MAXIMUM PRINCIPLES FOR
INTEGRO-DIFFERENTIAL PARABOLIC OPERATORS

MARIA GIOVANNA GARRONI
Università di Roma “La Sapienza”, Dipartimento di Matematica, 00185 Roma, Italy

José Luis Menaldi
Wayne State University, Department of Mathematics, Detroit, MI 48202

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Abstract. Several versions of the classical maximum principle, which are valid for parabolic differential problems, are proved to hold for second order integro-differential problems.

1. Introduction. We give here an analytic approach to the study of second order parabolic integro–differential operators. First we consider the operator in the whole space, then we pass first to a bounded domain and next to an unbounded domain.

We are interested in the integro-differential operator related with a diffusion process with jumps, i.e.,

\[ I\varphi(x, t) = \int_{\mathbb{R}^d} \left[ \varphi(x + z, t) - \varphi(x, t) - z \cdot \nabla \varphi(x, t) \right] M(x, t, dz), \tag{1.1} \]

where the Levy kernel \( M(x, t, dz) \) is a Radon measure on \( \mathbb{R}^d = \mathbb{R}^d \setminus \{0\} \) for any fixed \( x \in \mathbb{R}^d \), \( t \in [0, T] \), and such that at least

\[ \int_{\mathbb{R}^d} |z|^2 (1 + |z|)^{-1} M(x, t, dz) < \infty, \quad \forall x, t. \tag{1.2} \]

The reader is referred to the books of Gikhman and Skorokhod [14, p. 245], Bensoussan and Lions [3, p. 178] among others.

If the function \( \varphi \) is smooth then we can rewrite

\[ I\varphi(x, t) = \int_0^1 (1 - \theta) d\theta \int_{\mathbb{R}^d} [z \cdot \nabla^2 \varphi(x + \theta z, t) z] M(x, t, dz). \tag{1.3} \]

So, in view of (1.2) the expression \( I\varphi \) makes sense at least when the second order derivatives of \( \varphi \) in \( x \) (i.e., \( \nabla^2 \varphi \)) are continuous and bounded in \( \mathbb{R}^d \times [0, T] \).

A priori the integro-differential operator (1.1) is defined only for functions \( \varphi(x, t) \), with \( x \) in the whole space \( \mathbb{R}^d \) and \( t \) in \([0, T]\). However, we want to consider equations on an either bounded or unbounded region \( \Omega \) of \( \mathbb{R}^d \), with either Dirichlet or Neumann boundary conditions, even with oblique boundary conditions. Then we need to localize the operator into \( \Omega \), e.g., by extending the data \( \varphi \) outside of \( \Omega \). Thus (1.1) becomes

\[ I\varphi(x, t) = \int_{\mathbb{R}^d} [\tilde{\varphi}(x + z, t) - \varphi(x, t) - z \cdot \nabla \varphi(x, t)] M(x, t, dz), \tag{1.4} \]

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where \( \varphi \) is a function defined on \( \Omega \times [0, T] \), and \( \tilde{\varphi} \) is an extension of \( \varphi \) to the whole space \( \mathbb{R}^d \times [0, T] \).

If we are working with homogeneous Dirichlet boundary conditions, then it is natural to use the zero extension, i.e.,

\[
\tilde{\varphi}(x, t) = \begin{cases} 
\varphi(x, t) & \text{if } x \in \Omega, \ t \in [0, T], \\
0 & \text{otherwise}.
\end{cases}
\tag{1.5}
\]

From the probabilistic point of view, this corresponds to stopping the random process at the first exit time of the domain \( \Omega \).

Assuming \( \varphi \) smooth in \( \Omega \times [0, T] \), we can have only a global Lipschitz continuous zero extension \( \tilde{\varphi} \) because of the homogeneous Dirichlet boundary condition. However, we may need \( \nabla^2 \tilde{\varphi} \) in order to use expression (1.3). This is a delicate point which is not very clear in the literature (cf. Gimbert and Lions [15]).

One may use another extension, obviously under convenient hypotheses on \( \Omega \), say a smooth extension to \( \mathbb{R}^d \times [0, T] \), but this does not usually have a good probabilistic interpretation. We will make use of a condition under which the extension will not be necessary; see assumption (1.10) below.

For general theory and applications of this kind of operators we refer for instance to Amulova [1, 2], Bensoussan and Lions [3], Bony et al. [4], Chaleyat-Maurel et al. [5], Garroni and Menaldi [8, 9, 10], Garroni et al. [11, 12], Garroni and Vivaldi [13], Gikhman and Skorokhod [14], Gimbert and Lions [15], Komatsu [16], Lenhard [18], Lepeltier and Marchal [19], Menaldi [20], Menaldi and Robin [21, 22, 23], Protter [26], Stroock [27], Taira [28], among others.

We discuss here various forms of the maximum principle for classical solutions of second order parabolic differential which hold true for classical solutions of our integro-differential problems. This is fundamentally due to some properties of the integral operator \( I \varphi \), and in particular to its behavior, when \( \varphi(x, t) = \exp q(x, t) \). Since the maximum principles involve only estimates in \( L^\infty \), we will use the general expression (1.1) for the integro-differential operator.

Consider the integro-differential operator in a cylinder \( Q_T = \Omega \times (0, T) \), where \( \Omega \) is either a bounded or an unbounded domain in \( \mathbb{R}^d \), \( T \leq +\infty \),

\[
A(x, t, \partial_x, \partial_t)u(x, t) = \partial_t u(x, t) - Lu(x, t) - Iu(x, t)
\]
\[
= \partial_t u - a_{ij}(x)\partial_{ij} u + a_i \partial_i u + a_0 u - Iu.
\tag{1.6}
\]

We suppose, unless otherwise stated, that

\[
a_{ij}(x)\xi_i \xi_j \geq \mu |\xi|^2, \quad \forall (x, t) \in \overline{Q}_T, \quad \forall \xi \in \mathbb{R}^d, \ \mu > 0,
\tag{1.7}
\]
\[
a_{ij}, a_i, a_0 \in L^\infty(Q_T).
\tag{1.8}
\]

We assume (1.1), where the Levy kernel \( M(x, t, \cdot) \) is a Radon measure on \( \mathbb{R}^d \) for any \( (x, t) \) in \( \overline{Q}_T \), \( M(\cdot, \cdot, A) \) is Borel measurable for any Borel measurable subset \( A \) of \( \mathbb{R}^d \) and there exists a function \( r(\varepsilon) \), such that \( r(\varepsilon) \leq c_0 \), \( r(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \),

\[
\int_{|z| \leq \varepsilon} |z|^2 (1 + |z|)^{-1} M(x, t, dz) \leq r(\varepsilon), \quad \forall (x, t) \in \overline{Q}_T,
\tag{1.9}
\]
i.e., (1.2) holds uniformly in \( x, t \), and

\[
\int_{\mathbb{R}^d} \chi(x + z \notin \overline{\Omega}) M(x, t, dz) = 0, \quad \forall (x, t) \in \overline{Q}_T.
\tag{1.10}
\]
Also, the domain satisfies

\[
\begin{align*}
\text{there exists a function } k(x, z, \theta) \text{ defined for } x \text{ in } \Omega, \\
z \text{ in } \mathbb{R}^d, \ \theta \text{ in } [0, 1], \text{ with values in } \mathbb{R}^d \text{ such that} \\
|k(x, z, \theta)| \leq c_0|z|, \quad \int_0^1 k(x, z, \theta)d\theta = z, \\
\{x + \int_0^\tau k(x, z, \theta)d\theta : x \in \Omega, \ z \in \Omega, \ \tau \in [0, 1]\} \subset \Omega.
\end{align*}
\]

(1.11)

**Remark 1.1.** Notice that condition (1.9) is equivalent to

\[
\lim_{t \to 0} \sup_{(x,t) \in \Omega} \int_{|z| \leq \varepsilon} |z|^2 M(x, t, dz) = 0, \quad \sup_{(x,t) \in \Omega} \int_{|z| \geq 1} |z|M(x, t, dz) \leq c_0.
\]

(1.12)

**Remark 1.2.** Condition (1.11) is satisfied if the domain has \(C^1\) boundary, if it is convex and, more generally, if it satisfies the exterior cone condition.

**Plan of the paper.** In Section 2 we state only the properties of the integro-differential operator (of the general form (1.1)), which will be directly used to prove the maximum principles. Section 3 is devoted to showing that different versions of the maximum principle and various estimates for the classical solutions (say \(C^{2,1}\)) in bounded domains can be extended to integro-differential problems. Only some proofs (or outlines of the proofs) are given, especially when tools different from those of the differential case will be used. In Section 4 we prove the principal results of the paper, concerning unbounded domains. The unboundedness of the domain imposes the appropriate choice of the Levy kernel, according to the behavior of the solutions at infinity. If the solutions are bounded, then condition (1.9) is sufficient (see Theorem 4.1): if the solutions have either polynomial or exponential growth we have to add conditions (4.18) and (4.19) in order to define the new Levy kernel (see Theorem 4.3).

Finally, in Theorem 4.4 we prove a maximum principle (of Phragmèn–Lindelöf type): the solutions have a square exponential growth at infinity and condition (4.19) on the Levy kernel is replaced by condition (4.33).

We will assume throughout the paper that all functions on which the operator \(A\) acts belong to \(C^{2,1}\). Many of the results proved here can be obtained also assuming that the functions on which \(A\) acts belong to convenient Sobolev spaces.


**Lemma 2.1.** Under assumption (1.9), for every positive \(\varepsilon\), there exists a constant \(C(\varepsilon)\) such that

\[
\|I\varphi(\cdot, t)\|_{L^\infty(\Omega)} \leq \varepsilon \|\nabla^2 \varphi(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} + C(\varepsilon) \|\varphi(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} + \|\nabla \varphi(\cdot, t)\|_{L^\infty(\mathbb{R}^d)}
\]

(2.1)

for every smooth function \(\varphi\) in \(\mathbb{R}^d \times [0, T]\), any \(t\) in \([0, T]\), and where \(I\) is given by (1.1).

**Proof.** We write

\[
I = I^{(1)} + I^{(2)}, \quad 0 < \eta \leq 1,
\]

(2.2)

where

\[
I^{(1)}(\eta)\varphi(x, t) = \int_{|z| \leq \eta} |\varphi(x + z, t) - \varphi(x, t) - z \cdot \nabla \varphi(x, t)| M(x, t, dz)
\]

\[
= \int_0^1 (1 - \theta)d\theta \int_{|z| \leq \eta} |z \cdot \nabla^2 \varphi(x + \theta z, t)z| M(x, t, dz),
\]
and

\[ I^{(2)}_\eta \varphi(x, t) = \int_{|z|>\eta} [\varphi(x + z, t) - \varphi(x, t)]M(x, t, dz) = \int_{|z|>\eta}zM(x, t, dz) \cdot \nabla \varphi(x, t), \]

for any smooth (and bounded) function \( \varphi \) in \( \mathbb{R}^d \times [0, T] \). Notice that either \( x + z \) or \( x + \theta z \) may not be inside \( \Omega \).

It is clear now, that in view of the assumption (1.9) on the Levy kernel \( M(x, t, dz) \) we obtain (2.1). \( \square \)

If the domain \( \Omega \) is unbounded then sometimes we want to be able to apply the integro-differential operator \( I \) to an unbounded function \( \varphi \). Due to the “nonlocal” effect of the Levy kernel \( M(x, t, dz) \) some extra conditions are necessary. For instance, in order to consider functions with either exponential or polynomial growth we use the weight functions

\[
\begin{cases}
(i) \text{ either } w_r(x) = \exp[-r(1 + |x|^2)^{1/2}], \\
(ii) \text{ or } w_r(x) = (1 + |x|^2)^{-r/2},
\end{cases}
\]

for some \( r > 0 \), and we add the condition

\[
\begin{cases}
(i) \text{ either } \int_{|z|>1} \exp(r|z|)M(x, t, dz) \leq C, \quad \forall (x, t) \in Q_T, \\
(ii) \text{ or } \int_{|z|>1} (1 + |z|^2)^{r/2}M(x, t, dz) \leq C, \quad \forall (x, t) \in Q_T,
\end{cases}
\]

on the Levy kernel. We have

**Corollary 2.2.** Under assumptions (1.9) and (2.4) the expression \( I \varphi(x, t) \) is finite for every \( (x, t) \) and any smooth function \( \varphi \) with either exponential or polynomial growth. Moreover, for every positive \( \varepsilon \), there exists a constant \( C(\varepsilon) \) such that

\[
\|w_r \mathcal{I}_\varphi(\cdot, t)\|_{L^\infty(\Omega)} \leq \varepsilon \|w_r \nabla^2 \varphi(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} + C(\varepsilon)\|w_r \varphi(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} + \|w_r \nabla \varphi(\cdot, t)\|_{L^\infty(\mathbb{R}^d)},
\]

for every smooth function \( \varphi \) in \( \mathbb{R}^d \times [0, T] \), any \( t \) in \( [0, T] \), and where \( I \) and \( w_r \) are given by (1.1) and (2.3) respectively.

**Proof.** We use the decomposition (2.2) of the operator \( I \) into \( I^{(1)}_\eta + I^{(2)}_\eta \), \( 0 < \eta \leq 1 \). Since

\[
|\nabla^2 \varphi(x + \theta z, t)| \leq w_{(-r)}(x + \theta z)\|w_r \nabla^2 \varphi(\cdot, t)\|_{L^\infty(\mathbb{R}^d)},
\]

and

\[
\exp[r(1 + |x + \theta z|^2)^{1/2}] \leq \exp(r|\theta z|) \exp[r(1 + |x|^2)^{1/2}] ,
\]

\[
(1 + |x + \theta z|^2)^{r/2} \leq 2^{r/2}(1 + |\theta z|^2)^{r/2}(1 + |x|^2)^{r/2},
\]

we deduce that for some \( \eta \) sufficiently small we have

\[
|I^{(1)}_\eta \varphi(x, t)| \leq \varepsilon w_{(-r)}(x)\|w_r \nabla^2 \varphi(\cdot, t)\|_{L^\infty(\mathbb{R}^d)},
\]

after using assumption (1.9).

Similarly, by means of assumption (2.4), we can bound

\[
\int_{|z|>\eta} |\varphi(x + z, t)|M(x, t, dz) \leq \|w_r \varphi(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} \int_{|z|>\eta} |w_{(-r)}(x + z, t)|M(x, t, dz),
\]
\[
\int_{|z|>\eta}|w_{(-r)}(x+z,t)|M(x,t,dz) \leq 2^{r/2}w_r(x)\int_{|z|>\eta}|w_{(-r)}(z,t)|M(x,t,dz),
\]
and therefore we conclude the result.

**Remark 2.3.** If we assume that the domain \(\Omega\) is smooth, say \(C^2\), so that a \(C^2\) extension exists, then the assumption
\[
\int_{\mathbb{R}^d} \chi(x + z \notin \overline{\Omega})M(x,t,dz) = 0, \quad \forall (x,t) \in \overline{Q_T}
\]
implies that the integro-differential operator \(I\) given by (1.1) can be “localized” into \(\overline{\Omega}\), i.e., it acts on functions defined on \(\overline{\Omega}\), but we need to use the extension of \(\varphi\) to \(\mathbb{R}^d \times [0,T]\) to be able to write down decomposition (2.2) used in Lemma 2.1. If we suppose either
\[
\int_0^1 d\theta \int_{\mathbb{R}^d} \chi(x + \theta z \notin \overline{\Omega})M(x,t,dz) = 0, \quad \forall (x,t) \in \overline{Q_T}
\]
or \(\Omega\) convex [and (2.7)], then the extension of \(\varphi\) to the whole space is not necessary. Clearly, a more sophisticated assumption is (1.11). Thus under assumption (2.8), or under assumption (2.7) and with \(\Omega\) either smooth or convex, or under assumption (1.11), we deduce estimates (2.1) and (2.5) with \(\Omega\) replacing \(\mathbb{R}^d\) on the left hand side.

**Lemma 2.4.** Let \(\varphi, \psi\) be any arbitrary smooth (and bounded) functions on \(\mathbb{R}^d\). Then the integro-differential operator \(I\) given by (1.1) satisfies
\[
I(\varphi\psi) = (I\varphi)\psi + \varphi(I\psi) + [I\varphi,I\psi],
\]
where the commutator operator \([I,I]\) has the form
\[
[I\varphi,I\psi](x,t) = \int_{\mathbb{R}^d} [\varphi(x+z) - \varphi(x)][\psi(x+z) - \psi(x)]M(x,t,dz).
\]
Moreover, if \(\varphi\) has an exponential form, i.e., \(\varphi = \exp(\psi)\), then
\[
I\varphi \geq \varphi I\psi
\]
in \(\mathbb{R}^d \times [0,T]\).

**Proof.** In virtue of decomposition (2.2) of Lemma 2.1 we can write
\[
I(\varphi\psi)(x,t) = \int_{\mathbb{R}^d} \{\varphi(x+z)\psi(x+z) - \varphi(x)\psi(x)
\]
\[
\quad - z \cdot [\nabla \varphi(x)\psi(x) + \varphi(x)\nabla \psi(x)]\}M(x,t,dz)
\]
\[
= \int_{\mathbb{R}^d} [\varphi(x+z) - \varphi(x) - z \cdot \nabla \varphi(x)]\psi(x)M(x,t,dz)
\]
\[
+ \int_{\mathbb{R}^d} \varphi(x)[\psi(x+z) - \psi(x) - z \cdot \nabla \psi(x)]M(x,t,dz)
\]
\[
+ \int_{\mathbb{R}^d} [\varphi(x+z) - \varphi(x)][\psi(x+z) - \psi(x)]M(x,t,dz),
\]
i.e., (2.9) holds.

To show (2.11) we notice that the function $e^\lambda - 1 - \lambda$ attains its minimum value in $\mathbb{R}$ for $\lambda = 0$, thus

$$e^\lambda - 1 - c \geq \lambda - c, \quad \forall \lambda, c \in \mathbb{R}. \quad (2.12)$$

Hence, for $\varphi = \exp(\psi)$ we have

$$I_\varphi(x, t) = \varphi(x) \int_{\mathbb{R}^d} [\exp(\psi(x + z) - \psi(x)) - 1 - z \cdot \nabla \psi(x)] M(x, t, dz).$$

Therefore, by means of inequality (2.12) for $\lambda = \psi(x + z) - \psi(x)$ and $c = z \cdot \nabla \psi(x)$, we deduce inequality (2.11).

**Remark 2.5.** We may prefer to write expressions (2.9), (2.10) in the form

$$I(\varphi \psi) = I_\psi \varphi + D_\varphi \psi + (I \psi) \varphi, \quad (2.13)$$

where $I_\psi$ is an integro-differential operator with Levy kernel

$$M_\psi(x, t, A) = \int_A \psi(x + z) M(x, t, dz), \quad (2.14)$$

and $D_\varphi$ is a first order differential operator given by

$$D_\varphi \psi = \left( \int_{\mathbb{R}^d} z [\psi(x + z) - \psi(x)] M(x, t, dz) \right) \cdot \nabla \varphi. \quad (2.15)$$

This is well defined provided $\psi \geq 0$, so $M_\psi$ is a (positive) $\sigma$-measure.

**Remark 2.6.** Sometimes, we want to consider the integro-differential operator $I$ acting on smooth functions with a growth at infinity higher than the exponential, e.g., square exponential growth. Then we need to add more conditions on the Levy kernel (or measure) $M(x, t, dz)$ in order to get estimates of type (2.5) with another weight function $w_r$. Revising the proof of Corollary 2.2, we see that if we suppose that

$$\int_{|z| > 1} \exp(r|z|^2) M(x, t, dz) \leq C_r \exp(r|x|^2), \quad \forall (x, t) \in Q_T, \quad (2.16)$$

for some $r > 0$ and a constant $C_r$, then an estimate similar to (2.5) holds for a square exponential growth function $w_r$. Moreover, if we assume that the Levy measure has compact support, with a nonempty interior subset $R(x, t) \subset \mathbb{R}^d$, then the integro-differential operator $I$ makes sense for any smooth function $\varphi$, and even more, for every positive $\varepsilon$, there exists a constant $C(\varepsilon)$ such that

$$|I_\varphi(x, t)| \leq \varepsilon \|\nabla^2 \varphi(\cdot, t)\|_{L^\infty(x + R(x, t))} + C(\varepsilon) \|\varphi(\cdot, t)\|_{L^\infty(x + R(x, t))} + \|\nabla \varphi(\cdot, t)\|_{L^\infty(x + R(x, t))}, \quad (2.17)$$

where $R(x, t)$ is the support of the Levy measure $M(x, t, dz)$ in $\mathbb{R}^d$. Furthermore, the condition on growth either exponential or polynomial for the first and second derivatives of the test function $\varphi$ are only necessary to deduce the estimates, but a bound on the growth of the test function is always required (when the subset $R(x, t)$ is unbounded) to produce finite values for $I_\varphi(x, t)$ at each point $(x, t)$.

3. **Extension of classic maximum principles.** The following results are extensions of classic corresponding results for differential operators, cf. Friedman [6], Nirenberg [24], Protter and Weinberger [25] and references therein.
Lemma 3.1. Let (1.7), . . . , (1.11) hold. Suppose that a smooth function $u$ satisfies $\mathcal{A}u < 0$ in a domain $D$ contained in $Q = \Omega \times (t_1, t_2)$, $0 \leq t_1 \leq t_2 \leq T$, and that $M$ is the maximum of $u$ in $Q$. If $u$ is not constant in $Q$ and one of the following conditions holds:

$$\begin{align*}
\text{(i)} & \quad a_0 M = 0 \text{ in } D, \\
\text{(ii)} & \quad a_0 \geq 0 \text{ in } D, \quad M > 0,
\end{align*}$$

then the maximum value of $u$ in $Q$ cannot be attained at an interior point of $D$.

Proof. At an interior maximum point $P^0 = (x^0, t^0)$ we have

$$\sum_{i,j=1}^{d} a_{ij} \partial_{ij} u(x^0, t^0) \leq 0, \quad Iu(x^0, t^0) \leq 0,$$

$$\partial_t u(x^0, t^0) = \partial_t u(x^0, t^0) = 0$$

and from (3.1) we have $a_0(x^0, t^0)u(x^0, t^0) \geq 0$. Hence $\mathcal{A}u(x^0, t^0) \geq 0$ in violation of $\mathcal{A}u < 0$. So the lemma holds true. \qed

In order to extend the maximum principle to the solution of the integro differential inequality $\mathcal{A}u \leq 0$, some auxiliary exponential functions (as in the differential case) play a fundamental role. This will be clear in the following lemma. This lemma is one of the basic results in establishing the strong version of the maximum principle given by Theorem 3.5. Denote by $M$ the maximum of $u$ in $Q_T$.

Lemma 3.2. Let (1.7), . . . , (1.11) hold. Suppose that a smooth function $u$ satisfies $\mathcal{A}u \leq 0$ in a domain $D$ contained in $Q_T$. Let $B$ be a $(d+1)$-dimensional ball of radius $R$ and center $\overline{B} = (\overline{x}, \overline{t})$ and let $P^0 = (x^0, t^0)$ be a point such that

$$P^0 \in \partial B, \quad \overline{B} \subset D, \quad u = M \text{ at } P^0, \quad u < M \text{ in } \overline{B} \setminus \{P^0\}. \tag{3.2}$$

If (3.1) holds, then $x^0 = \overline{x}$.

Proof. We shall assume that $x^0 \neq \overline{x}$ and reach a contradiction.

We construct a ball $B_1$ with center at $P^0$ and radius $R_1 < |x^0 - \overline{x}|$ and also such that $B_1$ lies completely in $D$. Then, since $x^0 \neq \overline{x}$,

$$|x - \overline{x}| \geq |x^0 - \overline{x}| - R_1 = \delta > 0, \quad \text{for all } (x, t) \in B_1. \tag{3.3}$$

The boundary of $B_1$ is composed of a part $C'$ lying in $\overline{B}$ (i.e., $C' = \partial B_1 \cap \overline{B}$) and a part $C''$ lying outside $B$ (i.e., $C'' = \partial B_1 \setminus C'$). We can find a positive constant $\eta$ so that

$$u \leq M - \eta \quad \text{on } \overline{B} \setminus B_1. \tag{3.4}$$

Introduce the auxiliary function

$$v(x, t) = \exp(-cq(x, t)) - \exp(-c(1 + R^2)^{\frac{1}{2}}),$$

$$q(x, t) = [1 + (x - \overline{x})^2 + (t - \overline{t})^2]^{\frac{1}{2}}, \tag{3.5}$$

where $c$ is a positive constant to be determined. We have $v > 0$ in $B$, $v = 0$ on $\partial B$, $v < 0$ outside $B$.

Computation shows that

$$\mathcal{A}v = \exp(-cq(x, t)) \{-c^2 a_{ij}(x_i - \overline{x}_i)(x_j - \overline{x}_j)q(x, t)^{-3}$$

$$+ c[(a_{ii} - a_i(x_i - \overline{x}_i) - (t - \overline{t})q(x, t)^{-1}] + a_0\} - Iv - a_0 \exp(-c(1 + R^2)^{\frac{1}{2}}). \tag{3.6}$$
Observe now that 
\[ Iv = I(v + \exp(-c(1 + R^2)^{\frac{1}{2}})), \]

hence by Lemma 2.4 and taking into account (1.9),
\[
Iv \geq \exp(-cq(x,t))I(-cq(x,t))
\]
\[
= -c \exp(-cq(x,t)) \int_{\mathbb{R}^d_+} [q(x+z,t) - q(x,t) - z \cdot \nabla q(x,t)] M(x,t,dz).
\]

Proceeding as in Lemma 2.1, cf. (2.2), we have
\[
\int_{\mathbb{R}^d_+} [q(x+z,t) - q(x,t) - z \cdot \nabla q(x,t)] M(x,t,dz)
\]
\[
= \int_0^1 d\theta \int_{|z|>1} z \cdot [(x - \bar{x} + \partial z)(1 + |x - \bar{x} + \partial z|^2 + (t - \bar{t})^2)^{-\frac{1}{2}}
\]
\[
- (x - \bar{x})(1 + |x - \bar{x}|^2 + (t - \bar{t})^2)^{-\frac{1}{2}}] M(x,t,dz)
\]
\[
+ \int_0^1 (1 - \theta) d\theta \int_{|z|\leq 1} |z|^2(1 + |x - \bar{x} + \partial z|^2 + (t - \bar{t})^2)^{-\frac{1}{2}}
\]
\[
- z_i(x_i - \bar{x}_i + \partial z_i)(x_j - \bar{x}_j + \partial z_j)z_j
\]
\[
(1 + |x - \bar{x} + \partial z|^2 + (t - \bar{t})^2)^{-\frac{3}{2}}] M(x,t,dz)
\]
\[
\leq 2 \int_{|z|>1} |z| M(x,t,dz) + 2 \int_{|z|\leq 1} |z|^2 M(x,t,dz) \leq c_0,
\]

where \(c_0\) is a positive constant obtained from (1.9).

So we can conclude that for the function \(v\) given by (3.5) we have
\[
Iv \geq -cc_0 \exp(-cq(x,t)).
\]

Taking this into account, from (3.6) we derive that
\[
Av \leq c \exp(-cq(x,t))[-c\mu |x - \bar{x}|^2(1 + |x - \bar{x}|^2 + (t - \bar{t}))^{-\frac{3}{2}}
\]
\[
+ [a_{ii} - a_i(x_i - \bar{x}_i) - (t - \bar{t})](1 + |x - \bar{x}|^2 + (t - \bar{t}))^{-\frac{3}{2}}
\]
\[
- c_0 + a_n(c)(1 - \exp(-c(1 + R^2)^{\frac{1}{2}} - q(x,t))))].
\]

Using that
\[
0 < \delta \leq |x - \bar{x}| \leq R + R_1,
\]

we see that it is possible to choose \(c\) so large that
\[
(A + Q_0)v < 0 \quad \text{in} \quad B_1.
\]

Consider the function \(w = u + \varepsilon v - M\) where \(\varepsilon\) is a positive constant to be chosen.

By means of (3.4) we have, for \(\varepsilon\) sufficiently small, \(w < 0\) on \(\overline{B}\setminus B_1\). On \(Q_T\setminus B\), \(u \leq M\) and \(v < 0\), hence \(w < 0\) on \(Q_T\setminus B_1\). Thus \(w < 0\) on \(\partial B_1\), \((A + Q_0)v < 0\) in \(B_1\) if (3.1) holds.

On the other hand, since \(v = 0\) on \(\partial B\), we find \(w(x^0, t^0) = 0\). Hence the maximum of \(w\) in \(Q_T\) must occur at an interior point in \(B_1\). This fact contradicts Lemma 3.1; hence \(x^0 = \bar{x}\). \(\square\)

By means of Lemma 3.2 proceeding exactly as in the differential case (e.g., Lemma III.2 in Protter and Weinberger [25]) we can prove
Lemma 3.3. Let (1.7), . . . , (1.11) and (3.1) hold. Suppose that a smooth function satisfies $Au \leq 0$ in a domain $D$ contained in $Q_T$ and that $M$ is the maximum of $u$ in $\overline{Q}_T$. If there is a single interior point $P$ where $u = M$, then $u = M$ in any point $Q$ of $D$ which can be connected to $P$ by a path in $D$ consisting of horizontal segments (i.e., segments with constant $t$).

Finally we have

Lemma 3.4. Let (1.7), . . . , (1.11) and (3.1) hold. Suppose that $Au \leq 0$ in the lower half of a ball $B$ centered at $P = (x, \bar{t})$ contained in $\overline{Q}_T$, i.e.,
\[
B_\bar{t}(R) = \{(x, t) : (x - x)^2 + (t - \bar{t})^2 < R^2, \quad \bar{t} \leq \bar{t} \} \subset \overline{Q}_T,
\]
and that $M$ is the maximum of $u$ in $\overline{Q}_T$. If $u < M$ in the portion of $B$ where $t < \bar{t}$, then $u(P) < M$.

Proof. We introduce the auxiliary function
\[
v(x, t) = \exp(-q(x) - c(t - \bar{t})) - \exp(-1), \quad q(x) = (1 + |x - x|^2)^{\frac{1}{2}}, \quad (3.9)
\]
where $c$ is a positive constant to be determined.

We shall proceed in three steps.

Step 1: we show that we can choose the constant $c$ so large that
\[
(A + Q_0^v)v < 0 \quad \text{in} \quad B_\bar{t}(R). \quad (3.10)
\]

Indeed
\[
Av = \exp(-q(x) - c(t - \bar{t}))[-a_{ij}(x_i - x_i)(x_j - x_j)(q(x) - q(x)^2) + a_{ij}q(x)^{-1} - a_{ij}(x_i - x_i)q(x)^{-1} - c] - Iv + aqv. \quad (3.11)
\]

Proceeding as in the previous lemma, we have
\[
Iv \geq \exp(-q(x) - c(t - \bar{t}))I(-q) = -\exp(-q(x) - c(t - \bar{t})) \int_{\mathbb{R}^d} [q(x + z) - q(x) - z \cdot \nabla q(x)] \times M(x, t, dz) \geq -c_0 \exp(-q(x) - c(t - \bar{t})), \quad (3.12)
\]
where $c_0$ is a positive constant depending only on the Levy kernel.

From (3.11), taking into account (3.12) and choosing the constant $c$ large enough, inequality (3.10) follows.

Step 2: we remark that the surface
\[
(1 + |x - x|^2)^{\frac{1}{2}} + c(t - \bar{t}) = 1 \quad (3.13)
\]
is tangent to the hyperplane $t = \bar{t}$ at point $P$. We denote by $S'_r$ the portion of $\partial B(r)$, $r \leq R$, which is below the surface (3.13), by $S''_r$ the portion of the surface (3.13) located within the ball $B(r)$ and by $E_r$ the region included between $S'_r$ and $S''_r$. Fix $r < R$ such that $S'_r$ is above the hyperplane containing $S'_R \cap S''_R$. Clearly there exists a positive $\eta$ such that
\[
u \leq M - \eta \quad \text{on} \quad E_r \setminus E_r,
\]
Consider the function
\[ w(x, t) = u(x, t) + \varepsilon v(x, t), \]
where \( \varepsilon \) is a positive constant to be chosen and \( v \) is given by (3.9).

Notice that
\[ v = 0 \quad \text{on} \quad S''_R \quad \text{and} \quad v > 0 \quad \text{in} \quad E_R. \]

Thus we may choose \( \varepsilon \) so small that
\[
\begin{align*}
(\text{i}) & \quad (A + Q_-)w < 0 \quad \text{in} \quad E_r \\
(\text{ii}) & \quad w < M \quad \text{in} \quad S'_r \\
(\text{iii}) & \quad w < M \quad \text{on} \quad S''_r \setminus \overline{\mathcal{P}}, \quad w = M \quad \text{at} \quad \mathcal{P}.
\end{align*}
\]

From (3.14) taking into account Lemma 3.1 it follows that the maximum of \( w \) in
\( \overline{\Omega} \times [t - r, t] \) is \( M \) and it occurs at the point \( \mathcal{P} \). Then elementary considerations show that
\[ \partial_t w = \partial_t u - \varepsilon c \geq 0 \quad \text{at} \quad \mathcal{P}. \]

Thus, we conclude that \( \partial_t u > 0 \) at \( \mathcal{P} \); on the other hand
\[ \partial_t u = 0, \quad a_{ij} \partial_{ij} u \leq 0, \quad Iu \leq 0 \quad \text{at} \quad \mathcal{P}. \]

These inequalities and conditions (3.1) imply
\[ Au > a_0 M \geq 0 \quad \text{at} \quad \mathcal{P}. \]

This fact contradicts the hypothesis that \( Au \leq 0 \) in \( B_R(R) \), hence \( u(\mathcal{P}) < M. \)

At this point, as a consequence of the above results, the classic (or strong) maximum principle of Nirenberg type can be proved for our integro-differential operator. Similarly we can prove the Hopf boundary point lemma and the several versions of the maximum principle for classical solutions in a bounded domain.

Only some outline of the proofs will be given, especially when tools different from those of the differential case will be used. For the other ones we refer the reader to the classical proofs.

The maximum principle asserts:

**Theorem 3.5.** (Extension of the Classic Maximum Principle). Let (1.7), . . . , (1.11) hold. Suppose that a smooth function \( u \) satisfies \( Au \leq 0 \) in a domain \( D \) contained in \( \Omega_r \) and that the maximum of \( u \) in \( \Omega_r \) is \( M \) and is attained at some interior point \( P^0 = (x^0, t^0) \) of \( D \). If \( P \) is a point of \( D \) which can be connected to \( P^0 \) by a path in \( D \) consisting only of horizontal segments (i.e., \( t \) constant) and upward vertical segments (i.e., \( x \) constant and \( t \) increasing), then \( u = M \) in \( P \), whenever one of the following conditions holds:

\[
\begin{align*}
(\text{i}) & \quad \text{either } a_0 M = 0 \quad \text{in} \quad D, \\
(\text{ii}) & \quad \text{or } a_0 \geq 0 \quad \text{in} \quad D, \quad M > 0.
\end{align*}
\]

The conclusion is also valid if the point \( P^0 = (x^0, t^0) \) is on a horizontal component (say \( D(t^0) \)) of the boundary \( \partial D \) of \( D \), provided \( u, u_x, u_{x_i}, u_{x_i x_j} \) and \( u_t \) are all continuous on \( D \cup D(t^0) \).

**Proof.** Taking into account Lemma 3.3 we prove only the case of vertical line segments (i.e., segments with \( x \) constant). Suppose that \( u(x^0, t^1) < M \) and that the line \( l = \{(x, t) : x = x_0, t_1 \leq t \leq t_0\} \) lies in \( D \). Let \( \tau \) be the least upper bound of values of \( t < t^0 \) on \( l \) such that \( u(x^0, t) < M \). By continuity \( u(x^0, \tau) = M \), while \( u(x^0, t) < M \) for some interval \( \tau^1 < t < \tau \). Lemma 3.3 shows that there is an \( R > 0 \) such that \( u < M \) for \( |x - x_0| < R, \tau^1 < t < \tau \). This contradicts Lemma 3.4.
Theorem 3.6. (Extension of the Hopf Boundary Point Principle). Let conditions (1.7)–(1.11) and (3.15) hold. Suppose \( Au \leq 0 \) in \( D \) and that the maximum value \( M \) of \( u \) in \( Q_T \) is attained at a point \( P_0 = (x^0, t^0) \) on the boundary \( \partial D \), where \( u \) is differentiable. Assume that we can construct an open ball \( B \) with the following properties:

\[
\begin{align*}
(i) & \quad P_0 \in \partial B, \text{ which is tangent to } \partial D, \\
(ii) & \quad \{(x, t) \in B : t \leq t^0\} \subset \{(x, t) \in D : u(x, t) < M\}, \\
(iii) & \quad \text{if } (\bar{x}, \bar{t}) \text{ is the center of } B \text{ then } \bar{x} \neq x^0.
\end{align*}
\]

If \( \frac{\partial u}{\partial \ell} \) denotes any directional derivative in an outward direction from the region \( \{(x, t) \in D : t \leq t^0\} \) and one of the conditions in (3.15) holds, then

\[
\frac{\partial u}{\partial \ell} > 0 \text{ at } P_0.
\]

Proof. We introduce an auxiliary function as in (3.5). Since (3.8) holds we prove the claim as in the differential case, e.g., Friedman [6]. \( \square \)

By using Theorems 3.5 and 3.6 we can obtain an important version of the strong maximum principle in a cylindrical domain \( Q_T = \Omega \times (0, T) \). Suppose that the domain \( \Omega \) of \( \mathbb{R}^d \) is bounded with a \( C^1 \) boundary \( \partial \Omega \). Consider the operator \( B \) given by

\[
Bu = b_i(x, t)\partial_i u + b_0(x, t)u,
\]

satisfying

\[
b_i(x, t)\ell_i(x) \geq c_0 > 0
\]

and

\[
b_i \in C^0(\overline{Q_T}), \quad i = 0, \ldots, d.
\]

Theorem 3.7 (Extension of the Strong Maximum Principle). Let \( \Omega \) be a bounded smooth (i.e., with \( C^1 \) boundary) domain and (1.7), . . . , (1.11), (3.18) and (3.19) hold. Suppose that a smooth function \( u \) satisfies \( Au \leq 0 \) in \( \Omega \times (0, T] \), \( Bu \leq 0 \) on \( \Sigma_T \), and that \( M \) is the maximum value of \( u \) in \( Q_T \). If \( u \) is not constant and one of the following conditions holds:

\[
\begin{align*}
(i) & \quad \text{either } a_0M = 0 \quad \text{and } \quad b_0M = 0, \\
(ii) & \quad \text{or } \quad a_0 \geq 0, \quad b_0 \geq 0, \quad M > 0,
\end{align*}
\]

then the maximum value \( M \) of \( u \) in \( Q_T \) cannot be attained on \( (\Omega \times (0, T]) \cup \Sigma_T \). Moreover, if \( u \) is a constant then \( u \leq 0 \), whenever

\[
\max_{Q_T} a_0(x, t) + \max_{\Sigma_T} b_0(x, t) > 0.
\]

Proof. Clearly, this is a consequence of Theorems 3.5 and 3.6.

Remark 3.8. In all the above results the coefficients \( a_{ij} \) and \( a_i \), might have been taken in \( L^\infty(Q_T) \) instead of \( C^0(\overline{Q_T}) \) and \( a_0 \) bounded from below. On the other hand, similarly to the differential case, we can assume the mixed boundary condition

\[
h(x, t)\frac{\partial u}{\partial b} + b_0(x, t)u(x, t) = \psi(x) \quad \text{on } \Sigma_T,
\]
with \( b \equiv (b_1, \ldots, b_d) \) satisfying (3.18) and
\[
h(x, t) \geq 0, \quad h^2 + b_0^2 > 0.
\]
If we set \( \Gamma \equiv \{ x \in \partial \Omega : h(x, t) > 0 \} \), then the conclusions of Theorem 3.7 are also valid, when \( \Gamma \) is replacing \( \partial \Omega \).

We present here some inequalities for classical solutions of boundary value problems obtained by using only Lemma 3.1. The proofs are similar to the original proofs for differential operators found in Ladyženskaja et al. [17]; cf. also [10].

**Theorem 3.9.** Let \( \Omega \) be a bounded domain. Suppose that \( u(x, t) \) is the classical solution of the problem \( Au = f \) on \( \Omega \times (0, T) \), \( u(x, 0) = \varphi(x) \) in \( \Omega \), \( u(x, t) = \psi(x, t) \) in \( \Sigma_T \), where the coefficients and the free terms are bounded functions,
\[
a_{ij}(x, t)\xi_i\xi_j \geq 0, \quad (3.21)
\]
and \( I \) satisfies (1.9), \ldots , (1.11). Then for \( \tau \in [0, T] \) and \( \xi \in \overline{\Omega} \) the following estimate holds:
\[
u(\xi, \tau) \leq \inf_{\lambda > \lambda_0} \left\{ \left. 0 \right| \max_{\overline{\Omega}} \varphi(x)e^{\lambda \tau} \max_{\Sigma_T} \psi(x, t)e^{\lambda(\tau - t)} \max_{Q_\tau} \frac{f(x, t)e^{\lambda(\tau - t)}}{\lambda - \lambda_0} \right\}, \quad (3.22)
\]
where \( \lambda_0 = \max_{Q_\tau} (-a_{0}(x, t)) \).

From this theorem one obtains different formulations of the weak maximum principle.

**Corollary 3.10 (Extension of the Weak Maximum Principle).** Suppose that the conditions of Theorem 3.9 are satisfied.

If \( u \leq 0 \) in \( \partial Q_T \) and \( f \leq 0 \) then \( u \leq 0 \) in \( \overline{Q_T} \); (3.23)

If \( a_0(x, t) \geq 0 \) in \( Q_T \) and \( f \leq 0 \) then \( \max_{\partial Q_T} u \leq \max_{\partial Q_T} u_+ \); (3.24)

If \( a_0(x, t) \equiv 0 \) in \( Q_T \) and \( f \leq 0 \) then \( \max_{\overline{Q_T}} u \leq \max_{\partial Q_T} u_+ \); (3.25)

for any \( \tau \in [0, T] \).

We now consider the oblique boundary problem. The following theorem is relative to this problem.

**Theorem 3.11.** Suppose that \( u(x, t) \) is a classic solution of problem \( Au = f \) in \( \Omega \times (0, T) \), \( u(x, 0) = \varphi(x) \) in \( \Omega \), \( Bu = \psi \) in \( \Sigma_T \). Suppose that \( \Omega \) is bounded, that \( a_{ij}, a_i, a_0, b_i, b_0, f, \varphi \) and \( \psi \) are bounded, with \( a_{ij} \) satisfying (3.21), \( b_0/\Sigma_T > 0 \) and \( I \) satisfies (1.9), \ldots , (1.11). Then for \( \tau \in [0, T] \) and \( \xi \in \overline{\Omega} \),
\[
u(\xi, \tau) \leq \inf_{\lambda > \lambda_0} \left\{ \left. 0 \right| \max_{\Sigma_T} \varphi(x)e^{\lambda \tau} \max_{\overline{\Omega}} \psi(x, t)e^{\lambda(\tau - t)} \max_{Q_\tau} \frac{f(x, t)e^{\lambda(\tau - t)}}{\lambda - \lambda_0} \right\}, \quad (3.26)
\]
where \( \lambda_0 = \max_{Q_T} (-a_0(x, t)) \).

4. **Unbounded domain.** If \( Q_T \) is an unbounded domain and if we suppose that in \( Q_T \) the function \( u \) satisfies the integro-differential inequalities \( Au \leq 0 \), the maximum principle as given in Theorems 3.5 and 3.6 is still applicable to the function \( u \). However, since \( \Omega \) is unbounded, we cannot always conclude that the maximum of \( u \) occurs either at \( t = 0 \) or on \( \Sigma_T = \partial \Omega \times [0, T] \), as is the case of bounded domains (cf. Theorem 3.7), so that we can obtain uniqueness results only when the solutions are required to satisfy certain conditions at infinity. A first result concerns bounded solutions (see Ladyženskaja et al. [17] for the differential case).
Theorem 4.1. Let $\Omega$ be a possibly unbounded domain and $\Gamma$ be a part (possibly empty) of the boundary $\partial \Omega$. Suppose that a smooth bounded function $u$ satisfies

\[
\begin{cases}
\partial_t u - Lu - I u \leq f & \text{in } \Omega \times (0, T], \\
u(\cdot, 0) \leq \varphi & \text{in } \Omega, \\
u \leq \psi_1 & \text{on } (\partial \Omega \setminus \Gamma) \times (0, T], \\
Bu \leq \psi_2 & \text{on } \Gamma \times (0, T],
\end{cases}
\]

where the functions $\varphi, \psi_1, \psi_2$ and $f$ are nonnegative and bounded. Assume that $I$ satisfies (1.9), (1.11), that $a_0$ and $b_0$ are bounded from below and that the coefficients $a_{ij}, a_i, b_i$ are bounded, and satisfy

\[
a_{ij}(x,t)\xi_i \xi_j \geq 0, \quad \forall \xi \in \mathbb{R}^d, \quad (x,t) \in \overline{Q_T},
\]

\[
b_i(x,t)m_i(x) \geq c_0 > 0, \quad \forall (x,t) \in \Gamma \times [0, T],
\]

where $n = (n_1, \ldots, n_d)$ is the unit outward normal to $\partial \Omega$. The domain $\Omega$ is such that the distance $d(x)$ to the boundary $\partial \Omega$ is a function of class $C^2$ in a neighborhood of $\Gamma$. Then there exists a constant $C$ depending on the bounds of the coefficients $a_{ij}, a_i, b_i$, the bound from below of the coefficients $a_0$, $b_0$ and the bound of the first and second derivatives of the function $d(x)$, such that for any $(x,t)$ in $\overline{Q_T}$ we have

\[
u(x,t) \leq C\left[\max_{\Omega} \varphi \vee \max_{\Sigma_T} \psi \vee \max_{Q_T} f\right],
\]

where $\psi = \psi_1$ on $(\partial \Omega \setminus \Gamma) \times [0, T]$, $\psi = \psi_2$ on $\Gamma \times [0, T]$.

Proof. Denote by $\rho(x)$ a function of class $C^2_b(\mathbb{R}^d)$ such that $\rho(x) = d(x)$ on a neighborhood of $\Gamma$ so that

\[
\rho(x) = 0 \quad \text{and} \quad \nabla \rho(x) = n(x), \quad \forall x \in \Gamma,
\]

where $n(x)$ is the outward (unit) normal to $\partial \Omega$.

Consider the function

\[
v(x,t) = u(x,t)(1 + |x|^2)^{-r/2} \exp(-\alpha t - \beta \rho(x)),
\]

where $\alpha$, and $\beta$ are positive constants to be selected, and $0 < r < 1$. We have

\[
(1 + |x|^2)^{-r/2} \exp(-\alpha t - \beta \rho(x))(\partial_t - L - I)u = (\partial_t - \tilde{L} - \tilde{I})v,
\]

where the coefficients of the second order parabolic operator $\tilde{L}$ are $\tilde{a}_{ij}, \tilde{a}_i, \tilde{a}_0$ and the Levy kernel of the integro-differential operator $\tilde{I}$ is $\tilde{M}(x,t,dz)$. Setting

\[
w(x) = (1 + |x|^2)^{r/2} \exp(\beta \rho(x))
\]

and in view of Remark 2.5 we obtain that

\[
\tilde{a}_{ij}(x,t) = a_{ij}(x,t),
\]

\[
\tilde{a}_i(x,t) = a_i(x,t) + 2a_{ij}(x,t) \frac{\partial_j w(x)}{w(x)} + \int_{\mathbb{R}^d} z_i[w(x+z)/w(x) - 1]M(x,t,dz),
\]

\[
\tilde{a}_0(x,t) = \alpha - (Lw)(x,t)/w(x) - (Iw)(x,t)/w(x)
\]
and
\[ \tilde{M}(x, t, A) = \int_A w(x + z)/w(x)M(x, t, dz). \]

Notice that the coefficients of \( \tilde{L}, \tilde{I} \) satisfy the same assumptions as those of \( L, I \), since \( \rho \) is of class \( C^2_b(\mathbb{R}^d) \) and \( 0 < r \leq 1 \).

Similarly, we have
\[ (1 + |x|^2)^{-r/2} \exp(-\alpha t - \beta \rho(x))Bu = \tilde{B}v, \]
where the coefficients of the boundary first order differential operator \( \tilde{B} \) are given by
\[ \tilde{b}_i(x, t) = b_i(x, t), \quad \tilde{b}_0(x, t) = b_0(x, t) + b(x, t) \cdot \nabla w(x)/w(x). \]

We claim that we can choose \( \alpha, \beta \) sufficiently large so that for any \( 0 < r \leq 1 \) we have
\[ \begin{cases} \tilde{a}_0(x, t) \geq 1, & \forall (x, t) \in \overline{Q}_T, \\ \tilde{b}_0(x, t) \geq 1, & \forall (x, t) \in \Gamma \times [0, T]. \end{cases} \tag{4.7} \]
Indeed, we have
\[ \tilde{b}_0(x, t) = b_0(x, t) + \beta b(x, t) \cdot \nabla \rho(x) + rb_i(x, t)x_i(1 + |x|^2)^{-1} \]
\[ \geq b_0(x, t) + \beta c_0 - |b(x, t)|/(1 + |x|^2)^{-1/2}, \]
so the second part of (4.7) holds for a constant \( \beta \) depending only on the bounds of \( b_0, \beta \) and on the constant \( c_0 \) of assumption (4.3).

To prove the first part of claim (4.7), we proceed in two steps. First, computation shows that
\[ \partial_i w(x) = [\beta \partial_i \rho(x) + rx_i(1 + |x|^2)^{-1}]w(x), \]
\[ \partial_{ij} w(x) = [\beta \partial_{ij} \rho(x) + \beta^2 \partial_i \rho(x) \partial_j \rho(x) + r\delta_{ij}(1 + |x|^2)^{-1} \]
\[ + r\beta(x_j \partial_i \rho(x) + x_i \partial_j \rho(x))(1 + |x|^2)^{-1} + \]
\[ + r(r - 2)x_i - x_j(1 + |x|^2)^{-2}]w(x), \]
which implies that for any \( 0 < r \leq 1 \) we have
\[ |a_{ij}(x, t)\partial_{ij} w(x) + a_i(x, t)\partial_i w(x)| \leq C_1(\beta)w(x), \tag{4.8} \]
where the constant \( C_1(\beta) \) depends only on the bounds of the coefficients \( a_{ij}, a_i \), the bounds of the function \( \partial_i \rho, \partial_{ij} \rho \) and on the parameter \( \beta \).

The second step is the estimate of \( \tilde{I}w \). In view of the decomposition
\[ \tilde{I}w(x, t) = \int_{|z| > 1} [w(x + z) - w(x) - z \cdot \nabla w(x)]M(x, t, dz) \]
\[ + \int_0^1 (1 - \theta)d\theta \int_{|z| \leq 1} z \cdot \nabla^2 w(x + \theta z)zM(x, t, dz) \]
and the inequality
\[ (1 + |x + \theta z|^2)^{1/2} \leq \sqrt{2}(1 + |x|^2)^{1/2}(1 + |\theta z|^2)^{1/2}, \]
first we get for any $0 < r \leq 1$,
\[
|w(x + z)| \leq C_2(\beta)|z|w(x), \quad \text{if } |z| > 1,
\]
\[
|\nabla^2 w(x + \theta z)| \leq C_2(\beta)w(x), \quad \text{if } |z| \leq 1,
\]
where the constant $C_2(\beta)$ depends only on the bounds of the functions $\rho, \partial_1 \rho, \partial_2 \rho$ and $\beta$. Next, we deduce
\[
|Iw(x, t)| \leq C_3(\beta)w(x),
\]
where the constant $C_3(\beta)$ depends only on the bounds of the functions $\partial_1 \rho, \partial_2 \rho$, on the constant of the assumption $(1.9)$ on the Levy kernel $M(x, t, dz)$, and on $\beta$.

Thus to prove the claim $(4.4)$ we notice that from $(4.8)$ and $(4.9)$ we deduce
\[
\bar{a}_0(x, t) \geq \alpha - \alpha_0(\beta),
\]
where the constant $\alpha_0(\beta)$ depends only on the bounds of the coefficients $a_{ij}, a_i$, the bounds of the functions $\rho, \partial_1 \rho, \partial_2 \rho$, on the constant of the assumption $(1.12)$ on the Levy kernel and on $\beta$.

Hence, by virtue of the inequalities satisfied by the function $u$, the new function $v$ given by $(4.5)$ satisfies
\[
\begin{aligned}
&\begin{cases}
(\partial_t - \tilde{L} - \tilde{I})v \leq \tilde{f} & \text{in } \Omega \times (0, T], \\
v(\cdot, 0) \leq \tilde{\varphi} & \text{in } \Omega, \\
v \leq \tilde{\psi}_1 & \text{on } (\partial \Omega \setminus \Gamma) \times [0, T], \\
\tilde{B}u \leq \tilde{\psi}_2 & \text{on } \Gamma \times [0, T],
\end{cases}
\end{aligned}
\]
where the zero-order coefficients $\tilde{a}_0, \tilde{b}_0$ of $\tilde{L}, \tilde{B}$ satisfy $(4.7)$, and
\[
\begin{aligned}
&\begin{cases}
\tilde{f}(x, t) = (1 + |x|^2)^{-r/2} \exp(-\alpha t - \beta \rho(x))f(x, t), \\
\tilde{\varphi}(x) = (1 + |x|^2)^{-r/2} \exp(-\beta \rho(x))\varphi(x), \\
\tilde{\psi}_1(x, t) = (1 + |x|^2)^{-r/2} \exp(-\alpha t)\psi_1(x), \\
\tilde{\psi}_2(x, t) = (1 + |x|^2)^{-r/2} \exp(-\alpha t)\psi_2(x),
\end{cases}
\end{aligned}
\]
We claim that
\[
v(x, t) \leq (\sup_{\Omega_T} \tilde{f}) \vee (\sup_{\Sigma_T} \tilde{\varphi}) \vee (\sup_{\Sigma_T} \tilde{\psi}), \quad \forall (x, t) \in \Omega_T,
\]
where $\tilde{\psi} = \tilde{\psi}_1\chi_{\partial \Omega \setminus \Gamma} + \tilde{\psi}_2\chi_{\Gamma}$. Indeed, we denote by $C(\tilde{f}, \tilde{\varphi}, \tilde{\psi})$ the right hand side of $(4.13)$ and we set
\[
h(x, t) = v(x, t) - C(\tilde{f}, \tilde{\varphi}, \tilde{\psi});
\]
we obtain
\[
(\partial_t - \tilde{L} - \tilde{I})h \leq \tilde{f} - \tilde{a}_0C(\tilde{f}, \tilde{\varphi}, \tilde{\psi}) \leq 0 \quad \text{in } \Omega_T \cup (\Omega \times \{T\}),
\]
\[
h(\cdot, 0) \leq \tilde{\varphi} - C(\tilde{f}, \tilde{\varphi}, \tilde{\psi}) \leq 0 \quad \text{in } \Omega,
\]
\[
h \leq \tilde{\psi}_1 - C(\tilde{f}, \tilde{\varphi}, \tilde{\psi}) \leq 0 \quad \text{on } (\partial \Omega \setminus \Gamma) \times [0, T],
\]
and
\[
\tilde{B}h \leq \tilde{\psi}_2 - \tilde{b}_0C(\tilde{f}, \tilde{\varphi}, \tilde{\psi}) \leq 0 \quad \text{on } \Gamma \times [0, T].
\]
Therefore, because \( v \) vanishes at infinity (i.e., \( v(x, \cdot) \to 0 \) as \( |x| \to \infty, \ x \in \Omega \)) the maximum value of \( w \) should be attained over \( \overline{Q}_T \) if it is positive. But, if a positive maximum value is attained over \( \overline{Q}_T \), then it will contradict one of the four conditions (4.14), . . . , (4.17). Thus \( h \) is not positive and the claim (4.13) is proved.

Finally, we let \( r \) vanish in (4.13) to obtain the desired estimate (4.4).

**Remark 4.2.** The above theorem applies also when \( \Gamma = \emptyset \), in this case the domain \( \Omega \) need not be smooth and the function \( \rho \) is useless. Moreover, if \( \Omega = \mathbb{R}^d \) we obtain the classic estimates for the Cauchy problem. It is clear that if \( u \) is a solution of an equation, then we can use (4.4) for \(-u\) to deduce a bound on the \( L^\infty \) norm of \( u \).

Uniqueness results follow.

The case of unbounded functions with growth either polynomial or exponential is considered now.

**Theorem 4.3.** (A Priori Estimates). Let the assumptions be the same as in Theorem 4.1 except that the function \( u \) satisfying inequality (4.1) is smooth but not necessarily bounded, and it may grow either exponentially or polynomially at infinity (as well as the data \( f, \varphi, \psi \)), i.e., either (i) or (ii) below holds:

\[
\begin{align*}
(i) & \sup_{(x,t) \in Q_T} |u(x,t)| \exp(-r|x|) < \infty, \\
(ii) & \sup_{(x,t) \in Q_T} |u(x,t)|(1 + |x|^2)^{-r/2} < \infty,
\end{align*}
\]

for some positive constant \( r \). In each case, we add one extra condition on the Levy kernel of the integro-differential operator \( I \), namely

\[
\begin{align*}
(i) & \text{ either } \int_{|z|>1} |z| \exp(r|z|) M(x,t,dz) \leq C, \quad \forall (x,t) \in Q_T, \\
(ii) & \text{ or } \int_{|z|>1} |z|(1 + |z|^2)^{r/2} M(x,t,dz) \leq C, \quad \forall (x,t) \in Q_T,
\end{align*}
\]

for some constant \( C \). Then there exist positive constants \( \alpha, \beta \) such that with the weight functions

\[
\begin{align*}
\begin{cases}
(i) \text{ either } & w_r(x,t) = \exp[-\alpha t - \beta \rho(x)] - r(1 + |x|^2)^{1/2}, \\
(ii) \text{ or } & w_r(x,t) = (1 + |x|^2)^{-r/2} \exp[-\alpha t - \beta \rho(x)],
\end{cases}
\end{align*}
\]

we have the estimate

\[
\sup_{Q_T} u w_r \leq (\sup_{Q_T} f w_r) \vee (\sup_{\Omega \times \{0\}} \varphi w_r) \vee (\sup_{\Sigma_T} \psi w_r);
\]

the function \( \rho \) is the same function used in Theorem 4.1, i.e., a smooth extension of the distance function to the boundary \( \partial \Omega \) in a neighborhood of \( \Gamma \) (cf. condition (4.24) below).

**Proof.** First notice that the difference between (4.20) and (2.3) of Section 2 for the function \( w_r \) is a bounded multiplicative function, namely \( \exp[-\alpha t - \beta \rho(x)] \). Then by means of the trivial inequality \( 0 \leq (1 + |x|^2)^{1/2} - |x| \leq 1 \), we conclude that in both cases (i) and (ii) the function \( u(x,t)w_r(x,t) \) is bounded under assumption (4.18).

The case of polynomial growth is essentially identical to Theorem 4.1. The only point to remark on is the use of assumption (4.19) in order to define the new Levy kernel \( \tilde{M} \). Similarly, assumption (4.18) on the function \( u \) is used to give sense to the nonlocal expression \( Iu \). This can be deduced by splitting the operator \( I \) into the small and the large jumps; cf. proof of Lemma 2.1 and Remark 2.6.
For the case of exponential growth we revise the steps of Theorem 4.1 with the function \( w(x) \) defined by

\[
w(x) = \exp[\beta \rho(x) + r(1 + |x|^2)^{1/2}].
\]

(4.22)

We set

\[
v(x, t) = u(x, t) \exp[-\alpha t - \beta \rho(x) - r(1 + |x|^2)^{1/2}],
\]

(4.23)

where \( \rho \) is a function in \( \mathbb{R}^d \) such that

\[
\begin{cases}
\rho, \nabla \rho, \nabla^2 \rho \text{ are continuous and bounded,} \\
\rho(x) = 0 \text{ and } \nabla \rho(x) = n(x), \quad \forall x \in \Gamma,
\end{cases}
\]

(4.24)

where \( n(x) \) is the outward (unit) normal to \( \partial \Omega \). Since \( v = u w_r \), computation shows that

\[
w_r A u = \tilde{A} v \quad \text{and} \quad w_r B u = \tilde{B} v,
\]

(4.25)

where now the coefficients and the Levy measure are given as in Theorem 4.1, but with \( w(x) \) defined by (4.22). Next, in view of

\[
\begin{align*}
\partial_t w(x) &= [\beta \partial_t \rho(x) + rx_i(1 + |x|^2)^{-1/2}] w(x) \\
\partial_{ij} w(x) &= [\beta \partial_{ij} \rho(x) + \beta^2 \partial_i \rho(x) \partial_j \rho(x) + r \delta_{ij}(1 + |x|^2)^{-1/2} \\
&\quad + \beta r (x_j \partial_i \rho(x) + x_i \partial_j \rho(x))(1 + |x|^2)^{-1/2} \\
&\quad - rx_i x_j (1 + |x|^2)^{-3/2} + r^2 x_i x_j (1 + |x|^2)^{-1}] w(x),
\end{align*}
\]

(4.26)

we obtain that

\[
\left| \sum_{i,j=1}^d a_{ij}(x, t) \partial_{ij} w(x) + \sum_{i=1}^d a_i(x, t) \partial_i w(x) \right| \leq C_1(\beta) w(x),
\]

(4.27)

where the constant \( C_1(\beta) \) depends only on \( r \), the bounds of the coefficients \( a_{ij}, a_i \), the bounds of the functions \( \partial_i \rho, \partial_{ij} \rho \) and clearly on the parameter \( \beta \). By means of the inequality

\[
|(1 + |x + z|^2)^{1/2} - (1 + |x|^2)^{1/2}| \leq |z|
\]

(4.28)

first we get

\[
w(x + z) \leq C_2(\beta) \exp(r|z|) w(x),
\]

\[
|\nabla w(x)| \leq C_2(\beta) w(x),
\]

\[
|\nabla^2 w(x + z)| \leq C_2(\beta) w(x), \quad \text{if } |z| \leq 1,
\]

and then

\[
|I w(x, t)| \leq C_3(\beta) w(x),
\]

(4.29)

where the constant \( C_3(\beta) \) depends only on \( r \), the bounds of the functions \( \partial_i \rho, \partial_{ij} \rho \), the constants in assumptions (1.9), (4.19) on the Levy kernel \( M(x, t, dz) \) and clearly on the parameter \( \beta \).

The claim is that for any \( r > 0 \) we can choose \( \alpha, \beta \) sufficiently large so that

\[
\begin{cases}
\tilde{a}_0(x, t) \geq 1, & \forall (x, t) \in \overline{Q}_T, \\
\tilde{b}_0(x, t) \geq 1, & \forall (x, t) \in \Gamma \times [0, T].
\end{cases}
\]

(4.30)
Indeed, from the above calculation we deduce that
\[ \tilde{a}_0(x, t) \geq \alpha - a_0(\beta), \]
where the constant \( a_0(\beta) \) depends only on the constants \( C_1(\beta), \ C_3(\beta) \) and the lower bound of the coefficient \( a_0 \). This proves the first part of the claim. To prove the second part of the claim, we notice that
\[ \tilde{b}_0(x, t) = b_0(x, t) + \beta \sum_{i=1}^d b_i(x, t) \partial_i \rho(x) + r \sum_{i=1}^d b_i(x, t) x_i (1 + |x|^2)^{-1/2} \]
\[ \geq b_0(x, t) + \beta c_0 - r(\sum_{i=1}^d |b_i(x, t)|^2)^{1/2}, \]
where \( c_0 \) is the constant in assumption (4.3).

By virtue of the inequalities satisfied by the function \( u \), the new function \( v = u w_r \) satisfies
\[
\begin{align*}
\tilde{A}v &\leq f w_r \quad \text{in } \Omega \times (0, T], \\
v(\cdot, 0) &\leq \varphi w_r \quad \text{in } \Omega, \\
v &\leq \psi_1 w_r \quad \text{on } (\partial \Omega \setminus \Gamma) \times (0, T], \\
\tilde{B}v &\leq \psi_2 w_r \quad \text{on } \Gamma \times (0, T],
\end{align*}
\]
where the zero-order coefficients \( \tilde{a}_0, \tilde{b}_0 \) of \( \tilde{A}, \tilde{B} \) satisfy (4.30). Hence, an argument by contradiction similar to that of Theorem 4.1 implies estimate (4.21). \( \square \)

The previous technique does not apply to functions with square exponential growth, as we see in the following theorem which extends the maximum principle (of Phragmèn-Lindelöf type) to the integro-differential problems. This includes both the case of the boundary problems (\( \Omega \neq \mathbb{R}^d \)) and the case of the Cauchy problem (\( \Omega = \mathbb{R}^d \)): for differential operators we refer to Protter and Weinberg [25] and Friedman [6,7], respectively. This theorem requires stronger conditions on the Levy kernel.

**Theorem 4.4.** Suppose that \( \Omega \) is an unbounded domain in \( \mathbb{R}^d \) and that \( \mathcal{A}(= \partial_t - \mathcal{L} - I) \) is the integro-differential operator (1.6) satisfying (1.8), . . . , (1.11), and (4.2). Moreover we assume that
\[ \int_{|z| > 1} |z| \exp[\lambda(|z|^2 + 2z \cdot x)] M(x, t, dz) \leq C(\lambda), \quad \forall \lambda > 0, \] (4.33)
for any \( x \in \mathbb{R}^d \), \( t \in [0, T] \). Let \( u(x, t) \) be a smooth at most with square exponential growth at infinity\(^1\) function satisfying
\[ \partial_t u - Lu - Iv \leq 0 \quad \text{in } \Omega \times (0, T], \]
\[ \liminf_{r \to \infty} e^{-\alpha r^2} \left[ \max_{0 \leq t \leq T} u(x, t) \right] \leq 0 \] (4.35)
for some \( \alpha > 0 \).

If \( u \leq 0 \) on \( \overline{\Omega} \times \{0\} \) and \( u \leq 0 \) on \( \partial \Omega \times (0, T) \), (4.36)

\(^1\)This means \( C^2 \) in \( x \), \( C^1 \) in \( t \), and either the set of jumps is \( R(x, t) \) bounded for each point \( (x, t) \) or the function \( u(x, t) \exp(-c|z|^2) \) is bounded in \( (x, t) \) for some \( c > 0 \), so that \( Iu \) makes sense; cf. Remark 2.6.
then \( u \leq 0 \) in \( \Omega \times (0, T] \).

**Proof.** We set
\[
\psi(x, t) = \exp\left[ \frac{\alpha \gamma |x|^2}{\gamma - \alpha t} + \beta t \right],
\]
(4.37)
where \( \alpha \) is the constant in (4.35), and \( \beta, \gamma \) are constants to be determined later on. Consider the function
\[
u(x, t) = v(x, t)\psi(x, t).
\]
(4.38)
Computation shows that
\[
\partial_t u = (\partial_t \nu) \psi + v (\partial_t \psi) = \psi \partial_t v + \psi v \left[ \frac{\alpha^2 \gamma |x|^2}{(\gamma - \alpha t)^2} + \beta \right],
\]
\[
Lu = \psi Lv + \psi \left[ \frac{4 \alpha \gamma}{\gamma - \alpha t} \sum_{i,j=1}^{d} x_j a_{ij} \partial_i v \right]
\]
\[
+ \psi \left[ \frac{4 \alpha^2 \gamma^2}{(\gamma - \alpha t)^2} \sum_{i,j=1}^{d} x_i x_j a_{ij} + \frac{2 \alpha \gamma}{\gamma - \alpha t} \sum_{i=1}^{d} (a_{ii} - x_i a_i) \right] v, \tag{4.39}
\]
and
\[
Iu = \psi Iv + v I \psi + \psi \int_{\mathbb{R}^d} \left[ v(x + z, t) - v(x, t) \right] \left[ \frac{\psi(x + z, t)}{\psi(x, t)} - 1 \right] M(x, t, dz).
\]
Therefore, as in Remark 2.5, we obtain
\[
(\partial_t u - Lu - Iu)/\psi = \partial_t v - L \psi v - I \psi v,
\]
where the coefficients of \( L_\psi \) and \( I_\psi \) are defined as
\[
\begin{align*}
a^\psi_i(x, t) &= a_i(x, t) - \frac{4 \alpha \gamma}{\gamma - \alpha t} \sum_{i=1}^{d} x_i a_{ij}(x, t) - \int_{\mathbb{R}^d} z_i [\psi_0(x, t, z) - 1] M(x, t, dz), \\
a^\psi_0(x, t) &= a_0(x, t) + \frac{\alpha^2 \gamma |x|^2}{(\gamma - \alpha t)^2} + \beta - \frac{4 \alpha^2 \gamma^2}{(\gamma - \alpha t)^2} \sum_{i,j=1}^{d} x_i x_j a_{ij}(x, t) \\
&- \frac{2 \alpha \gamma}{\gamma - \alpha t} \sum_{i=1}^{d} (a_{ii}(x, t) - x_i a_i(x, t)) \\
&- \int_{\mathbb{R}^d} [\psi_0(x, t, z) - 1 - \frac{2 \alpha \gamma}{\gamma - \alpha t} x.z] M(x, t, dz), \\
a^\psi_{ij}(x, t) &= a_{ij}(x, t),
\end{align*}
\]
and the Levy measure
\[
M^\psi(x, t, dz) = \psi_0(x, t, z) M(x, t, dz),
\]
where
\[
\psi_0(x, t, z) = \frac{\psi(x + z, t)}{\psi(x, t)} = \exp\left[ \frac{\alpha \gamma}{\gamma - \alpha t} (|z|^2 + 2x.z) \right].
\]
Notice that in view of assumption (4.33) the new Levy kernel is well defined.
Since the coefficients $a_{ij}$ and $a_i$ are bounded there exist positive constants $C_0, C_1$ and $C_2$ such that
\begin{align}
\sum_{i,j=1}^{d} x_i x_j a_{ij}(x, t) &\leq C_2 |x|^2, \\
\sum_{i=1}^{d} a_{ii}(x, t) &\leq C_2,
\end{align}
(4.40)

for any $x, t$. On the other hand, by means of assumptions satisfied by the Levy measure, and the inequality
\[
\psi_0(x, t, z) - 1 - \frac{2a \gamma}{\gamma - \alpha t} x . z \leq \psi_0(x, t, z) + \psi_0(x, t, -z),
\]
we get a positive constant $C_3$ such that
\[
- \int_{R^d} [\psi_0(x, t, z) - 1 - \frac{2a \gamma}{\gamma - \alpha t} x . z] M(x, t, dz) \geq -C_3 \frac{\alpha \gamma}{\gamma - \alpha t}.
\]
(4.41)

Then, we deduce that
\[
a_0^\psi(x, t) \geq \frac{\alpha^2 \gamma |x|^2}{(\gamma - \alpha t)^2} \left[ 1 - C_0 \frac{(\gamma - \alpha t)^2}{\alpha^2 \gamma} - C_2 4 \gamma - C_1 2 \frac{\gamma - \alpha t}{\alpha} \right] + \left[ \beta - C_0 - \frac{\alpha \gamma}{\gamma - \alpha t} (2C_1 + 2C_2 + C_3) \right].
\]
Recalling that $\alpha$ is the constant in assumption (4.35), we can select $\gamma$ so small and $\beta$ so large that
\[
1 - \gamma \left( \frac{C_0}{\alpha^2} + \frac{2C_1}{\alpha} + 4C_2 \right) \geq \frac{1}{2} \quad C_0 + 2\alpha(2C_1 + 2C_2 + C_3) \leq \frac{\beta}{2}.
\]
(4.42)
Hence, we obtain
\[
a_0^\psi(x, t) \geq \frac{\alpha^2}{2\gamma} |x|^2 + \frac{\beta}{2} > 0,
\]
(4.43)
for any $(x, t) \in \Omega \times [0, \frac{\gamma}{2\alpha}]$. We set
\[
\Omega_r = \{ x \in \Omega : |x| < r \}, \quad r > 0.
\]

By the maximum principle on the bounded region $\Omega_r \times [0, \frac{\gamma}{2\alpha}]$ for the operator $\partial_t - L_\psi - J_\psi$ the function $v$ cannot have a positive maximum at an interior point (cf. (4.34), (4.39) and (4.43)). Since condition (4.35) implies that $v \leq \varepsilon$ on $(\Omega \setminus \Omega_r) \times [0, \frac{\gamma}{2\alpha}]$, for $r$ sufficiently large, and $v \leq 0$ for $t = 0$, then we deduce that for any positive $\varepsilon$ there exists $r$ such that
\[
v \leq \varepsilon \quad \text{in} \quad \Omega_r \times [0, \frac{\gamma}{2\alpha}],
\]
(4.44)
Therefore we arrive at
\[
v \leq 0 \quad \text{in} \quad \Omega \times [0, \frac{\gamma}{2\alpha}].
\]
In particular $v(x, \frac{t}{T}) \leq 0$, for any $x$ in $\Omega$. At this point, the entire above argument may be repeated with $t = \frac{t}{T}$ as the initial surface instead of $t = 0$. In this way we obtain $v \leq 0$ in $\Omega \times [\frac{t}{T}, 2(\frac{t}{T})]$. In a finite number of steps we obtain $v \leq 0$ in $\Omega \times [0, T]$, which gives us the conclusion. □

**Remark 4.5.** If condition (4.33) is satisfied only for some $\lambda > \alpha$ and the coefficients $a_{ij}$ and $a_t$ are locally bounded and $a_0$ is locally bounded from below [instead of being in $L^\infty$, as in (1.8)] and satisfy (4.40) for some constants $C_0$, $C_1$ and $C_2$, then the conclusions of Theorem 4.4 hold. As in the differential case, the Phragmén-Lindelöf type results can be extended to solutions of parabolic integro-differential problems with more general boundary conditions.

**REFERENCES**