^*=0$. Thus, we have, from (28), $E(t)\leq CD(t)^{2\kappa_2}$ or

$$E(t)^{(r+2)/2\kappa_2}\leq C\left(E(t)-E(t+1)\right), \tag{31}$$

where we set

$$\kappa_2=\min\{1,(q+1)(r+2)/(q+2)(1+\nu_2),(p+1)(r+1)/(p+2)\}.$$

We obtain, by Lemma 2,

$$E(t)\leq C(1+t)^{-\eta_2}, \tag{32}$$

$$(E(t)\leq C\exp(-\lambda t),\ \lambda>0,\ \text{ if }\ r=0\text{ and }(N-4)q\leq 4)$$

with

$$\eta_2=2\min\{1/r,2(q+1)/((N-4)q-4)^+\}.$$

Here, we note that $\eta_1=\eta_2$ if $Nr=(N-2)\alpha$. Thus the proof of Theorem 1 is now complete.

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