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A DIRECT APPROACH TO INFINITE DIMENSIONAL HAMILTON–JACOBI EQUATIONS AND APPLICATIONS TO CONVEX CONTROL WITH STATE CONSTRAINTS*

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1. Introduction. For several reasons convex optimal control plays a special role in the theory of Distributed Parameter Systems. For instance, convex control is one of the few examples of nonlinear control problems possessing a smooth value function. As well known, this fact is essential to constructing optimal feedback strategies by the Dynamic Programming approach. Moreover, such a function can obtained by a direct method, solving a first order nonlinear partial differential equation, the so–called Hamilton–Jacobi–Bellman equation (see [1]).

The present paper is devoted to the analysis of Hamilton–Jacobi equations, when related to control problems with constraints on the state.

To fix ideas, let X and U be separable real Hilbert spaces, $0 \le t \le T$, and consider the problem of minimizing, overall controls $u \in L^2(t,T;U)$, the cost functional

$$f(y(T)) + \int_t^T \big[\frac{1}{2}\|u(s)\|_U^2 + g(y(s))\big]ds, \tag{1.1}$$

where $y \in C([t,T];X)$ is the mild solution of the state equation

$$\begin{cases} y'(s) = Ay(s) + Bu(s), & t \le s \le T \\ y(t) = x. \end{cases} \tag{1.2}$$

Here, we assume the following conditions, that are typical in convex control:

$$\begin{array}{ll} \text{(i)} & f,g : X \to [0,+\infty \text{ are continuous and convex functions such that} \\ & f(0) = 0 = g(0); \\ \text{(ii)} & A : D(A) \subset X \to X \text{ is the infinitesimal generator of a } C_0\text{-semigroup} \\ & e^{tA} \text{ satisfying } \|e^{tA}x\|_X \le e^{\alpha t}\|x\|_X \text{ for some } \alpha \in \mathbb{R}; \\ \text{(iii)} & B \in \mathcal{L}(U,X). \end{array} \tag{1.3}$$

A control $\overline{u}$ at which the above functional attains its minimum is said to be *optimal*. The pair $\{\overline{y},\overline{u}\}$, where $\overline{y}$ is the corresponding solution of equation (1.2), is called an optimal pair.

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The Dynamic Programming approach to problem (1.1)–(1.2) is based on the properties of the value function V, defined as

$$V(t,x) = \inf_{u \in L^2(t,T;U)} \Big\{ f(y(T)) + \int_t^T \big[\frac{1}{2}\|u(s)\|_U^2 + g(y(s))\big] ds \Big\}.$$

Knowing this function, one can construct optimal controls u^* by the feedback law

$$u^*(s) \in -B^* \partial_x V(s, y^*(s)), \tag{1.4}$$

where y^* can be recovered solving the closed loop equation

$$\begin{cases} \dfrac{d}{ds} y^*(s) \in Ay^*(s) - BB^* \partial_x V(s, y^*(s)), \quad t \le s \le T \\ y^*(t) = x. \end{cases} \tag{1.5}$$

Moreover, the function $v(t,x) = V(T-t,x)$ is a solution of the infinite dimensional Hamilton–Jacobi equation

$$v_t(t,x) + \tfrac{1}{2}\|B^* \nabla_x v(t,x)\|_U^2 - \langle Ax, \nabla_x v(t,x)\rangle_X = g(x), \quad v(0,x) = f(x), \tag{1.6}$$

where B^* is the adjoint of B and the subscript x denotes the Fréchet derivative with respect to x.

Motivated by the above applications, direct methods for solving (1.6) have been developed by various authors both in the finite and in the infinite dimensional case. For infinite dimensional convex problems we refer the reader to [1] and [6]. Without assuming convexity, Hamilton–Jacobi equations have, in general, no classical solutions. In this case, equation (1.6) has been studied in [5], [2], and [9] using viscosity solutions.

Of particular interest for applications are optimal control problems with constraints on the state. One can model these constraints prescribing a set $K \subset X$ and restricting the class of admissible controls to those leaving the state $y(s)$ inside K.

In finite dimensions, Hamilton–Jacobi equations for constrained problems have been studied in [8] and [4] by the viscosity solution method, assuming that the constraint set K be the closure of an open domain in $\mathbb{R}^n$ with smooth boundary. This approach has been extended to infinite dimensions in [3], when A in (1.2) is a bounded operator in X.

However, the optimal control of a system governed by a partial differential equation naturally leads to considering equations of type (1.2) in which A is an unbounded operator satisfying (1.3)(ii). Moreover, significant examples of state constraints, like those in obstacle problems, can only be modeled allowing K to have an empty interior.

The purpose of the present paper is to study the Dynamic Programming equation for the problem of minimizing the functional (1.1) overall admissible controls $u \in \mathcal{U}(t,x)$, where

$$\mathcal{U}(t,x) = \{u \in L^2(t,T;U) \ : \ y(s) \in K, \ \forall s \in [t,T]\}$$

and

$$K \subset X \text{ is a closed convex set containing } 0, \tag{1.7}$$

possibly having an empty interior.

In order to insert the constraints into the cost functional, we set

$$\phi(x) = \begin{cases} f(x), & \text{if } x \in K \\ +\infty, & \text{if } x \notin K \end{cases} \qquad \psi(x) = \begin{cases} g(x), & \text{if } x \in K \\ +\infty, & \text{if } x \notin K \end{cases} \tag{1.8}$$

and consider the corresponding value function

$$V(t,x) = \inf_{u \in L^2(t,T;U)} \Big\{ \phi(y(T)) + \int_t^T \frac{1}{2} \left[\|u(s)\|_U^2 + \psi(y(s)) \right] ds \Big\}. \tag{1.9}$$

The related Hamilton–Jacobi–Bellman equation reads as follows

$$v_t(t,x) + \tfrac{1}{2}\|B^* \nabla_x v(t,x)\|_U^2 - \langle Ax, \nabla_x v(t,x)\rangle_X = \psi(x), \quad v(0,x) = \phi(x), \tag{1.10}$$

where $v(t,x) = V(T-t,x)$. Notice that the data of problem (1.10) above are still convex functions defined in the whole space X, but they may now take the value $+\infty$.

In §3 of this paper we show that (1.10) has a unique solution if and only if at every point of $[0,T] \times K$, there exists at least an admissible control, i.e., $\mathcal{U}(t,x) \neq \emptyset$ for all $(t,x) \in [0,T] \times K$. This solution is constructed as a limit of classical solutions of regularized problems, that are obtained replacing ϕ and ψ by their Yosida approximations. Indeed, the condition $\mathcal{U}(t,x) \neq \emptyset$ for all $(t,x) \in [0,T] \times K$ is shown to be equivalent to the convergence of such approximate solutions (see Theorems 3.4 and 3.7). The existence of admissible controls is part of the data of the control problem. In Examples 3.9 and 3.10 we describe two typical situations in which $\mathcal{U}(t,x) \neq \emptyset$ for all $(t,x) \in [0,T] \times K$.

In §4 we derive a feedback law like (1.4) for optimal controls (Theorem 4.4). Then, we investigate the smoothness of solutions to (1.10). In the first place, we show that any weak solution w is locally Lipschitz continuous in the interior of $[0,T] \times K$. If K has an empty interior but operator B is invertible, we prove the continuity of w on $[0,T] \times K$. Moreover, if e^{tA} is compact for $t > 0$, we show that w is weakly continuous.

Applications of the above results to the analysis of the closed loop equation (1.5) are discussed in §5. Examples concerning the optimal control of the heat equation under state constraints are described in §6.

2. Preliminaries, the regularized problem. In what follows we denote by $C_2(X)$ the space of continuous functions $w : X \to \mathbb{R}$, verifying

$$\|w\|_2 := \sup_{x \in X} \frac{|w(x)|}{1 + \|x\|^2} < +\infty,$$

endowed with the norm $\|\cdot\|_2$. Moreover we denote by $C^1_{Lip}(X)$ the space of Fréchet differentiable functions $w : X \to \mathbb{R}$, such that the gradient ∇w is Lipschitz continuous on X. Moreover we denote by Σ the set defined as

$$\Sigma = \{ w \in C_2(X) : w \text{ is convex},\ w(0) = 0, \partial w(0) \ni 0 \},$$

where ∂w denotes the subdifferential of w.

Given $a, b \in \mathbb{R}$, $a < b$, and a Banach space Y, we denote by $C([a,b];Y)$ the Banach space of all continuous functions w from $[a,b]$ into Y; moreover we denote by $L^2(a,b;Y)$ the Banach space of all measurable functions w from $]a,b[$ into Y such that $\|w\|_Y$ is integrable on $]a,b[$.

In this section we consider the following problem: for given $\phi, \psi \in \Sigma$,

$$w_t(t,x) + \tfrac{1}{2}\|B^*\nabla_x w(t,x)\|_U^2 - \langle Ax, \nabla_x w(t,x)\rangle_X = \psi(x), \quad w(0,x) = \phi(x). \quad (2.1)$$

Since ϕ, ψ are convex and continuous we have that there exists a solution of problem (2.1). We refer to Section 1.4 of [1] or Section 3 of [6] for a description of the methods which can be used without essential modifications in the present situation, and quote now the results that will be needed in the sequel.

Theorem 2.1. *For each $\phi, \psi \in \Sigma$ there exists a unique continuous function $w : [0,T] \times X \to \mathbb{R}$ satisfying the following properties:*

(i) *$w(t,\cdot) \in \Sigma$ for all $t \in [0,T]$;*
(ii) *$w(\cdot,x)$ is Lipschitz continuous on $[0,T]$ for each $x \in D(A)$;*
(iii) *for almost every $t \in [0,T]$ and for all $x \in D(A)$ w satisfies*

$$w_t(t,x) + \tfrac{1}{2}\|B^*\partial_x w(t,x)\|_U^2 - \langle Ax, \partial_x w(t,x)\rangle_X \ni \psi(x)$$

and

$$w(0,x) = \phi(x)\ .$$

If in addition $\phi, \psi \in C^1_{Lip}(X)$, then

(iv) *$w(t,\cdot) \in C^1_{Lip}(X)$.*

The function w is called the strong solution of (2.1).

Now, for $u \in L^2(t,T;U)$, let $y(\cdot;t,x,u) \in C([t,T];X)$ be the mild solution of the state equation

$$y'(s) = Ay(s) + Bu(s), \quad y(t) = x, \quad t \le s \le Ty(t) = x\ . \quad (2.2)$$

Given $\phi, \psi \in \Sigma$, consider the problem of minimizing the cost functional

$$J(t,x;u) = \phi(y(T;t,x,u)) + \int_t^T \big[\frac{1}{2}\|u(s)\|_U^2 + \psi(y(s;t,x,u))\big]ds \quad (2.3)$$

overall controls $u \in L^2(t,T;U)$. As usual, a control u^* which minimizes $J(t,x;u)$ is called an *optimal control*, and the corresponding pair $\{u^*, y^*\}$ an *optimal pair*.

The following result concerns the connection between strong solutions of (2.1) and problem (2.2), (2.3).

Theorem 2.2. *Let w be the strong solution of* (2.1), *then*

$$w(T-t,x) = \inf_{u \in L^2(t,T;U)} J(t,x;u). \quad (2.4)$$

Moreover the unique optimal pair $\{y^*, u^*\}$ satisfies

$$u^*(s) \in -B^*\partial_x w(T-s, y^*(s)).$$

For computational purposes the following approximating result will be useful in the sequel.

Proposition 2.3. *Let w be the strong solution of* (2.1). *Then there exist* $\{\phi_n\}, \{\psi_n\} \subset \Sigma$ *and* $\{w_n\}$ *satisfying:*

(i) $w_n \in C([0,T]; C^1_{Lip}(X))$, $w_n(t,\cdot) \in \Sigma$, *for each* $t \in [0,T]$, $w_n(\cdot,x) \in C^1([0,T])$, *for each* $x \in D(A)$;

(ii) *for each* $t \in [0,T]$ *and* $x \in D(A)$, w_n *satisfies*

$$\partial_t w_n(t,x) + \tfrac{1}{2}\|B^*\nabla_x w_n(t,x)\|^2_U - \langle Ax, \nabla_x w_n t, x)\rangle_X = \psi_n(x), \quad w(0,x) = \phi_n(x);$$

(iii) $\phi_n \to \phi, \psi_n \to \psi$ *in* $C_2(X)$ *and* $w_n \to w$ *in* $C([0,T]; C_2(X))$;

(iv) *for each* $\{y,u\}$ *satisfying* (2.2) *we have*

$$w_n(T-t,x) + \int_t^T \|B^*\nabla_x w_n(T-s, y(s)) + u(s)\|^2_U ds$$
$$\leq J_n(t,x;u) := \phi_n(y(T)) + \int_t^T \big[\frac{1}{2}\|u(s)\|^2_U + \psi_n(y(s))\big] ds \ .$$

If in addition $\phi, \psi \in C^1_{Lip}(X)$, *then*

(v) $\nabla w_n(t,x) \rightharpoonup \nabla w(t,x)$, *for each* $t \in [0,T]$.

Finally, the following result concerns monotonicity of the solutions with respect to the data .

Corollary 2.4. *Let w and $\overline{w}$ be the strong solutions of* (2.1) *with data* $\{\phi, \psi\}$ *and* $\{\overline{\phi}, \overline{\psi}\}$, *respectively. If* $\phi \leq \overline{\phi}$ *and* $\psi \leq \overline{\psi}$, *then*

$$w(t,x) \leq \overline{w}(t,x) \ .$$

3. Solutions to the Hamilton-Jacobi equation.

In order to study the Hamilton–Jacobi equation (1.10), we need to extend the theory described in the previous section to equations with more general convex coefficients ϕ and ψ, possibly taking the value $+\infty$. To fix ideas, let $K \subset X$ be a closed convex set containing 0, and define the class Σ_K by $\Sigma_K = \{w : X \to [0,+\infty] : w$ is l.s.c. and convex,$K \subset D(w)$, $w(0) = 0$, $\partial w(0) \ni 0\}$. Here $D(w)$ is the effective domain of w. We note that, if f, g and K satisfy (1.3)(i), (1.7) as in §1, then the functions ϕ and ψ defined in (1.8) belong to Σ_K.

Definition 3.1. Let $\phi, \psi \in \Sigma_K$. We say that a function $w : [0,T] \times X \to [0,+\infty]$ is a *weak* solution of the problem

$$w_t(t,x) + \tfrac{1}{2}\|B^*\nabla_x w(t,x)\|^2_U - \langle Ax, \nabla_x w(t,x)\rangle_X = \psi(x), \;\; w(0,x) = \phi(x) \quad (3.1)$$

if

(i) $w(t,\cdot) \in \Sigma_K$,

(ii) there exist $\{\phi_n, \psi_n\} \subset \Sigma$ and $\{w_n\}$ such that w_n is the strong solution of the problem

$$w_t(t,x) + \frac{\|}{B}^* \nabla_x w(t,x)\|^2_U - \langle Ax, \nabla_x w(t,x)\rangle_X = \psi_n, \quad w(0,x) = \phi_n(x)$$

and

$$\phi_n(x) \uparrow \phi(x) \ , \ \psi_n(x) \uparrow \psi(x) \ , \ w_n(t,x) \uparrow w(t,x).$$

Remark 3.2. From property (ii) above and the continuity of w_n in $[0,T]\times X$, it follows that any weak solution of (3.1) is lower semi-continuous in $[0,T]\times X$.

For $\phi,\psi\in\Sigma_K$ let us consider the problem of minimizing the cost functional

$$J(t,x;u)=\phi(y(T;t,x,u))+\int_t^T\big[\frac{1}{2}\|u(s)\|_U^2+\psi(y(s;t,x,u))\big]ds \tag{3.2}$$

overall controls $u\in L^2(t,T;U)$, where $y(\cdot;t,x,u)\in C([t,T];X)$ is the mild solution of the state equation (2.2).

Definition 3.3. For any $(t,x)\in[0,T]\times K$ we say that $u\in L^2(t,T;U)$ is an *admissible* control at (t,x) if $J(t,x;u)<+\infty$, and denote by $\mathcal{U}(t,x)$ the class of all admissible controls at (t,x). An admissible control u^* is *optimal* at (t,x) if $J(t,x;u^*)=\inf_{\mathcal{U}(t,x)}J(t,x;u)$.

The following result concerns the connection between the weak solutions of (3.1) and problem (3.2).

Theorem 3.4. *Let there exist a weak solution w of* (3.1). *Then*

$$w(T-t,x)=\inf_{u\in\mathcal{U}(t,x)}J(t,x;u),\ \forall(t,x)\in[0,T]\times K. \tag{3.3}$$

Moreover for each $(t,x)\in[0,T]\times K$ there exists an optimal pair $\{y^*,u^*\}$

$$w(T-t,x)=J(t,x;u^*)\ . \tag{3.4}$$

Moreover

$$y^*=\lim_{n\to+\infty}y_n^*\ ,\quad in\ \ C([t,T];X)$$

and

$$u^*=\lim_{n\to+\infty}u_n^*\ ,\quad in\ \ L^2(t,T;U),$$

where $\{y_n^*,u_n^*\}$ is the optimal pair at (t,x) for the functional

$$J_n(t,x;u)=\phi_n(y(T;t,x,u))+\int_t^T\big[\frac{1}{2}\|u(s)\|_U^2+\psi_n(y(s;t,x,u))\big]ds$$

and $\{\phi_n\}$, $\{\psi_n\}$ are given by Definition 3.1-(ii).

Proof. Let $\{\phi_n\}$, $\{\psi_n\}$ and $\{w_n\}$ be given by Definition 3.1 (ii). By Corollary 2.2 we have for each trajectory–control pair $\{y,u\}$

$$w_n(T-t,x)\le J_n(t,x;u)\le J(t,x;u)\ .$$

Therefore, letting $n\to\infty$ we get

$$w(T-t,x)\le J(t,x;u)\ . \tag{3.5}$$

To complete the proof of (3.3) let $x\in K$ and let $\{y_n^*,u_n^*\}$ be the optimal pair for $J_n(t,x;u)$

$$w_n(T-t,x)=J_n(t,x;u_n^*)\ . \tag{3.6}$$

Since $x\in K$ and $w_n(T-t,x)\uparrow w(T-t,x)$, we have that there exists M_x verifying

$$J_n(t,x;u_n^*)=w_n(T-t,x)\le w(T-t,x)\le M_x\ .$$

This in turn implies

$$\int_t^T \|u_n^*(s)\|_U^2 ds \leq M_x.$$

Therefore, there exist $\{n_k\}$ and $u^* \in L^2(t, T; U)$ such that

$$u_{n_k}^* \rightharpoonup u^* \ .$$

Moreover, from (2.2) we get

$$y_{n_k}^*(t) \rightharpoonup y^*(t),$$

where y^* is the solution of (2.2) with u replaced by u^*.

To prove that $\{y^*, u^*\}$ is optimal, let $\overline{n}$ be fixed. Since $\{\phi_n\}$ is nondecreasing we have for $n_k > \overline{n}$

$$\phi_{n_k}(y_{n_k}^*(T)) \geq \phi_{\overline{n}}(y_{n_k}^*(T)).$$

By the convexity of $\phi_{\overline{n}}$,

$$\liminf_{k\to+\infty} \phi_{n_k}(y_{n_k}^*(T)) \geq \phi_{\overline{n}}(y^*(T))$$

and so, letting $\overline{n} \to \infty$,

$$\liminf_{k\to+\infty} \phi_{n_k}(y_{n_k}^*(T)) \geq \phi(y^*(T)). \tag{3.7}$$

In the same way we find

$$\liminf_{k\to+\infty} \psi_{n_k}(y_{n_k}^*(s)) \geq \psi(y^*(s)). \tag{3.8}$$

Therefore, from (3.6)

$$w(T-t, x) = \lim_{k\to\infty} J_{n_k}(t, x; u_{n_k}^*) \geq J(t, x; u^*) \tag{3.9}$$

so that (3.5) yields

$$w(T-t, x) = \lim_{k\to\infty} J_{n_k}(t, x; u_{n_k}^*) = J(t, x; u^*) \tag{3.10}$$

and the optimality of $\{y^*, u^*\}$. Moreover from (3.7), (3.8) and (3.10) we get

$$\liminf_{k\to+\infty} \int_t^T \|u_{n_k}^*(s)\|^2 ds = \int_t^T \|u^*(s)\|^2 ds,$$

from which it follows that there exist a subsequence, again denoted by $\{u_{n_k}^*\}$, verifying

$$u_{n_k}^* \to u^*, \quad \text{in} \quad L^2(t, T; U),$$

so that

$$y_{n_k}^* \to y^* \ , \quad \text{in} \quad C([t, T]; X).$$

Since the optimal pair is unique, by the same arguments used above we prove that each subsequence of $\{y_n^*, u_n^*\}$ contains a subsequence $\{y_{n_k}^*, u_{n_k}^*\}$ which converges strongly to $\{y^*, u^*\}$. □

An immediate consequence of (3.3) is the uniqueness of weak solutions.

Corollary 3.5. *The weak solution of* (3.1) *is unique.*

Another consequence of (3.3) and of convexity is the weak lower semicontinuity of weak solutions.

Corollary 3.6. *The weak solution of* (3.1) *is sequentially weakly lower semicontinuous in* $[0,T]\times K$.

The proof of the above corollary is standard. We provide it for the reader's convenience.

Proof. Let $(t_n,x_n)\in[0,T]\times K$ be such that

$$t_n\to t_0,\quad x_n\rightharpoonup x_0\quad\text{as}\quad n\to+\infty.$$

If $\liminf_{n\to+\infty}V(t_n,x_n)=+\infty$, then the conclusion is trivial. Otherwise, let (t_{k_n},x_{k_n}) be a subsequence of (t_n,x_n) such that

$$\lim_{n\to+\infty}V(t_{k_n},x_{k_n})=\liminf_{n\to+\infty}V(t_n,x_n)<+\infty.$$

For each $n\in\mathbb{N}$ let $\{y_n,u_n\}$ be an optimal pair at (t_{k_n},x_{k_n}). Clearly, y_n and u_n may be extended on the whole interval $[0,T]$, defining $y_n(s)=x_{k_n}$ and $u_n(s)=0$ on $[0,t_{k_n}]$. Since $V(t_{k_n},x_{k_n})$ is bounded, $\{u_n\}$ is bounded in $L^2(0,T;U)$. Therefore, passing to a subsequence (still denoted by the same symbol), we may assume that

$$u_n\rightharpoonup u_0\ \text{ in }\ L^2(0,T;U)\ \text{ as }\ n\to+\infty.$$

Passing to the limit in the integral equation

$$y_n(s)=e^{(s-t_{k_n})A}x_{k_n}+\int_{t_{k_n}}^{s}e^{(s-\sigma)A}Bu_n(\sigma)\,d\sigma,$$

it is easy to see that $y_n(s)\rightharpoonup y_0(s)=y(s;t_0,x_0,u_0)$ for all $s\in[t_0,T]$. Therefore, by the convexity of ϕ and ψ, we conclude that

$$\begin{aligned}\lim_{n\to\infty}V(t_{k_n},x_{k_n})&=\lim_{n\to\infty}\Big[\phi(y_n(T))+\int_{t_{k_n}}^{T}\big(\frac{1}{2}\|u_n(s)\|_U^2+\psi(y_n(s))\big)ds\Big]\\&\geq\phi(y_0(T))+\int_{t_0}^{T}\big(\frac{1}{2}\|u_0(s)\|_U^2+\psi(y_0(s))\big)ds\geq V(t_0,x_0),\end{aligned}$$

and the proof is complete. □

Concerning the existence of weak solutions we have the following result.

Theorem 3.7. *Let* $\phi,\psi\in\Sigma_K$ *and assume that for each* $(t,x)\in[0,T]\times K$ *there exists an admissible control. Then there exists a weak solution of problem* (3.1).

Proof. For given $\epsilon>0$ let $\phi_\epsilon,\psi_\epsilon$ be defined as

$$\phi_\epsilon(x)=\inf_{y\in X}\big\{\phi(y)+\tfrac{1}{2\epsilon}\|x-y\|_X^2\big\},\quad\psi_\epsilon(x)=\inf_{y\in X}\big\{\psi(y)+\tfrac{1}{2\epsilon}\|x-y\|_X^2\big\}.\qquad(3.11)$$

Then it is known that $\phi_\epsilon,\psi_\epsilon\in C^1_{Lip}(X)$ and that $\phi_\epsilon(x)\uparrow\phi(x),\psi_\epsilon(x)\uparrow\psi(x)$. Therefore, by Theorem 2.1, there exists a unique strong solution w_ϵ of problem (2.1) with data $\{\phi_\epsilon,\psi_\epsilon\}$. Moreover, by Corollary 2.4 there exists w such that

$$w_\epsilon(t,x)\uparrow w(t,x),$$

from which it follows that $w(t,\cdot)$ is convex and l.s.c. Now let $x\in K$; by assumption and Corollary 2.2, we get that there exists M_x verifying

$$w_\epsilon(t,x)\leq J(t,x;u)\leq M_x,$$

so that

$$w(t,x)\leq M_x$$

and hence $w(t,\cdot)\in\Sigma_K$. Therefore w is a weak solution of (3.1). □

Corollary 3.8. *Let $\phi, \psi \in \Sigma_K$. Then the following properties are equivalent:*

(i) *There exists a (unique) weak solution w of problem* (3.1).

(ii) *At each $(t,x) \in [0,T] \times K$ there exists an admissible control.*

Moreover, if (i) *or* (ii) *holds then there exists a unique optimal pair $\{y^*, u^*\}$, and we have*

$$w(T-t,x) = J(t,x;u^*) \ .$$

We now discuss some sufficient conditions guaranteeing existence of admissible pairs for each $(t,x) \in [0,T] \times K$.

Example 3.9. The simplest example of existence of admissible pairs is when K is invariant for the semigroup generated by A, i.e., $e^{tA}K \subset K$, for all $t \geq 0$. Then, for each $x \in K$ the 0 control is admissible, as $e^{(s-t)A}x \in K, \forall s \geq t$. In particular, this is the case when K is a ball and e^{tA} a contraction semigroup.

Example 3.10. Let $B^{-1} \in \mathcal{L}(X,U)$ and suppose that there exists ω satisfying $e^{t(A-\omega I)}K \subset K$ for every $t \geq 0$. Then, for a given $x \in K$, it suffices to take as admissible trajectory–control pair

$$y(s) = e^{(s-t)(A-\omega I)}x, \qquad u(s) = -\omega B^{-1} e^{(s-t)(A-\omega I)}x.$$

4. Properties of solutions of the Hamilton–Jacobi equation. This section is devoted to qualitative properties of weak solutions to (3.1). We begin with the monotonicity of solutions with respect to data.

Theorem 4.1. *Let w and $\overline{w}$ be the weak solutions of* (3.1) *with data $\{\phi, \psi\}$ and $\{\overline{\phi}, \overline{\psi}\}$, respectively. If $\phi \leq \overline{\phi}$ and $\psi \leq \overline{\psi}$, then*

$$w(t,x) \leq \overline{w}(t,x).$$

Proof. From Theorem 3.7, Corollary 3.8 and uniqueness, we have that

$$w_\epsilon(t,x) \uparrow w(t,x),$$

where w_ϵ is the strong solution of (3.1) with data $\{\phi_\epsilon, \psi_\epsilon\}$ given by (3.11), and similarly for $\overline{w}$. Hence the assertion follows from Corollary 2.4 and from the fact that $\phi_\epsilon \leq \overline{\phi}_\epsilon$ and $\psi_\epsilon \leq \overline{\psi}_\epsilon$. □

The following result concerns continuous dependence of the solutions upon the data.

Theorem 4.2. *Let w and w_n be the weak solutions of* (3.1) *with data $\{\phi, \psi\}$ and $\{\phi_n, \psi_n\}$, respectively. If $\phi_n \uparrow \phi$ and $\psi_n \uparrow \psi$, then*

$$w_n(t,x) \uparrow w(t,x).$$

Proof. From Theorem 4.1 we have that $\{w_n\}$ is nondecreasing. Therefore there exists v verifying

$$w_n(t,x) \uparrow v(t,x). \tag{4.1}$$

Furthermore, from (3.3) we have

$$w_n(T-t,x) \leq \phi_n(y(T)) + \int_t^T \big[\frac{1}{2}\|u(s)\|_U^2 + \psi_n(y(s))\big]ds \tag{4.2}$$

for all trajectory–control pairs $\{y, u\}$, whereas we have

$$w_n(T-t,x) = \phi_n(y_n^*(T)) + \int_t^T \big[\frac{1}{2}\|u_n^*(s)\|_U^2 + \psi_n(y_n^*(s))\big]ds \tag{4.3}$$

if $x \in K$ and $\{y_n^*, u_n^*\}$ is optimal. Now (4.1) and (4.2) imply

$$v(T-t,x) \le \phi(y(T)) + \int_t^T \big[\frac{1}{2}\|u(s)\|_U^2 + \psi(y(s))\big]ds \ , \tag{4.4}$$

and hence from (3.3), $v(t,x) \le w(t,x)$. Moreover, let $x \in K$. Using (4.3) and a computation similar to the one used in proving (3.9), we find that there exists a trajectory–control pair $\{y^*, u^*\}$ verifying

$$v(T-t,x) = \lim_{n\to\infty} w_n(T-t,x) \ge \phi(y^*(T)) + \int_t^T \big[\frac{1}{2}\|u^*(s)\|_U^2 + \psi(y^*(s))\big]ds,$$

and hence by (4.4)

$$v(T-t,x) = \inf J(t,x;u).$$

Hence using (3.3) we get $v = w$ and the assertion is proved. □

Remark 4.3. Without essential modification it can be proved that the assertion of Theorem 4.2 is replaced by $w_n(t,x) \downarrow w(t,x)$ if the assumptions $\phi_n \uparrow \phi, \psi_n \uparrow \psi$ are replaced by $\phi_n \downarrow \phi, \psi_n \downarrow \psi$.

The following result holds under the additional assumption that the operator B is invertible.

Theorem 4.4. *Let w be the weak solution of* (3.1) *and let $B^{-1} \in \mathcal{L}(X,U)$. Then the optimal pair $\{y^*, u^*\}$ satisfies the feedback formula for almost every $s \in [t,T]$*

$$u^*(s) \in -B^*\partial_x w(T-s, y^*(s)).$$

Moreover we have for almost every $s \in [t,T]$

$$u^*(s) = -\lim_{\epsilon\to 0} B^*\nabla_x w_\epsilon(T-s, y^*(s)) \ .$$

Proof. By assumption there exists the weak solution w of problem (3.1). Moreover from Theorem 3.7, Corollary 3.8 and uniqueness we have that

$$w_\epsilon(t,x) \uparrow w(t,x),$$

where w_ϵ is the strong solution of (3.1) with data $\{\phi_\epsilon, \psi_\epsilon\}$ given by (3.11). By Proposition 2.3(iv) and (v) we have

$$w_\epsilon(T-t,x) + \int_t^T \|B^*\nabla_x w_\epsilon(T-s, y^*(s)) + u^*(s)\|_U^2 ds \le J(t,x;u^*).$$

Since $\{y^*, u^*\}$ is optimal we have $w(T-t,x) = J(t,x;u^*)$. Hence there exists $\{\epsilon_k\}$ verifying, for almost every $s \in [t,T]$

$$-\lim_{k\to+\infty} B^*\nabla_x w_{\epsilon_k}(T-s, y^*(s)) = u^*(s) \ .$$

This in turn implies that there exist $z(s) \in \partial_x w(T-s, y^*(s))$ such that

$$\lim_{k\to+\infty} \nabla_x w_{\epsilon_k}(T-s, y^*(s)) = z(s).$$

Hence

$$-B^*z(s) = u^*(s) \ .$$

Since the optimal pair is unique the result follows. □

We now study the continuity of weak solutions. We begin with an interior continuity result.

Theorem 4.5. *Let A satisfy the property $e^{tA}K \subset K$ and let w be the weak solution of* (3.1) *with ϕ and ψ given by* (1.8)*. Then*

(i) *$w(t,\cdot)$ is locally Lipschitz continuous on $\overset{\circ}{K}$ for all $t \in [0,T]$;*

(ii) *for any $x \in D(A) \cap \overset{\circ}{K}$ the function $w(\cdot, x)$ is Lipschitz continuous on $[0,T]$.*

Proof. Let $x \in \overset{\circ}{K}$. By Example 3.9 and (3.3) we get

$$w(T-t,x) \le f(e^{(T-t)A}x) + \int_t^T g(e^{(s-t)A}x)ds.$$

Hence, since f and g are continuous we have that there exists a neighborhood V_0 of x such that $w(t,\cdot)$ is bounded in V_0, uniformly for $t \in [0,T]$. Therefore, assertion (i) follows from the fact that $w(t,\cdot)$ is convex. To prove (ii) let $\{\phi_\epsilon\}, \{\psi_\epsilon\}$ be the functions introduced in the proof of Theorem 3.7, and let $\{w_\epsilon\}$ be the solutions of (3.1) with data $\{\phi_\epsilon\}, \{\psi_\epsilon\}$. Since $w_\epsilon \uparrow w$, we have that $w_\epsilon(t,\cdot)$ are uniformly bounded in V_0. This in turn implies that $\nabla_x w_\epsilon$ is bounded on V_0 for all $\epsilon > 0$. Hence, if $x \in D(A)$ we have from (3.1) that $\frac{\partial}{\partial t} w_\epsilon(\cdot, x)$ is uniformly bounded, so that $w_\epsilon(\cdot,x)$ is uniformly Lipschitz continuous on $[0,T]$ for any $\epsilon > 0$. Thus, (ii) follows letting $\epsilon \downarrow 0$. □

Under a stronger controllability assumption, we can show that weak solutions are continuous up to the boundary of their effective domain.

Theorem 4.6. *Assume that $e^{t(A-\omega I)}K \subset K$ for all $t \ge 0$ and some $\omega \in \mathbb{R}$, and suppose that $B^{-1} \in \mathcal{L}(X,U)$. Let w be the weak solution of* (3.1) *with ϕ and ψ given by* (1.8)*. Then w is continuous in $[0,T] \times K$.*

The proof relies on the following technical result.

Lemma 4.7. *Assume that $e^{t(A-\omega I)}K \subset K$ for all $t \ge 0$ and some $\omega \in \mathbb{R}$, and suppose that $B^{-1} \in \mathcal{L}(X,U)$. Let $(t_0, x_0) \in [0,T] \times K$ and $\{y_0, u_0\}$ be an admissible trajectory–control pair at (t_0,x_0). Then, for any $(t,x) \in [t_0,T] \times K$ and any $r > 0$, there exists a trajectory–control pair $\{y_r, u_r\}$ which is admissible at (t,x) and such that, for all $s \ge t$,*

$$\|y_r(s) - y_0(s)\|_X \le C\Big[\big\|e^{(s-t_0)A}x_0 - e^{(s-t)A}x\big\|_X + r\|x\|_X + \int_{t_0}^s \|u_0(\sigma)\|_U \, d\sigma\Big] \quad (4.5)$$

and

$$\int_t^T (\|u_r(s)\|_U^2 - \|u_0(s)\|_U^2)ds \le Cr\|x\|_X^2 + \frac{C}{r^2}\int_t^{T\wedge(t+r)} \big\|e^{(s-t_0)A}x_0 - e^{(s-t)A}x\big\|_X^2 + C\big(1 + \frac{t-t_0}{r}\big)^2 \int_{t_0}^{T\wedge(t+r)} \|u_0(s)\|_U^2 \, ds, \quad (4.6)$$

where $C = C(\alpha, \omega, \|B\|, \|B^{-1}\|, T)$. Moreover, if $t + r \le T$, then

$$y_r(s) = y_0(s), \quad u_r(s) = u_0(s) \quad \forall s \in [t+r, T].$$

Proof. For any $(t,x) \in [t_0,T] \times K$ we denote by $\{y_{t,x}, u_{t,x}\}$ the admissible pair introduced in Example 3.10

$$y_{t,x}(s) = e^{(s-t)(A-\omega I)}x, \quad u_{t,x}(s) = -\omega B^{-1} e^{(s-t)(A-\omega I)}x \quad (4.7)$$

for all $t \le s \le T$. We define, for any $r > 0$,

$$y_r(s) = \lambda_r(s)y_0(s) + [1-\lambda_r(s)]y_{t,x}(s), \quad t \le s \le T, \tag{4.8}$$

where $\lambda_r : \mathbb{R} \to \mathbb{R}$ is given by

$$\lambda_r(s) = \begin{cases} 0 & s \le t \\ \frac{s-t}{r} & t \le s \le t+r \\ 1 & t+r \le s. \end{cases}$$

We note that $y_r = y(\cdot; t, x, u_r)$, with

$$u_r(s) = \lambda_r(s)u_0(s) + [1-\lambda_r(s)]u_{t,x}(s) + \lambda_r'(s)B^{-1}[y_0(s) - y_{t,x}(s)]. \tag{4.9}$$

Since K is convex,

$$y_r(s) \in K, \quad \forall s \in [t, T]$$

and so u_r is admissible at (t, x). Moreover, if $t + r \le T$, then

$$y_r(s) = y_0(s), \quad u_r(s) = u_0(s) \quad \forall s \in [t+r, T]$$

by construction. For $s \ge t$, we have

$$\begin{aligned} &\|y_0(s) - y_{t,x}(s)\|_X \\ &= \left\| e^{(s-t_0)A}x_0 - e^{(s-t)(A-\omega I)}x + \int_{t_0}^s e^{(s-\sigma)A}Bu_0(\sigma)d\sigma \right\|_X \\ &\le \left\| e^{(s-t_0)A}x_0 - e^{(s-t)A}x \right\|_X + e^{\alpha T}\left[1 - e^{-\omega(s-t)}\right]\|x\|_X + e^{\alpha T}\|B\| \int_{t_0}^s \|u_0(\sigma)\|_U d\sigma \\ &\le C\Big[\left\| e^{(s-t_0)A}x_0 - e^{(s-t)A}x \right\|_X + (s-t)\|x\|_X + \int_{t_0}^s \|u_0(\sigma)\|_U d\sigma \Big]. \end{aligned} \tag{4.10}$$

Estimate (4.5) easily follows from (4.8) and (4.10).

Next, to derive (4.6) we note that (4.9) yields

$$\begin{aligned} &\int_t^T \left(\|u_r(s)\|_U^2 - \|u_0(s)\|_U^2\right) ds \le \int_t^{T\wedge(t+r)} \|u_r(s)\|_U^2 ds \\ &\le 3\int_t^{T\wedge(t+r)} \Big[\|u_0(s)\|_U^2 + \|u_{t,x}(s)\|_U^2 + \frac{\|B^{-1}\|^2}{r^2}\|y_0(s) - y_{t,x}(s)\|_X^2\Big] ds. \end{aligned} \tag{4.11}$$

Now, by (4.10) and Hölder's inequality we obtain

$$\begin{aligned} &\int_t^{T\wedge(t+r)} \|y_0(s) - y_{t,x}(s)\|_X^2 ds \\ &\le C\int_t^{T\wedge(t+r)} \Big[(s-t)^2\|x\|_X^2 + \left\| e^{(s-t_0)A}x_0 - e^{(s-t)A}x \right\|_X^2 \\ &\qquad + (s-t_0)\int_{t_0}^s \|u_0(\sigma)\|_U^2 d\sigma\Big] ds \\ &\le C\Big[r^3\|x\|_X^2 + \int_t^{T\wedge(t+r)} \left\| e^{(s-t_0)A}x_0 - e^{(s-t)A}x \right\|_X^2 ds \\ &\qquad + (t-t_0+r)^2\int_{t_0}^{T\wedge(t+r)} \|u_0(\sigma)\|_U^2 d\sigma\Big]. \end{aligned} \tag{4.12}$$

Therefore, estimate (4.6) follows by inserting (4.12) into (4.11). □

Proof of Theorem 4.6. The lower semi–continuity of w in $[0,T]\times X$ was already noted in Remark 3.2. Therefore, it suffices to show that w is upper semi–continuous in $[0,T]\times K$. We will exploit the fact that $V(t,x)=w(T-t,x)$ is the value function of problem (3.2).

Let $(t_0,x_0)\in[0,T]\times K$ and let $\{y_0,u_0\}$ be an optimal pair at (t_0,x_0). For $R>0$ we denote by Γ_R the tube about y_0 defined by

$$\Gamma_R=\bigcup_{s\in[t_0,T]}\{x\in X\ :\ \|x-y_0(s)\|_X\le R\}.$$

In view of assumption (1.3)(i), f and g are Lipschitz continuous on Γ_{R_0} for some $R_0>0$, and we denote by L_0 a Lipschitz constant for f and g on Γ_{R_0}.

Now, let $t\ge t_0, x\in X$ and $r>0$. Let $\{y_r,u_r\}$ be the admissible pair at (t,x) constructed by Lemma 4.7. Estimate 4.5 implies that

$$y_r(s)\in\Gamma_{R_0}\ ,\quad \forall s\in[t,T] \tag{4.13}$$

if r is sufficiently small and (t,x) sufficiently close to (t_0,x_0).

Then, by the definition of V,

$$\begin{aligned}&V(t,x)-V(t_0,x_0)\le J(t,x;u_r)-J(t_0,x_0;u_0)\le \qquad (4.14)\\ &\int_t^T\big[\tfrac12\big(\|u_r(s)\|_U^2-\|u_0(s)\|_U^2\big)+g(y_r(s))-g(y_0(s))\big]ds+f(y_r(T))-f(y_0(T))\end{aligned}$$

as $g\ge 0$. From the Lipschitz continuity of f and g on Γ_{R_0} and (4.5) we obtain

$$\begin{aligned}&\int_t^T[g(y_r(s))-g(y_0(s))]\,ds+f(y_r(T))-f(y_0(T))\\ &\le L_0\int_t^T\|y_r(s)-y_0(s)\|_X\,ds+\|y_r(T)-y_0(T)\|_X\\ &\le C\Big[\Big\|e^{(t-t_0)A}x_0-x\Big\|_X+r\|x\|_X+\int_{t_0}^{T\wedge(t+r)}\|u_0(\sigma)\|_U d\sigma\Big]\end{aligned}\tag{4.15}$$

for some constant $C(\alpha,\omega,T,R,\|B\|)>0$.

Using (4.6) and (4.15) to estimate the right–hand side of (4.14), we obtain

$$\begin{aligned}V(t,x)-V(t_0,x_0)&\le C\Big[\Big\|e^{(t-t_0)A}x_0-x\Big\|_X+r\|x\|_X+\int_{t_0}^{T\wedge(t+r)}\|u_0(\sigma)\|_U d\sigma\Big]\\ &+C\Big[\frac1r\Big\|e^{(t-t_0)A}x_0-x\Big\|_X^2+r\|x\|_U^2+\big(1+\frac{t-t_0}{r}\big)^2\int_{t_0}^{T\wedge(t+r)}\|u_0(\sigma)\|_U^2 d\sigma\Big]ds.\end{aligned}\tag{4.16}$$

Therefore,

$$\begin{aligned}\limsup_{t\downarrow t_0,x\to x_0}V(t,x)&\le V(t_0,x_0)+C\Big[r\|x_0\|_X+\int_{t_0}^{T\wedge(t_0+r)}\|u_0(\sigma)\|_U d\sigma\Big]\\ &+C\Big[r\|x_0\|_X^2+\int_{t_0}^{T\wedge(t_0+r)}\|u_0(\sigma)\|_U^2 d\sigma\Big]ds.\end{aligned}$$

Since $r > 0$ is arbitrary, we conclude that

$$\limsup_{t\downarrow t_0, x\to x_0} V(t,x) \le V(t_0,x_0).$$

In order to complete our reasoning, it remains to analyze the case of $t < t_0$. By the Dynamic Programming principle we have

$$V(t,x) \le V(t_0, y_{t,x}(t_0)) + \int_t^{t_0} \big[g(y_{t,x}(s)) + \frac{1}{2}\|u_{t,x}(s)\|_U^2\big]ds.$$

Now, using (4.16),

$$\begin{aligned} V(t_0, y_{t,x}(t_0)) \le V(t_0,x_0) + C\Big[\, \|x_0 - y_{t,x}(t_0)\|_U + r\|y_{t,x}(t_0)\|_X \\ + \int_{t_0}^{T\wedge(t_0+r)} \|u_0(\sigma)\|_U d\sigma\Big] \\ + C\Big[\frac{1}{r}\,\|x_0 - y_{t,x}(t_0)\|_U^2 + r\|y_{t,x}(t_0)\|_U^2 + \int_{t_0}^{T\wedge(t_0+r)} \|u_0(\sigma)\|^2 d\sigma\Big]ds \end{aligned}$$

for any $r > 0$. Hence,

$$\begin{aligned} \limsup_{t\uparrow t_0, x\to x_0} V(t,x) \le V(t_0,x_0) + C\Big[r\|x_0\|_X + \int_{t_0}^{T\wedge(t_0+r)} \|u_0(\sigma)\|_U d\sigma\Big] \\ + C\Big[r\|x_0\|_X^2 + \int_{t_0}^{T\wedge(t_0+r)} \|u_0(\sigma)\|_U^2 d\sigma\Big]ds \end{aligned}$$

and the conclusion follows as $r \downarrow 0$. □

The same technique as in the proof above yields the weak continuity of solutions if the semigroup generated by A is compact.

Theorem 4.8. *Assume that e^{tA} is compact for $t > 0$, that $e^{t(A-\omega I)}K \subset K$ for all $t \ge 0$ and some $\omega \in \mathbb{R}$, that $B^{-1} \in \mathcal{L}(X,U)$, and suppose that g is bounded on the bounded subsets of K. Let w be the weak solution of* (3.1) *with ϕ and ψ given by* (1.8)*. Then w is sequentially weakly continuous in $]0,T] \times K$. Furthermore, if f is sequentially weakly continuous, then w is sequentially weakly continuous in $[0,T] \times K$.*

Proof. The weak lower semi–continuity of w in $[0,T] \times X$ was already proved in Corollary 3.6. Therefore, it suffices to show that w is weakly upper semi–continuous in $]0,T] \times K$. Again, we will exploit the fact that $V(t,x) = w(T-t,x)$ is the value function of problem (3.2), and prove that V is weakly upper semi–continuous in $[0,T[\times K$.

Let $(t_0,x_0) \in [0,T[\times K$ and $(t_n,x_n) \in [0,T[\times K$ be such that

$$t_n \ge t_0, \quad t_n \to t_0, \quad x_n \rightharpoonup x_0 \quad \text{as} \quad n \to +\infty.$$

Let, moreover, $\{y_0,u_0\}$ be an optimal pair at (t_0,x_0) and fix $0 < r < 1 \wedge (T-t_0)$. For any $n \in \mathbb{N}$ we denote by$\{y_n,u_n\}$ the admissible pair at (t_n,x_n) constructed by Lemma 4.7.

For sufficiently large n we have that $t_n + r < T$, so

$$y_n(s) = y_0(s), \quad u_n(s) = u_0(s), \quad \forall s \in [t_n + r, T].$$

Hence, by the definition of V and the fact that $g \geq 0$,

$$V(t_n, x_n) - V(t_0, x_0) \leq \int_{t_n}^{t_n+r} \Big[\frac{1}{2}\left(\|u_n(s)\|_U^2 - \|u_0(s)\|_U^2\right) + g(y_n(s)) - g(y_0(s))\Big] ds \,.$$

Now, by estimate (4.6) and the fact that y_n is bounded, uniformly with respect to n, we obtain

$$\begin{aligned} V(t_n, x_n) - V(t_0, x_0) \leq C\Big[r + \frac{1}{r^2}\int_{t_n}^{t_n+r} \Big\|e^{(s-t_0)A}x_0 - e^{(s-t_n)A}x_n\Big\|_X^2 ds \\ + \big(1 + \frac{t_n - t_0}{r}\big)^2 \int_{t_0}^{t_n+r} \|u_0(\sigma)\|_U^2 d\sigma\Big]. \end{aligned} \tag{4.17}$$

Therefore,

$$\begin{aligned} \limsup_{n\to+\infty} V(t_n, x_n) \leq V(t_0, x_0) + C\Big[r + \int_{t_0}^{t_0+r} \|u_0(\sigma)\|_U^2 d\sigma\Big] \\ + \frac{C}{r^2}\limsup_{n\to+\infty}\int_{t_n}^{t_n+r} \Big\|e^{(s-t_0)A}x_0 - e^{(s-t_n)A}x_n\Big\|_X^2 ds. \end{aligned} \tag{4.18}$$

Since $t_n < t_0 + \frac{r^3}{2}$ for sufficiently large n, we have that

$$\begin{aligned} &\int_{t_n}^{t_n+r} \Big\|e^{(s-t_0)A}x_0 - e^{(s-t_n)A}x_n\Big\|_X^2 ds \\ &\leq \int_{t_n}^{t_0+r^3} \Big\|e^{(s-t_0)A}x_0 - e^{(s-t_n)A}x_n\Big\|_X^2 ds + \int_{t_0+r^3}^{t_n+r} \Big\|e^{(s-t_0)A}x_0 - e^{(s-t_n)A}x_n\Big\|_X^2 ds \\ &\leq C\Big[r^3 + \int_{t_0+r^3}^{T} \Big\|\Big[e^{(s-t_0)A} - e^{(s-t_n)A}\Big]x_0\Big\|_X^2 ds + \int_{t_0+r^3}^{T} \Big\|e^{\frac{r^3}{2}A}(x_0 - x_n)\Big\|_X^2 ds\Big]. \end{aligned}$$

Recalling that e^{tA} is strongly continuous and compact for $t > 0$, from the above inequality and (4.18) we obtain

$$\limsup_{n\to+\infty} V(t_n, x_n) \leq V(t_0, x_0) + C\Big[r + \int_{t_0}^{t_0+r} \|u_0(\sigma)\|_U^2 d\sigma\Big].$$

Then,

$$\limsup_{n\to\infty} V(t_n, x_n) \leq V(t_0, x_0) \tag{4.19}$$

as $r > 0$ is arbitrary.

To treat the case of $t_n < t_0$ we argue as in the proof of Theorem 4.6. By the Dynamic Programming principle we have

$$V(t_n, x_n) \leq V(t_0, y_{t_n,x_n}(t_0)) + \int_{t_n}^{t_0} \big[g(y_{t_n,x_n}(s)) + \frac{1}{2}\|u_{t_n,x_n}(s)\|_U^2\big] ds, \tag{4.20}$$

where y_{t_n,x_n} and u_{t_n,x_n} are defined in (4.7). Now, $y_{t_n,x_n}(t_0) \rightharpoonup x_0$ as $n \to +\infty$, and so, by the first part of the proof,

$$\limsup_{n\to\infty} V(t_0, y_{t_n,x_n}(t_0)) \leq V(t_0, x_0).$$

Therefore, estimate (4.20) yields the conclusion (4.19).

By a similar argument one can show weak continuity up to time T, provided that f is sequentially weakly continuous. □

Remark 4.9. Under the assumptions of Theorem 4.8, we have that the approximating sequence $\{w_n\}$ in Definition 3.1 may be assumed to converge to w uniformly on the bounded subsets of $[a, T] \times K$, for each $a > 0$. This fact follows, by Dini's Theorem, from the weak continuity of w_n and w.

5. The closed loop equation. In this section we assume that there exists the weak solution w of problem (3.1) for fixed $\phi, \psi \in \Sigma_K$ and that $B^{-1} \in \mathcal{L}(X, U)$. Without loss of generality we may assume $B = I$.

We want to study the properties of the solutions of the closed loop equation

$$y'(s) \in Ay(s) - \partial_x w(T-s, y(s)), \quad y(t) = x \in K. \tag{5.1}$$

As usual we say that $y(\cdot; x) \in C([t,T]; X)$ is a *mild solution* of (5.1) if there exists $u(\cdot; x) \in L^2(t, T; X)$ such that the function $u(s; x) \in -\partial_x w(T-s, y(s))$ for almost every $s \in [t, T]$, and

$$y(s;x) = e^{(s-t)A}x - \int_t^s e^{(s-\sigma)A} u(\sigma; x) d\sigma \ . \tag{5.2}$$

It will turn out that if $y(\cdot; x)$ is a mild solution of (5.1), then the pair $\{y(\cdot; x), u(\cdot; x)\}$ is optimal for the functional (3.2).

We begin with the following result concerning continuous dependence upon the initial data.

Theorem 5.1. *Let $x_1, x_2 \in K$. Then for each $s \in [t, T]$ we have*

$$\|y(s; x_1) - y(s; x_2)\|_X \le e^{\alpha(s-t)} \|x_1 - x_2\|_X,$$

where α satisfies (1.3)(ii), *i.e.,*

$$\|e^{tA}x\|_X \le e^{\alpha t}\|x\|_X.$$

Proof. For $i = 1, 2$ we set

$$y_i(s) = y(s; x_i), \quad u_i(s) = u(s, x_i)$$

and

$$x_{i,n} = n(n-A)^{-1}x_i \ , \ u_{i,n} = n(n-A)^{-1}u_i \ .$$

Then we have

$$x_{i,n} \in D(A) \ , \ u_{i,n} \in L^2(t, T; D(A))$$

and

$$\lim_{n\to\infty} x_{i,n} = x_i \quad \text{in } X \ ,$$

$$\lim_{n\to\infty} u_{i,n} = u_i \quad \text{in } L^2(t, T; X).$$

Therefore, there exists $y_{i,n} \in W^{1,2}(0, T; X) \cap L^2(0, T; D(A))$ verifying

$$y'_{i,n}(s) = Ay_{i,n}(s) + u_{i,n}(s), \quad y_{i,n}(t) = x_{i,n} \ , \tag{5.3}$$

and moreover,

$$\lim_{n\to\infty} y_{i,n} = y_i \quad \text{in } C([t,T];X)\ .$$

From (5.3) we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{ds}\|y_{1,n}(s)-y_{2,n}(s)\|_X^2 = \langle Ay_{1,n}(s)-Ay_{2,n}(s), y_{1,n}(s)-y_{2,n}(s)\rangle_X + \\ &\langle u_{1,n}(s)-u_{2,n}(s), y_{1,n}(s)-y_{2,n}(s)\rangle_X \\ \le\ & \alpha\|y_{1,n}(s)-y_{2,n}(s)\|_X^2 + \langle u_{1,n}(s)-u_{2,n}(s), y_{1,n}(s)-y_{2,n}(s)\rangle_X .\end{aligned}$$

Integrating over $[t,\tau]$ we get

$$\begin{aligned}&\frac{1}{2}\|y_{1,n}(\tau)-y_{2,n}(\tau)\|_X^2 e^{-2\alpha(\tau-t)} \\ \le\ & \frac{1}{2}\|x_{1,n}-x_{2,n}\|_X^2 + \int_t^\tau e^{-2\alpha(s-t)}\langle u_{1,n}(s)-u_{2,n}(s), y_{1,n}(s)-y_{2,n}(s)\rangle_X\, ds.\end{aligned}$$

Therefore, letting $n\to\infty$, we get

$$\begin{aligned}&\|y_1(\tau)-y_2(\tau)\|_X^2 \\ \le\ & e^{2\alpha(\tau-t)}\|x_1-x_2\|_X^2 + 2\int_t^\tau e^{2\alpha(\tau-s)}\langle u_1(s)-u_2(s), y_1(s)-y_2(s)\rangle_X\, ds.\end{aligned}$$

Now the convexity of $w(t,\cdot)$ implies

$$\langle u_1(s)-u_2(s), y_1(s)-y_2(s)\rangle_X \le 0\ ,$$

and the result follows. □

As a consequence we have the following uniqueness result:

Corollary 5.2. *The mild solution of* (5.1) *is unique.*

If A generates a contraction semigroup we have:

Corollary 5.3. *Let $\alpha = 0$ and assume that the convex set K is given by*

$$K = \{\ \|x\|_X \le R\,\}$$

for some $R>0$. Then if $x\in \mathring{K}$ and y is the mild solution of (5.1), *then $y(s)\in\mathring{K}$, for each $s\in[t,T]$.*

We now study the existence of a solution of (5.1). To this end we consider the following approximating problem

$$y'(s) = Ay(s) - \nabla_x w_\epsilon(T-s, y(s)), \quad y(t) = x\in K, \tag{5.4}$$

where w_ϵ is the strong solution of (3.1) with data $\{\phi_\epsilon\}$, $\{\psi_\epsilon\}$ given by (3.11).

We have:

Theorem 5.4. *Assume that problem* (3.1) *has the weak solution* w. *Then, for each* $x \in K$ *there exists a unique mild solution* $y(\cdot;x)$ *of* (5.1). *Moreover, we have*

$$y(s;x) = \lim_{\epsilon \to 0} y_\epsilon(s;x) \ , \ \text{in } X,$$

where $y(\cdot;x)_\epsilon$ *is the mild solution of* (5.4). *Moreover, the pair* $\{y(\cdot;x), u(\cdot;x)\}$, *where*

$$u(s;x) = -\lim_{\epsilon \to 0} \nabla_x w_\epsilon(T-s, y_\epsilon(s;x))$$

is the unique optimal pair for the functional (3.2).

Proof. In light of Proposition 2.3 we have that the function $s \to \nabla_x w_\epsilon(T-s, y(s))$ is continuous in $[t,T]$ for each $y \in C([t,T];X)$. Moreover, since $w_\epsilon(t,\cdot) \in C^1_{Lip}(x)$, we have that there exists a (unique) mild solution of (5.4) , i.e., a function $y_\epsilon(\cdot;x) \in C([t,T];X)$ satisfying

$$y_\epsilon(s;x) = e^{(s-t)A}x - \int_t^s e^{(s-\sigma)A} \nabla_x w_\epsilon(T-\sigma, y_\epsilon(\sigma;x)) d\sigma. \tag{5.5}$$

Now set

$$u_\epsilon(s;x) = -\nabla_x w_\epsilon(T-s, y_\epsilon(s;x)).$$

Then, Theorem 3.4 implies that there exists $\{y^*, u^*\}$ such that $y_\epsilon(\cdot;x) \to y^*$ in $C([t,T];X)$ and $u_\epsilon(\cdot;x) \to u^*$ in $L^2(t,T;X)$. Moreover, $\{y^*, u^*\}$ is the optimal pair for the functional (3.2). Furthermore, from Theorem 4.4 we have

$$u^*(s) \in -\partial_x w(T-s, y^*(s)) \ .$$

Therefore, y^* is the mild solution of (5.1). □

We conclude this section with the continuous dependence of optimal controls on the state, in the strong L^2 topology.

Theorem 5.5. *Assume that* $e^{t(A-\omega I)}K \subset K$ *for all* $t \geq 0$ *and some* $\omega \in \mathbb{R}$. *Let* $t \in [0,T], x \in X$ *and let* $y(\cdot;x)$ *be the mild solution of equation* (5.1), $u(\cdot;x)$ *being the corresponding optimal control. Then, the map*

$$\Lambda : K \to L^2(t,T;X) \ , \quad \Lambda(x) = u(\cdot;x)$$

is continuous.

Proof. Let $x_n \in K$, $x_n \to x$ as $n \to \infty$. We will show that $u(\cdot;x_n) \to u(\cdot;x)$ in $L^2(t,T;X)$. Indeed, Theorem 5.1 implies that $y(\cdot;x_n) \to y(\cdot;x)$ in $C([t,T];X)$. Therefore, recalling Theorem 4.6,

$$\int_t^T \left[|u(s;x_n)|^2 - |u(s;x)|^2\right] ds = 2[w(T-t,x_n) - w(T-t,x)]$$
$$-2\int_t^T [g(y(s;x_n)) - g(y(s;x))]\, ds + 2\,[f(y(T;x_n)) - f(y(T;x))] = 0.$$

Since we may assume, without loss of generality, that $u(\cdot;x_n) \rightharpoonup u(\cdot;x)$ weakly in $L^2(t,T;X)$, the conclusion follows. □

6. Examples. Let Ω be an open bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and let $B \in \mathcal{L}(L^2(\Omega))$. Given $u \in L^2(]t_0, T[\times\Omega)$ we consider the following initial boundary value problem

$$\begin{cases} y_t(t,\xi) = \Delta y(t,\xi) + (Bu)(t,\xi)\ ;\ t_0 \le t \le T, \xi \in \Omega \\ y(t,\xi) = 0\ ;\ t_0 \le t \le T, \xi \in \partial\Omega \\ y(t_0,\xi) = y_0(\xi)\ ; \xi \in \Omega\ , \end{cases} \tag{6.1}$$

where Δ is the Laplace operator.

Furthermore we introduce the functional

$$J(t_0, y_0; u) = \frac{1}{2}\int_\Omega |y(T,\xi)|^2 d\xi + \frac{1}{2}\int_{t_0}^T dt \int_\Omega \left(|y(t,\xi)|^2 + |u(t,\xi)|^2\right) d\xi. \tag{6.2}$$

We want to describe some optimal control problems which can be studied by means of the results of the preceding sections.

We begin with the problem of minimizing the functional (6.2) overall $\{y, u\}$ verifying (6.1) and the state constraint

$$\int_\Omega |y(t,\xi)|^2 d\xi \le R; \quad t_0 \le t \le T, \tag{6.3}$$

where R is a given positive constant.

We get the following:

Theorem 6.1. *Let $y_0 \in L^2(\Omega)$ satisfy the property*

$$\int_\Omega |y_0(\xi)|^2 d\xi \le R.$$

Then there exists a unique optimal pair $\{y^*, u^*\}$ at (t_0, y_0) for problem (6.1), (6.2) *and* (6.3). *Moreover, we have*

$$J(t_0, y_0; u^*) = w(T - t_0, y_0),$$

where w is the weak solution of the problem

$$\begin{cases} w_t(t,x) + \frac{1}{2}\int_\Omega |B^*\nabla_x w(t,x)(\xi)|^2 d\xi - \int_\Omega \Delta x(\xi)(\nabla_x w(t,x))(\xi) d\xi = \psi(x) \\ x(\xi) = 0\ ,\ \xi \in \partial\Omega \\ w(0,x) = \phi(x). \end{cases} \tag{6.4}$$

Here ψ is defined as

$$\psi(x) = \begin{cases} \frac{1}{2}\int_\Omega |x(\xi)|^2 d\xi, & \text{if } \int_\Omega |x(\xi)|^2 d\xi \le R \\ +\infty, & \text{if } \int_\Omega |x(\xi)|^2 d\xi > R \end{cases}$$

and ϕ is similarly defined.

Moreover, w satisfies the following properties:

(i) *$x \to w(t,x)$ is Lipschitz continuous, uniformly for $t \in [0,T]$, for*

$$\int_\Omega |x(\xi)|^2 d\xi \le R', \tag{6.5}$$

where $R' < R$

(ii) *$w(\cdot, x)$ is Lipschitz continuous on $[0,T]$ for each x satisfying* (6.5) *and*

$$x \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega).$$

Moreover, let ψ_ϵ and ϕ_ϵ denote the Yosida approximation of ψ and ϕ given by (3.11)*, let w_ϵ denote the solution of* (6.4) *with ψ, ϕ replaced by $\psi_\epsilon, \phi_\epsilon$, and let y^*_ϵ be the strong solution of*

$$y^*_t(t,\xi) = \Delta y(t,\xi) - BB^*\nabla_x w_\epsilon(T-t, y(t,\cdot))(\xi), \quad y(t_0,\xi) = y_0(\xi).$$

Then we have

(iii)

$$y^* = \lim y^*_\epsilon \quad in \quad C([t_0,T]; L^2(\Omega))$$

and

$$u^* = \lim B^*\nabla_x w_\epsilon(T-\cdot, y^*_\epsilon) \quad in \quad L^2(]t_0,T[\times\Omega).$$

If in addition $B^{-1} \in \mathcal{L}(L^2(\Omega))$, then we have

(iv) *$w_\epsilon(t,x) \to w(t,x)$, uniformly for $t \in [a,T]$ for each $a > 0$ and x on bounded subsets of K, where*

$$K = \{x \in L^2(\Omega) : \textstyle\int_\Omega |x(\xi)|^2 d\xi \le R\},$$

(v) *w is continuous in $[0,T] \times K$, and sequentially weakly lower semicontinuous on $]0,T] \times K$,*

(vi) *the optimal pair satisfies the feedback formula*

$$u^*(t,\xi) \in -B^*\partial_x w(T-t, y^*(t,\cdot))(\xi).$$

Finally we have the following result concerning continuous dependence of optimal pairs upon the initial data

(vii) *Let $\{x_n\} \subset K$ be such that $x_n \to x$, in $L^2(\Omega)$ and let $\{y^*_n, u^*_n\}$ and $\{y^*, u^*\}$ be the optimal pairs at (t_0, x_n) and (t_0, x), respectively, then*

$$y^*_n \to y^* \quad in \quad C([t_0,T]; L^2(\Omega)) \quad and \quad u^*_n \to u^* \quad in \quad L^2(]t_0,T[\times\Omega)\ .$$

Proof. Let A denote the operator in $X = L^2(\Omega)$ defined as

$$D(A) = W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega), \quad Ax = \Delta x. \tag{6.6}$$

Then it is known that A generates a compact semigroup e^{tA} in X, for $t > 0$, verifying

$$\|e^{tA}x\|_X \le \|x\|_X.$$

Therefore the results follow from Example 3.9, from Theorems 4.4, 4.5, 4.6, 4.8, and from Remark 4.9. ☐

Another interesting problem is the following: minimize the functional (6.2) overall $\{y, u\}$ verifying (6.1) and the state constraint

$$y(t,\xi) \ge 0; \quad t_0 \le t \le T. \tag{6.7}$$

We have the following result:

Theorem 6.2. *Let $y_0 \in L^2(\Omega)$ satisfy the property*

$$y_0(\xi) \geq 0.$$

Then there exists an optimal pair $\{y^*, u^*\}$ at (t_0, y_0) for problem (6.1), (6.2), (6.7). *Moreover, we have*

$$J(t_0, y_0; u^*) = w(T - t_0, y_0),$$

where w is the weak solution of problem (6.4) *with ψ defined as*

$$\psi(x) = \begin{cases} \frac{1}{2}\int_\Omega |x(\xi)|^2 d\xi, & \text{if } x(\xi) \geq 0 \\ +\infty, & \text{if } x(\xi) < 0 \end{cases}$$

and ϕ is similarly defined. Moreover, assertions (iii)-(vii) *of Theorem* 6.1 *remain valid with K replaced by*

$$K = \{x \in L^2(\Omega) : x(\xi) \geq 0\}.$$

Proof. Let A denote the operator defined by (6.6). Then it is known that A satisfies $e^{tA}K \subset K$. Therefore the results follow from Example 3.9, from Theorems 4.4, 4.6, 4.8 and from Remark 4.9. □

Another problem which can be studied is the following: minimize the functional (6.1) overall $\{y, u\}$ verifying (6.2) and the state constraint

$$\|y(t, \cdot)\|_{L^\infty(\Omega)} \leq R; \quad t_0 \leq t \leq T, \tag{6.8}$$

where R is a given constant. Then we have the following result:

Theorem 6.3. *Let $y_0 \in L^\infty(\Omega)$ satisfy the property*

$$\text{ess sup}\, |y_0(\xi)| \leq R.$$

Then there exists an optimal pair $\{y^*, u^*\}$ at (t_0, y_0) for problem (6.2), (6.1), (6.8). *Moreover, we have*

$$J(t_0, y_0; u^*) = w(T - t_0, y_0),$$

where w is the weak solution of problem (6.4) *with ψ is defined as*

$$\psi(x) = \begin{cases} \frac{1}{2}\int_\Omega |x(\xi)|^2 d\xi, & \text{if } \text{ess sup}\, |x(\xi)| \leq R \\ +\infty, & \text{if } \text{ess sup}\, |x(\xi)| > R \end{cases}$$

and ϕ is similarly defined. Moreover, assertions (iii)-(vii) *of Theorem* 6.1 *remain valid with K replaced by*

$$K = \{x \in L^\infty(\Omega) : \text{ess sup}\, |x(\xi)| \leq R\}.$$

Proof. Let A denote the operator defined by (6.6). Then by maximum principle for elliptic operators it is known that A satisfies $e^{tA}K \subset K$. Therefore the results follow from Example 3.9, from Theorems 4.4, 4.6 and 4.8 and from Remark 4.9. □

Finally we can study the problem of minimizing the functional (6.2) overall $\{y, u\}$ verifying (6.1) and the state constraint

$$\int_\Omega |\nabla y(t, \xi)|^2 d\xi \leq R; \quad t_0 \leq t \leq T, \tag{6.9}$$

where R is a given constant.

We have the following result:

Theorem 6.4. *Let* $y_0 \in W_0^{1,2}(\Omega)$ *satisfy the property*

$$\int_\Omega |\nabla y_0(\xi)|^2 d\xi \le R.$$

Then there exists an optimal pair $\{y^*, u^*\}$ *at* (t_0, y_0) *for problem* (6.2), (6.1), (6.9). *Moreover, we have*

$$J(t_0, y_0; u^*) = w(T - t_0, y_0),$$

where w *is the weak solution of problem* (6.4) *with* ψ *is defined as*

$$\psi(x) = \begin{cases} \frac{1}{2}\int_\Omega |x(\xi)|^2 d\xi, & \text{if } \int_\Omega |\nabla x(\xi)|^2 d\xi \le R \\ +\infty, & \text{if } \int_\Omega |\nabla x(\xi)|^2 d\xi > R \end{cases}$$

and ϕ *is similarly defined. Moreover, assertions* (iii)-(vii) *of Theorem* 6.1 *remain valid with* K *replaced by*

$$K = \{x \in W_0^{1,2}(\Omega) : \textstyle\int_\Omega |\nabla x(\xi)|^2 d\xi \le R\}.$$

Proof. Let A denote the operator defined by (6.6). Then it is known that

$$W_0^{1,2}(\Omega) = (D(A), L^2(\Omega))_{1/2,2},$$

where $(D(A), L^2(\Omega))_{1/2,2}$ denotes the real interpolation space between $D(A)$ and $L^2(\Omega)$. Therefore it is known that if $x \in W_0^{1,2}(\Omega)$, then the function $z(t) = e^{tA}x$ satisfies

$$z \in L^2(0, T; D(A)) \cap W^{1,2}(0, T; X) \subset C([0, T]; W_0^{1,2}(\Omega)) \ .$$

Moreover, we have

$$-\frac{1}{2}\frac{d}{dt}\|\nabla z(t)\|_X^2 = \langle z'(t), Az(t)\rangle_X = \|Az(t)\|_X^2 \ge 0 \ ,$$

so that A satisfies $e^{tA}K \subset K$. Therefore the results follow from Example 3.9, from Theorems 4.4, 4.6, and 4.8 and from Remark 4.9. □

Remark 6.5. It can be seen that results similar to those of Theorems 6.1, 6.2, 6.3 and 6.4 remain valid if the Dirichlet boundary condition $y(t, \xi) = 0$ in (6.1) is replaced by the Neumann condition $\frac{\partial}{\partial \nu} y(t, \xi) = 0$.

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