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ALMOST PERIODIC SOLUTIONS OF INFINITE DIMENSIONAL RICCATI EQUATIONS

Giuseppe Da Prato
Scuola Normale Superiore di Pisa, Piazza dei Cavalieri 6, 56126 Pisa, Italy

Constantin Tudor†
Faculty of Mathematics, University of Bucharest, Str. Academiei 14, 70109 Bucharest, Romania

Abstract. We consider an infinite dimensional linear controlled Ito system with almost periodic coefficients and two–sided average cost. Under stabilizability and detectability assumptions we prove that the unique bounded solution of the associated Riccati equation is almost periodic.

1. Introduction. We assume that H, U, Y are separable Hilbert spaces and $(\Omega, \mathcal{F}, \mu)$ is a probability space. We denote by $L(H,U)$ ($L(H)$ for short if $H = U$) the Banach space of all linear bounded operators from H into U and by $L_2(H,U)$ the space of all Hilbert–Schmidt operators acting between H and U equipped with the Hilbert–Schmidt norm $\|\cdot\|_2$. We shall denote by $C_s(\mathbb{R}, L(H,Y))$ the set of all strongly continuous mappings $f : \mathbb{R} \to L(H,Y)$, that is $f(\cdot)u : \mathbb{R} \to Y$ is continuous for all $u \in U$.

Given a symmetric nonnegative operator $Q \in L(U)$ with finite trace, we assume that $\{W(t)\}_{t\in\mathbb{R}}$ is a Q–Wiener process defined on $(\Omega, \mathcal{F}, \mu)$ and with values in H. Recall that W can be obtained as follows: let $\{W_i(t)\}_{t\geq 0},\ i = 1, 2$, be two independent H–valued Q–Wiener processes, then

$$W(t) = \begin{cases} W_1(t) & \text{if } t \geq 0 \\ W_2(-t) & \text{if } t \leq 0 \end{cases}$$

is a Q–Wiener process with $\mathbb{R}$ as time parameter. We put $\mathcal{F}_t = \sigma\{W(s) : s \leq t\}$.

Let $H_0 = Q^{1/2}H$ and $L_2^0 = L_2(H_0, Y)$ which is a separable Hilbert space with respect to the norm

$$\|\Psi\|^2_{L_2^0} = \|\Psi Q^{1/2}\|_2^2 = \text{Tr } (\Psi Q \Psi^*),$$

where Tr denotes the trace.

We consider the standard linear quadratic control problem in Y with the dynamics

$$dx(t) = [A(t)x(t) + B(t)u(t)) + f(t)]dt + [G(t)x(t) + g(t)]dW(t),\ \ t \in \mathbb{R}, \tag{1.1}$$

and the two–sided cost functional

$$J(u) = \limsup_{T\to\infty} \frac{1}{2T} \mathbb{E} \int_{-T}^{T} \left[\|M(t)x(t)\|^2 + < H(t)u(t), u(t) >\right] dt, \tag{1.2}$$

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where we assume

Hypothesis 1: (i) There exists a strongly continuous mapping

$$U(\cdot,\cdot) : \{(t,s) \in \mathbb{R}^2 : t \geq s\} \to L(Y),\ (t,s) \to U(t,s)$$

such that

$$\frac{\partial}{\partial t}U(t,s)x = A(t)U(t,s)x,\ \ U(s,s)x = x,\ \forall\, x \in D(A(t)),\ \ 0 \leq s \leq t \leq T.$$

(ii) The mapping

$$U^*(\cdot,\cdot) : \{(t,s) \in \mathbb{R}^2 : t \geq s\} \to L(Y),\ \ (t,s) \to U^*(t,s)$$

is strongly continuous, $U^*(t,s)$ being the adjoint of $U(t,s)$.

Hypothesis 2: (i) $B \in C_s(\mathbb{R}; L(U,Y)),\ f \in C_s(\mathbb{R}, Y)$.
(ii) $G \in C_s(\mathbb{R}; L(H, L_2^0)),\ f \in C_s(\mathbb{R}, L_2^0)$.
(iii) $M \in C_s(\mathbb{R}; L(Y))$, $H \in C_s(\mathbb{R}, L(U))$, and $H(t) \geq \varepsilon > 0$ for all $t \in \mathbb{R}$ and some $\varepsilon > 0$.
(iv) B, f, G, M and H are bounded.

We say that a $\mathcal{F}_t$–progressively measurable Y–valued process $\{x(t)\}_{t\in\mathbb{R}}$ is a *mild solution* to equation (1.1) if for every $t \geq r$, $t, s \in \mathbb{R}$ we have

$$\begin{aligned} x(t) = U(t,s)x(s) &+ \int_s^t U(t,\sigma)[B(\sigma)u(\sigma) + f(\sigma)]d\sigma \\ &+ \int_s^t U(t,\sigma)[G(\sigma)x(\sigma) + g(\sigma)]dW(\sigma). \end{aligned} \tag{1.3}$$

For the definition of the stochastic integral above see for instance [6].

As a set of admissible controls, we choose the set $\mathcal{U}$ of all $\mathcal{F}_t$–progressively measurable U–valued process $\{u(t)\}_{t\in\mathbb{R}}$, that are L^2–bounded, i.e.,

$$\sup_{t\in\mathbb{R}} \mathbb{E}\left(|u(t)|_U^2\right) < \infty,$$

and the corresponding solution of (1.1) is also L^2–bounded.

We consider the optimization problem

$$\text{Minimize } J(u) \text{ over all } u \in \mathcal{U}.$$

Let us now introduce for any $R \in L(Y)$ the operator $\Delta(R) \in L(Y)$ defined by

$$\langle \Delta(R)x, y\rangle = \operatorname{Tr}[RG(x)QG^*(x)],\ \ x, y \in Y. \tag{1.4}$$

To solve the above optimization problem the following backward Riccati equation is useful:

$$P' + A^*P + PA - PBH^{-1}B^*P + \Delta(P) + M^*M = 0. \tag{1.5}$$

We say that $P \in C_s(\mathbb{R}, L(Y))$ is a *mild solution* of (1.5) if for every $t \leq s,\ x \in Y$ we have

$$\begin{aligned} P(t)x &= U^*(s,t)P(s)U(s,t)x + \int_t^s U(r,t)M^*(r)M(r)U^*(r,t)xdr \\ &\quad - \int_t^s \big[U(r,t)P(r)B(r)H^{-1}(r)B^*(r)P(r) + \Delta(P(r))U^*(r,t)\big]xdr. \end{aligned} \tag{1.6}$$

A mild solution $P(\cdot)$ of (1.5) is said to be *bounded* if

$$\sup_{t\in\mathbb{R}} \|P(t)\| < +\infty.$$

The above mentioned optimization problem has been considered in [3, 4, 5], where under Hypotheses 1, 2 and a suitable stabilizability and detectability assumption, is proved that there exists a unique bounded mild solution of (1.5). Moreover, the solution is T–periodic whenever all the coefficients are T–periodic.

For some time almost periodicity of the solution (if coefficients are almost periodic), was an open problem. Recently Morozan, [12] gave a positive answer in finite dimensions.

In the present paper, we prove almost periodicity of the mild solution P of the Riccati equation if A is periodic and all the remaining coefficients B, f, G, M are almost periodic and $\{P(t)\}_{t\in\mathbb{R}}$ is sequentially relatively compact in the strong topology (Theorem 3.8). Then we characterize the optimal control (Theorem 3.14).

2. Preliminaries concerning almost periodic functions.

Let (X, d) be a separable and complete metric space, $f, g : \mathbb{R} \to X$ be continuous and $\alpha = \{\alpha_k\} \subset \mathbb{R}$. The notation $T_\alpha f = g$ means that $\lim_{n\to\infty} f(t+\alpha_n) = g(t)$. The mode of convergence will be specified at each use of the symbol.

Definition 2.1 (Bohr)**.** A continuous function $f : \mathbb{R} \to X$ ($f \in C(\mathbb{R}, X)$ for short), is said to be almost periodic if, for all $\varepsilon > 0$, there exists a number $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains $\tau \in \mathbb{R}$ such that

$$d(f(t+\tau), f(t)) \leq \varepsilon \quad \text{for all } t \in \mathbb{R}.$$

We have the following characterization of almost periodicity.

Theorem 2.2 ([7])**.** *Let $f \in C(\mathbb{R}, X)$. Then the following statements are equivalent.*

(i) *f is almost periodic.*
(ii) *The set of translates $\{f(t+\cdot)\}_{t\in\mathbb{R}}$ is sequentially relatively compact in $C(\mathbb{R}; X)$ with respect to the uniform norm.*
(iii) *For every pair of sequences $\alpha' = \{\alpha'_n\} \subset \mathbb{R}$ and $\beta' = \{\beta'_n\} \subset \mathbb{R}$ there are common subsequences $\alpha \subset \alpha'$ and $\beta \subset \beta'$ such that*

$$T_{\alpha+\beta} f = T_\alpha T_\beta f \tag{2.1}$$

uniformly on $\mathbb{R}$.
(iv) *As in* (iii) *but with pointwise convergence in* (2.1) *instead of uniform convergence.*

Let X_1 and X_2 be separable Banach spaces.

Definition 2.3. $f \in C_s(\mathbb{R}, L(X_1, X_2))$ is said to be strongly almost periodic if for every $x \in X_1$ the function $f(\cdot)x \in C(\mathbb{R}, X_2)$ is almost periodic.

Remark 2.4. It is an easy consequence of the uniform boundedness principle that $f \in C_s(\mathbb{R}, L(X_1, X_2))$ is almost strongly periodic if and only if for every sequence $\alpha' = \{\alpha'_n\} \subset \mathbb{R}$ there exists $\alpha = \{\alpha_n\} \subset \alpha'$ and $\widetilde{f} \in C_s(\mathbb{R}, L(X_1, X_2))$ such that $T_\alpha f = \widetilde{f}$ strongly, uniformly on $\mathbb{R}$.

From Theorem 2.2 it follows:

Theorem 2.5. *Let $f \in C_s(\mathbb{R}, L(X_1, X_2))$. Then the following statements are equivalent.*

(i) *f is strongly almost periodic.*
(ii) *For every pair of sequences $\alpha' = \{\alpha'_n\} \subset \mathbb{R}$ and $\beta' = \{\beta'_n\} \subset \mathbb{R}$ there are common subsequences $\alpha \subset \alpha'$ and $\beta \subset \beta'$ such that*

$$T_{\alpha+\beta} f = T_\alpha T_\beta f \tag{2.2}$$

strongly uniformly on $\mathbb{R}$.
(iii) *As in* (ii) *but with pointwise convergence in* (2.2) *instead of uniform convergence.*

3. Almost periodic solutions of Riccati equations. Consider the linear evolution equation

$$dx(t) = A(t)x(t)dt + G(t)x(t)dW(t), \quad t \in \mathbb{R}, \tag{3.1}$$

and denote by $\{X(t,x)\}_{t\geq s}$ the corresponding stochastic evolution operator.

Definition 3.1. We say that equation (3.1) (or the pair (A, G)) is mean square stable if there are constants $M > 0, \beta > 0$ such that

$$\mathbb{E}\left(\|X(t,s)x\|^2\right) \leq Me^{-\beta(t-s)}\|x\|^2, \quad \text{for all } t \geq s,\ x \in Y. \tag{3.2}$$

Remark 3.2. Equivalent conditions for square mean stability in terms of Liapunov equations are given in [2].

Definition 3.3. We say that the triplet (A, B, G) is stabilizable if there exists

$$K \in C_s(\mathbb{R}, L(Y, U))$$

bounded, such that $(A - BK, G)$ is stable.

Remark 3.4. If (A, B, G) is stabilizable then the equation (1.1), with $u = -Kx$, has a pathwise unique L^2- bounded solution, see [8], and, in particular u belongs to the set of admissible controls $\mathcal{U}$.

Definition 3.5. We say that the triplet (A, M, G) is detectable if there exists

$$K_1 \in C_s(\mathbb{R}, L(Y))$$

bounded, such that $(A - K_1 M, G)$ is stable.

The following result is proved in [5].

Theorem 3.6. *Assume that Hypotheses* 1 *and* 2 *hold. If* (A, B, G) *is stabilizable and* (A, M, G) *is detectable, then equation* (1.5) *has a unique bounded solution* $\{P(t)\}_{t\in\mathbb{R}}$ *which is stable, that is such that* $(A - BH^{-1}B^*P, G)$ *is stable. Moreover if all coefficients of* (1.5) *are* τ*–periodic, then* P *is* τ*–periodic.*

Remark 3.7. Under the hypotheses of Theorem 3.6 we have $\mathcal{U} \neq \emptyset$ and

$$u = -H^{-1}B^*Px \in \mathcal{U}.$$

The main result of the paper is the following.

Theorem 3.8. *Assume, besides the hypotheses of Theorem* 3.6, *the following:*

(i) A *is periodic,*
(ii) B, B^*, M, M^*, H^{-1} *are strongly almost periodic,*
(iii) K, K_1 *from Definitions* 3.3, 3.5, *are strongly almost periodic,*
(iv) *if* P *is the unique bounded solution to* (1.5), *then* $\{P(t)\}_{t\in\mathbb{R}}$ *is sequentially relatively compact in the strong topology,*

Then P *is strongly almost periodic.*

In order to prove Theorem 3.8 we need some auxiliary results. The next lemma is an infinite dimensional version of a similar result in finite dimensions, see [13].

Lemma 3.9. *Assume that* (A, G) *fulfills Hypotheses* 1 *and* 2 *and that it is stable. Then*

(a) *For every* $\sigma \in \mathbb{R}$, $A(\cdot + \sigma), G(\cdot + \sigma)$ *is stable with the same constants* M, β *in* (3.2).
(b) *If* $T_\alpha U = \widetilde{U}$ *strongly pointwise, where* $\widetilde{U}$ *is an evolution operator generated by a family* $\{\widetilde{A}(t)\}_{t\in\mathbb{R}}$ *satisfying Hypothesis* 1 *and* $T_\alpha G = \widetilde{G}$ *strongly, pointwise, then* $(\widetilde{A}, \widetilde{G})$ *is stable with the same constants* M, β.

Proof. (a) If $X_\sigma(t, s)$ is the stochastic fundamental solution associated with $(A(\cdot + \sigma), G(\cdot+\sigma))$, then as a consequence of the uniqueness in law we have that $(X_\sigma(t, s))_{t\geq s}$ and $(X(t + \sigma, s + \sigma))_{t\geq s}$ have the same repartition. In particular, it follows that

$$\mathbb{E}(|X_\sigma(t, s)x|^2) = \mathbb{E}(|X(t + \sigma, s + \sigma)x|^2)$$

and the conclusion follows easily.

(b) From the local inversion theorem (see Lemma 9.2 of [6]) we have that

$$X_{\alpha_n}(t, s)x \to \widetilde{X}(t, s)x, \ \text{ in } L^2 \ \text{ for all } t \geq s, \ x \in Y,$$

where $\widetilde{X}(t, s)$ is the stochastic fundamental solution corresponding to (A_α, G_α). Now the conclusion follows. $\square$

Proposition 3.10. *Assume that the hypotheses of Theorem* 3.8 *are satisfied. Let* $\alpha = \{\alpha_n\} \subset \mathbb{R}$ *be such that*

$$T_\alpha Z = \widetilde{Z} \quad \textit{strongly, uniformly on } \mathbb{R} \tag{3.3}$$

for $Z = B,\ B^*,\ M,\ M^*,\ G,\ G^*,\ H^{-1},\ K,\ K_1$, *and*

$$U(t + \alpha_n, s + \alpha_n) \to U(t + \sigma, s + \sigma), \qquad U^*(t + \alpha_n, s + \alpha_n) \to U^*(t + \sigma, s + \sigma)$$

strongly, for all $t \geq s$ *(by periodicity and strong continuity of* U, U^**.) Let* P *(resp.* $\widetilde{P}$*) be the mild bounded solution to* (1.5) *corresponding to the coefficients* A, B, M, H^{-1}, G *(resp.* $\widetilde{A}(\cdot) = A(\cdot + \sigma), \widetilde{B}, \widetilde{M}, \widetilde{H}^{-1}, \widetilde{G}$*). Then* $T_\alpha P = \widetilde{P}$ *strongly pointwise.*

Proof. First observe that by Lemma 3.9 $(\widetilde{A}, \widetilde{B}, \widetilde{G})$ is stabilizable and $(\widetilde{A}, \widetilde{M}, \widetilde{G})$ is detectable, so that by Theorem 3.6 $\widetilde{P}$ is well defined. It is enough to prove that every subsequence $\beta = \{\beta_n\} \subset \alpha$ contains a subsequence $\gamma = \{\gamma_n\} \subset \beta$ such that

$$T_\gamma P = P_\alpha \ \text{ strongly, for every } t \in \mathbb{R}.$$

For simplicity let $\beta = \alpha$. Then for every $r \geq 1$, $\{P(t+\alpha_n)\}_{t\leq r}$ satisfies

$$\begin{aligned} P(t+\alpha_n)x &= U^*(r+\alpha_n, t+\alpha_n)P(r+\alpha_n)U(r+\alpha_n, t+\alpha_n) \\ &\quad + \int_t^r U^*(s+\alpha_n, t+\alpha_n)[M^*(s+\alpha_n)M(s+\alpha_n) \\ &\quad - P(s+\alpha_n)B(s+\alpha_n)H^{-1}(s+\alpha_n)B^*(s+\alpha_n)P(s+\alpha_n) \\ &\quad + \Delta(P(s+\alpha_n))]U(r+\alpha_n, t+\alpha_n)xds, \ \ \forall\, x \in Y. \end{aligned}$$

By assumption (iv) of Theorem 3.8 there exists $\gamma_r = \{\gamma_{n,r}\} \subset \alpha$ such that

$$P(r+\gamma_{n,r}) \to P_r \ \text{ strongly as } \ n \to \infty,$$

for any r.

Let $\{P_r(t)\}_{t\leq r}$be the mild solution of the Riccati equation

$$P_r' + A_\alpha^* P_r + P_r A_\alpha - P_r B_\alpha H_\alpha^{-1} B_\alpha^* P_r + \Delta(P_r) + M_\alpha^* M_\alpha = 0, \quad P(r) = P_r(r). \tag{3.4}$$

By (3.3) and the continuous dependence with respect to the data for Riccati equations (see Theorem 2.2 of [1]) we have

$$P(\cdot + \gamma_{r,n}) \to P_r(\cdot) \quad \text{in } C_s([-r,r], L(Y)).$$

Let $\gamma_n = \gamma_{n,n}$. Then

$$\begin{aligned} P(\cdot+\gamma_n) &\to P_r(\cdot) \qquad \text{in } C_s([-r,r], L(Y), \\ P(\cdot+\gamma_n) &\to P_{r+1}(\cdot) \quad \text{in } C_s([-r-1, r+1], L(Y)). \end{aligned}$$

Therefore, $P_r(\cdot) = P_{r+1}(\cdot)$ on $[-r,r]$ and we can define without ambiguity

$$\widehat{P}(t) = P_r(t) \quad \text{if } \ t \leq r.$$

Of course $\widehat{P} \geq 0, \widehat{P} \in C_s(\mathbb{R}, L(Y))$ and since

$$\sup_{t\leq r} \|P_r(t)\| \leq \sup_{t\in\mathbb{R}} \|P(t)\| < +\infty,$$

it follows that $\sup_{t\in\mathbb{R}} \|\widehat{P}(t)\| < +\infty$. Since $\widehat{P}$ is a mild solution of (1.5) corresponding to the coefficients $(\widetilde{A}, \widetilde{B}, \widetilde{M}, \widetilde{H}^{-1}, \widetilde{G})$ we have by the uniqueness that $\widehat{P} = \widetilde{P}$. The proof is complete.

Proof of Theorem 3.8. Let $\alpha' = \{\alpha'_n\} \subset \mathbb{R}$, $\beta' = \{\beta'_n\} \subset \mathbb{R}$ and choose by Theorem 2.5 common subsequences $\alpha = \{\alpha_n\} \subset \alpha'$, $\beta = \{\beta_n\} \subset \beta'$ such that

$$T_{\alpha+\beta} Z = T_\beta T_\alpha Z, \ \text{ strongly uniformly on } \mathbb{R} \tag{3.5}$$

for $Z = (B, B^* H^{-1}, M, M^*, G, G^*, K, K_1)$, and

$$U(t + \alpha_n, s + \alpha_n) \to U(t + \sigma_1, s + \sigma_1) \tag{3.6}$$

$$U^*(t + \alpha_n, s + \alpha_n) \to U^*(t + \sigma_1, s + \sigma_1) \tag{3.7}$$

$$U(t + \beta_n + \sigma_1, s + \beta_n + \sigma_1) \to U(t + \sigma_1 + \sigma_2, s + \sigma_1 + \sigma_2) \tag{3.8}$$

$$U^*(t + \beta_n + \sigma_1, s + \beta_n + \sigma_1) \to U^*(t + \sigma_1 + \sigma_2, s + \sigma_1 + \sigma_2) \tag{3.9}$$

$$U(t + \alpha_n + \beta_n, s + \alpha_n + \beta_n) \to U(t + \sigma_1 + \sigma_2, s + \sigma_1 + \sigma_2) \tag{3.10}$$

$$U^*(t + \alpha_n + \beta_n, s + \alpha_n + \beta_n) \to U^*(t + \sigma_1 + \sigma_2, s + \sigma_1 + \sigma_2), \tag{3.11}$$

strongly, for all $t \geq s$ (by periodicity and strong continuity of U, U^*.)

By using (3.5)–(3.11) and Proposition 3.10 applied successively to the coefficients (A, B, M, H^{-1}, G) and sequence α, then to the coefficients

$$(T_\alpha A, T_\alpha B, T_\alpha M, T_\alpha H^{-1}, T_\alpha G),$$

and sequence β and finally to (A, B, M, H^{-1}, G) and sequence $\alpha + \beta$, we obtain that

$$T_{\alpha+\beta} P = T_\alpha T_\beta P \ \text{ strongly, for all } t \in \mathbb{R}.$$

We now conclude by use of Theorem 2.5. □

In order to obtain the validity of assumption (d) of Theorem 3.8, we introduce the following:

Hypothesis 3: (i) There exists a Hilbert space $D \subset Y$ such that $D(A(t)) = D,\ t \in \mathbb{R}$. Moreover, the embedding $D \subset H$ is continuous, compact and dense.

(ii) For some positive constants M_1, M_2 we have

$$M_1 \|x\|_D \leq \|A(t)x\|_Y \leq M_2 \|x\|_D, \ \ x \in D, \ x \in \mathbb{R}.$$

(iii) The operators $A(t), t \in \mathbb{R}$ generate analytic semigroups on Y and there exist $\alpha \in]0, 1]$ such that $A(\cdot) \in C^\alpha(\mathbb{R}, L(D, Y))$.

(iv) A is T–periodic.

The following result is proved in [11].

Proposition 3.11. *Assume Hypothesis* 3 *and that there exists* $\omega > 0$ *such that*

$$\langle A(t)x, x\rangle \leq -\omega \|x\|^2, \ \ \forall\, x \in D, t \in \mathbb{R}.$$

Then $A(\cdot)$ *generates an evolution operator* $U(t, s)$ *such that*

(i) $\|U(t, s)\| \leq e^{-\omega(t-s)}, \ \ t \geq s.$

(ii) *U is T–periodic, that is,* $U(t + T, s + T) = U(t, s), \ t \geq s.$

(iii) *For all* $x \in Y, t \in \mathbb{R}$ *and* $\theta \in]0, 1[$ *we have* $U(t, s)x \in (D, Y)_{1-\theta,2}$. *Moreover there exists a positive constant* C_θ *such that*

$$\|U(t, s)x\|_{(D,Y)_{1-\theta,2}} \leq \frac{C_\theta}{(t-s)^{1-\theta}} e^{-\omega(t-s)}, \ t \geq s.$$

We recall that $(D, Y)_{1-\theta,2}$ is the real interpolation space defined in [10].

Remark 3.12. If $A(t) = A_0$ and $A(t)$ is self–adjoint, $A(t) \leq -\omega$ and A^{-1} is nuclear, then the above hypothesis is fulfilled.

Proposition 3.13. *Let the assumptions of Proposition* 3.11 *be satisfied, and let* U^* *be strongly continuous. Let* P *be the unique bounded mild solution of* (1.5). *Then for all* $\theta \in]0, 1/2[$ *we have*

$$\sup_{t\in\mathbb{R}} \|P(t)x\|_{(D,Y)_{1-\theta,2}} < \infty, \quad \text{for all } x \in Y. \tag{3.12}$$

In particular the function $\{P(t)\}_{t\in\mathbb{R}}$ *is sequentially strongly relatively compact.*

Proof. It suffices to prove that there exists a constant C, independent of r, such that

$$\|Q_r(t)x\|_{(D,Y)_{1-\theta,2}} \leq C, \quad \text{for all } x \in Y,$$

where $\{Q_r(\cdot)\}_{t\leq r}$ is the mild solution of (1.5) on the interval $]-\infty, r]$ with $Q_r(r) = 0$. It is not difficult to show that $Q_r(t) \uparrow P(t)$ strongly for every $t \in \mathbb{R}$; thus there exists a constant $C_1 > 0$ such that

$$\|Q_r(t)\| \leq C_1, \quad t \in]-\infty, r].$$

Moreover, Q_r satisfies the equality

$$\begin{aligned} Q_r(t)x = \int_t^r & U^*(s,t)[M^*(s)M(s) \\ -Q_r(s)B(s)H^{-1}(s)B^*(s)Q_rP(s) &+ \Delta(Q_r(s))]U(s,t)x ds, \quad t \leq r, x \in Y. \end{aligned}$$

Then, by using Proposition 3.11 we have

$$\begin{aligned} \|Q_r(t)x\|_{(D,Y)_{1-\theta.2}} &\leq C_\theta \int_r^t \frac{e^{-2\omega(s-t)}}{(s-t)^\theta} \left[\frac{C_1^2}{\varepsilon}\|M\|_\infty^2 + C_1\|\Delta\|\right] \|x\| ds \\ &\leq C_{1,\theta} \int_r^t \frac{e^{-2\omega(s-t)}}{(s-t)^\theta} \leq C_{1,\theta} \int_0^{+\infty} e^{-2\omega\sigma} \frac{d\sigma}{\sigma^\theta}. \end{aligned}$$

Now it is easy to conclude by using the compactness of the embedding $(D, Y)_{1-\theta,2} \subset Y$. □

Now, by proceeding as in [5], we have

Theorem 3.14. *Let the assumptions of Theorem* 3.8 *be satisfied and assume that* f, g *are almost periodic. Then the optimal control is given by*

$$u = -H^{-1}B^*(P\overline{y} + r),$$

where P *is the unique almost periodic mild solution of* (1.5) *and* r *is the almost periodic function*

$$r(t) = \int_t^{+\infty} U_L^*(s,t)P(s)ds, \quad L = A - BH^{-1}B^*P.$$

The optimal cost is given by

$$J(\overline{u}) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} 2\langle r, f\rangle - \|H^{-1/2}B^*r\|^2 + \text{Tr}\,[GQG^*P]dt.$$

The optimal trajectory $\overline{y}$ is the unique L^2–bounded mild solution of

$$d\overline{y} = [(A - BH^{-1}B^*P)\overline{y} + f - BH^{-1}B^*r]dt + (G\overline{y} + g)dW(t), \quad t \in \mathbb{R}.$$

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