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APPROXIMATION OF AN INVERSE PROBLEM FOR VARIATIONAL INEQUALITIES

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Abstract. Inverse problems of dynamical identification of distributed control in hyperbolic variational inequalities are investigated. A numerical solution method based on constructions of the theory of positional control is described.

1. Introduction. Consider the hyperbolic system described by the variational inequalities

$$\begin{gathered}(\ddot{x}(t) - Bu(t) - f(t), \dot{x}(t) - z)_H + \langle Ax(t), \dot{x}(t) - z\rangle + \varphi(\dot{x}(t)) - \varphi(z) \le 0 \\ \text{a.e. } t \in T = [t_0, \theta], \quad \forall z \in V, \quad x(t_0) = x_{10} \in V, \quad \dot{x}(t_0) = x_0 \in H. \end{gathered} \tag{1.1}$$

Here H and V are real Hilbert spaces with norms $|\cdot|_H$ and $|\cdot|_V$ resp., $H = H^*$, $V \subset H$, V is densely and continuously imbedded in H, $(\cdot\,,\cdot)_H$ is the scalar product in H, $\langle\cdot\,,\cdot\rangle$ is the duality between V and V^*, $f(\cdot) \in L_2(T; H)$ is a given function, $A : V \to V^*$ is a linear, continuous ($A \in L(V; V^*)$) and symmetric operator satisfying the condition

$$\langle Ay, y\rangle \ge \omega |y|_V^2, \quad \forall y \in V$$

for certain $\omega > 0$, $(U, |\cdot|_U)$ is a uniformly convex Banach space, $\varphi : H \to \bar{\mathbb{R}} = \mathbb{R}^+ \cup \{+\infty\}$ is a convex and lower semicontinuous (l.s.c.) proper function, $\mathbb{R}^+ = \{r \in \mathbb{R} : r \ge 0\}$, $0 \in D(\varphi) = \{x \in H : \varphi(x) < +\infty\}$, $B \in L(U; V)$ is a linear continuous operator, $u(t) \in P$ for almost every $t \in T$, $P \subset U$ is a convex, closed and bounded set.

We assume that for any control $u(\cdot) \in P(\cdot) = \{u(\cdot) \in L_2(T; U) : u(t) \in P \text{ a.e. } t \in T\}$ there is a single solution of the system (1.1), $x(\cdot) = x(\cdot; t_0, x_0, x_{10}, u(\cdot)) \in C(T, V)$, such that $\dot{x}(\cdot) \in C(T; H) \cap L_\infty(T; V)$, $\ddot{x}(\cdot) \in L_2(T; H)$. This holds for example if $H = L_2(\Omega)$, $V = H_0^1(\Omega)$, $f(\cdot) \in L_2(T; V)$, $x_{10} \in V$, $\Delta_L x_{10} \in H$, $x(\eta) \in \operatorname{dom}(\partial j)$ for almost every $\eta \in \Omega$, $\Omega \subset \mathbb{R}^n$ is a bounded open set with smooth boundary, $j : \mathbb{R} \to \bar{\mathbb{R}}$ is a convex, proper and l.s.c. function, ∂j is a subdifferential of j, $\varphi(y) = \int_\Omega j(y(\eta))\, d\eta$, if $y \in H$, $\eta \to j(y(\eta)) \in L_1(\Omega)$, $\varphi(y) \to +\infty$, in the opposite case, and

$$\langle Ax, y\rangle = \int_\Omega \nabla x(\eta) \nabla y(\eta)\, d\eta, \quad \forall x, y \in V$$

[3, p. 279]. Here Δ_L is the Laplace operator.

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The problem in question can be explained as follows. The motion $x(\cdot) = x(\cdot; t_0, x_0, x_{10}, u(\cdot))$ of the system (1.1) depending on a time-varying unknown control $u(\cdot) \in P(\cdot)$ proceeds at the time interval T. At time instants $\tau_i \in \Delta = \{\tau_i\}_{i=0}^m$, $\tau_{i+1} = \tau_i + \delta$, $\tau_0 = t_0$, $\tau_m = \theta$, the coordinates $\dot{x}(\tau_i)$ of the system (1.1) are measured approximately. A motion $x(\cdot)$ is unknown. Let $U_*(x(\cdot))$ be the set of all controls from $P(\cdot)$ generating $x(\cdot) : U_*(x(\cdot)) = \{v(\cdot) \in P(\cdot) : x(\cdot\,; t_0, x_0, x_{10}, v(\cdot)) = x(\cdot)\}$. The problem is to calculate an approximation to a certain element $u(\cdot) \in U_*(x(\cdot))$ synchronously with the process, basing on nonaccurate measurements of $\dot{x}(\tau_i)$.

An approach to the problems of the above type based on the ideas of positional control theory [11,12] and the theory of ill-posed problems [10, 22] was suggested in [13] and developed in [14–21]. Below, an algorithm solving the problem stable with respect to informational and computational disturbances is constructed. For certain classes of hyperbolic variational inequalities, the above problems are investigated in [20, 17]. The basic difference between [20,17] and the present paper is that the later utilizes finite-dimensional models. The analogous constructions for parabolic variational inequalities were announced in [16]. Note that inverse problems for variational inequalities are considered in [4–6, 8, 9].

2. The main result. Let triples $\{V_\varepsilon, p_\varepsilon, r_\varepsilon\}$ and $\{U_\varepsilon, q_\varepsilon, s_\varepsilon\}$ ($\varepsilon \in H_0, H_0$ is a neighborhood of zero in $\mathbb{R}^n$) form inner approximations [2, 7, 1] to the spaces V and U, i.e.,

1^0 $V_\varepsilon, U_\varepsilon$ are finite-dimensional spaces whose norms $|\cdot|_\varepsilon$ and $|\,\|\cdot\|\,|_\varepsilon$ are generated by the norms $|\cdot|_H$ and $|\cdot|_U$:

$$|y|_\varepsilon = |p_\varepsilon y|_H, \quad \forall y \in V_\varepsilon, \quad |\,\|y\|\,|_\varepsilon = |q_\varepsilon y|_U, \quad \forall y \in U_\varepsilon;$$

2^0 $p_\varepsilon : V_\varepsilon \to V$, $r_\varepsilon : V \to V_\varepsilon$, $q_\varepsilon : U_\varepsilon \to U$ and $s_\varepsilon : U \to U_\varepsilon$ are linear and continuous operators, p_ε and q_ε are one-to-one;

3^0 $p_\varepsilon r_\varepsilon y \to y$ in V as $\varepsilon \to 0$, $\quad \forall y \in V$;

4^0 $q_\varepsilon s_\varepsilon u \to u$ in U as $\varepsilon \to 0$, $\quad \forall u \in P$;

5^0 $|q_\varepsilon s_\varepsilon u|_U \le c_1$, $\quad \forall u \in P$.

Let $(\cdot\,,\cdot)_\varepsilon$ be the inner product in V_ε defined by $(x_\varepsilon, y_\varepsilon)_\varepsilon = (p_\varepsilon x_\varepsilon, p_\varepsilon y_\varepsilon)_H, \forall x_\varepsilon, y_\varepsilon \in V_\varepsilon$. In the space V_ε we also introduce the norm $\|\cdot\|_\varepsilon$ corresponding to the norm in V : $\|x_\varepsilon\|_\varepsilon = |p_\varepsilon x_\varepsilon|_V$, $\forall x_\varepsilon \in V_\varepsilon$. For any $\varepsilon \in H_0$, $\varepsilon \neq 0$, we define operators $A_\varepsilon : V_\varepsilon \to V_\varepsilon$ and $B_\varepsilon : U_\varepsilon \to V_\varepsilon$ as

$$(A_\varepsilon y_\varepsilon, z_\varepsilon)_\varepsilon = \langle A p_\varepsilon y_\varepsilon, p_\varepsilon z_\varepsilon \rangle, \quad \forall y_\varepsilon, z_\varepsilon \in V_\varepsilon, \tag{2.1}$$

$$(B_\varepsilon u_\varepsilon, y_\varepsilon)_\varepsilon = (B q_\varepsilon u_\varepsilon, p_\varepsilon y_\varepsilon), \quad \forall u_\varepsilon \in U_\varepsilon, y_\varepsilon \in V_\varepsilon. \tag{2.2}$$

It is easy to see that $A_\varepsilon \in L(V_\varepsilon; V_\varepsilon)$, $B_\varepsilon \in L(U_\varepsilon; V_\varepsilon)$ and $|B_\varepsilon|_{L(U_\varepsilon;V_\varepsilon)} \le |B|_{L(U;H)}$ for any $\varepsilon \in H_0$, $\varepsilon \neq 0$. Let a convex, proper and l.s.c. function $\varphi_\varepsilon : V_\varepsilon \to (-\infty, +\infty]$ be given by $\varphi_\varepsilon(y_\varepsilon) = \varphi(p_\varepsilon y_\varepsilon)$, $\forall y_\varepsilon \in V_\varepsilon$ and a function $f_\varepsilon(\cdot) \in L_2(T; V_\varepsilon)$ be given by

$$(f_\varepsilon(t), y_\varepsilon)_\varepsilon = (f(t), p_\varepsilon y_\varepsilon)_H \quad \text{a.e. } t \in T, \quad \forall y_\varepsilon \in V_\varepsilon.$$

Let Δ_h be a partition of the interval T with diameter $\delta(h)$, $\Delta_h = \{\tau_{h,i}\}_{i=0}^{m_h}$, $m_h = m(\delta(h))$, $\delta(h) = \tau_{h,i+1} - \tau_{h,i}$, $\tau_{h,0} = t_0$, $\tau_{h,m_h} = \theta$; $P_\varepsilon \subset U_\varepsilon$ ($\varepsilon \in H_0, \varepsilon \neq 0$) be a family of convex, closed and bounded sets with the property $æ(P_\varepsilon, s_\varepsilon P) \le \gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$, $\varepsilon \in H_0$; $æ(E_1, E_2)$ be the Hausdorff distance between $E_1, E_2 \subset U_\varepsilon$ corresponding to the norm $|\,\|\cdot\|\,|_\varepsilon$; X_T be the bundle of motions $X_T = \{x(\cdot; t_0, x_0, x_{10}, u(\cdot)) : u(\cdot) \in P(\cdot)\}$; $u_*(\cdot; x(\cdot))$ be the element from $U_*(x(\cdot))$ whose $L_2(T;U)$-norm is minimal; and

$$\nu(t;\varepsilon,x(\cdot)) = |(I-p_\varepsilon r_\varepsilon)x_{10}|_V + |(I-p_\varepsilon r_\varepsilon)x_{10}|_V^2 + \int_{t_0}^{t} \{|(I-p_\varepsilon r_\varepsilon)\dot{x}(\tau)|_V + |(I-p_\varepsilon r_\varepsilon)\dot{x}(\tau)|_V^2\}\,d\tau \quad (x(\cdot) \in X_T);$$

$$\mu(t;h,\delta,\varepsilon,x(\cdot),u(\cdot)) = \int_{t_0}^{t} |(I-q_\varepsilon s_\varepsilon)u(\tau)|_U d\tau + \delta^{1/2} + h + \gamma(\varepsilon) + \nu(t;\varepsilon,x(\cdot)).$$

The problem consists in designing an algorithm for approximate calculation of the unknown control $u_*(\cdot;x(\cdot))$ on the basis of observation of the values $\xi_i^{h,\varepsilon} \in V_\varepsilon$ at time instants $\tau_i = \tau_{h,i}$, $i \in [0 : m_h - 1]$, with the property

$$|\dot{x}(\tau_i) - p_\varepsilon \xi_i^{h,\varepsilon}|_{V^*} \le h,\ i \in [1 : m_h - 1],\ |\dot{x}(t_0) - p_\varepsilon \xi_0^{h,\varepsilon}|_H \le h. \tag{2.3}$$

At the time t_0 the value $\xi_{10}^{h,\varepsilon} \in V_\varepsilon$, $|x_{10} - p_\varepsilon \xi_{10}^{h,\varepsilon}|_V \le h$ becomes available. Assume that the following conditions are fulfilled:

6^0 $\int_{t_0}^{\theta} |(I-p_\varepsilon r_\varepsilon)\dot{x}(\tau)|_V^2 d\tau \to 0$ as $\varepsilon \to 0$, $\quad \forall x(\cdot) \in X_T$,

7^0 $|q_\varepsilon p_\varepsilon|_U \le c_2$, $\quad \forall \varepsilon \in H_0$,

8^0 $\sup\{|y|_{V^*} : y \in \partial\varphi(x), x \in X\} < +\infty$ for every bounded set $X \subset H$.

If $|p_\varepsilon z_\varepsilon|_{L(V;V)} \le c$, $\forall \varepsilon \in H_0$ then 6^0 follows from 2^0, 3^0 and the Lebesgue's limit theorem.

Introduce functions $\varepsilon(h)$, $\alpha(h)$ and a family of partitions $\{\Delta_h\}$ with diameters $\delta(h)$ such that

$$\varepsilon(h) \in H_0,\ \varepsilon(h) \ne 0\ \forall h \in (0,1),\ \alpha(h) > 0,\ \delta(h) \to 0,\ \alpha(h) \to 0,\ \varepsilon(h) \to 0,$$

$$\mu(\theta; h, \delta(h), \varepsilon(h), x(\,\cdot\,), u_*(\,\cdot\,; x(\,\cdot\,)))/\alpha(h) \to 0 \quad \text{a.e.} \quad h \to 0.$$

Let us describe an algorithm approximating the unknown control $u_*(\,\cdot\,;x(\,\cdot\,))$. Before the initial time of the process, values $h \in (0,1)$, $\varepsilon \in H_0$, $\varepsilon \ne 0$ and partition $\Delta = \Delta_h$ are fixed. After that an auxiliary system (a model) functioning synchronously with the real system is chosen. The model is described by the finite-dimensional variational inequality

$$\begin{gathered}(\ddot{w}_\varepsilon(t) + A_\varepsilon w_\varepsilon(t) - B_\varepsilon v_\varepsilon^h(t) - f_\varepsilon(t),\, \dot{w}_\varepsilon(t) - z)_\varepsilon + \varphi_\varepsilon(\dot{w}_\varepsilon(t)) - \varphi_\varepsilon(z) \le 0 \\ \text{a.e.} \quad t \in T, \quad \forall z \in V, \quad w_\varepsilon(t_0) = \xi_{10}^{h,\varepsilon}, \quad \dot{w}_\varepsilon(t_0) = \xi_0^{h,\varepsilon}.\end{gathered} \tag{2.4}$$

We assume that for $\xi_{10}^{h,\varepsilon}$, $\xi_0^{h\varepsilon}$ and by piecewise constant control $v_\varepsilon^h(t) \in P_\varepsilon$ for almost every $t \in T$, there is the single solution of the system (2.4),

$$w_\varepsilon(\,\cdot\,) = w_\varepsilon(\,\cdot\,; t_0, \xi_{10}^{h,\varepsilon}, \xi_0^{h,\varepsilon}\ v_\varepsilon^h(\,\cdot\,)) \in C(T; V_\varepsilon)$$

such that $\dot w_\varepsilon(\,\cdot\,) \in W^{1,2}(T;V_\varepsilon)$, $w_\varepsilon(\tau_{i+1}) = w_\varepsilon(\tau_{i+1};\tau_i, w_\varepsilon(\tau_i), \dot w_\varepsilon(\tau_i), v^h_\varepsilon(\,\cdot\,))$. The work of the algorithm starting at time t_0 is decomposed into $m_h - 1$ steps. At the ith step carried out during the time interval $\delta_i = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{h,i}$, we determine the control

$$v^h_\varepsilon(t) = v^h_i = \arg\min\{(B_\varepsilon v,\, s^{(i)}_\varepsilon)_\varepsilon + \alpha(h)|\,\|v\|\,|^2_\varepsilon : v \in P_\varepsilon\},\quad s^{(i)}_\varepsilon = \dot w_\varepsilon(\tau_i) - \xi^{h,\varepsilon}_i. \tag{2.5}$$

After that we transform the state $\{\dot w_\varepsilon(\tau_i),\, w_\varepsilon(\tau_i)\}$ of the model into $\{\dot w_\varepsilon(\tau_{i+1}),$ $w_\varepsilon(\tau_{i+1})\}$. The procedure stops at time θ.

Denote $u^{h,\varepsilon}(t) = q_\varepsilon v^h_\varepsilon(t)$ for almost every $t \in T$ $(\varepsilon = \varepsilon(h))$.

Theorem 1. *If $h \to 0$, then $\|u^{h,\varepsilon}(\cdot) - u_*(,\cdot; x(\cdot))\|_{L_2(T;U)} \to 0$.*

3. Auxiliary lemmas. The Theorem follows from three lemmas.

Let $\Xi(x(,\cdot), \varepsilon, h)$ be the set of all piecewise functions $\xi^{h,\varepsilon}(\,\cdot\,) : T \to V_\varepsilon$ such that (2.3) hold for $\xi^{h,\varepsilon}_i = \xi^{h,\varepsilon}(\tau_{h,i})$.

Lemma 1. *There exists a $\nu \in (0, +\infty)$ such that the bounds*

$$\int_{t_0}^t |p_\varepsilon \ddot w_\varepsilon(\tau)|^2_{v^*}\, d\tau + |\dot w_\varepsilon(t)|_\varepsilon + \|w_\varepsilon(t))\|_\varepsilon + \varphi(p_\varepsilon \dot w_\varepsilon(t)) \le \nu$$

hold uniformly with respect to all $t \in T$, $h \in (0,1)$, $\varepsilon \in H_0$, $\varepsilon \ne 0$, $\xi^{h,\varepsilon}_0$, $\xi^{h,\varepsilon}_{10} \in V_\varepsilon$, $|x_{10} - p_\varepsilon \xi^{h,\varepsilon}_{10}|_v \le h$, $|\dot x(t_0) - p_\varepsilon \xi^{h,\varepsilon}_0|_H \le h$, $w_\varepsilon(\cdot) \in W_\varepsilon = \{w_\varepsilon(\cdot; t_0, \xi^{h,\varepsilon}_{10}, \xi^{h,\varepsilon}_0, v(\cdot)) :$ $v(t) \in P_\varepsilon$ a.e. $t \in T\}$.

Proof. Let $z = 0$. Then from (2.4) we deduce

$$\frac{1}{2}\frac{d}{dt}\{|\dot w_\varepsilon(t)|^2_\varepsilon + (A_\varepsilon w_\varepsilon(t),\, \dot w_\varepsilon(t))_\varepsilon\} + \phi_\varepsilon(\ddot w_\varepsilon(t)) \le \phi(0) + (B_\varepsilon v^h_\varepsilon(t) + f_\varepsilon(t),\, \dot w_\varepsilon(t))_\varepsilon.$$

Taking into account this inequality and $\phi \ge 0$, we have

$$\begin{aligned}&|\dot w_\varepsilon(t)|^2_\varepsilon + \omega\|w_\varepsilon(t)\|^2_\varepsilon \le |\dot w_\varepsilon(t)|^2_\varepsilon + (A_\varepsilon w_\varepsilon(t), w_\varepsilon(t))_\varepsilon\\ &\le 2\int_{t_0}^t \{|Bq_\varepsilon v^h_\varepsilon(t) + f(t)|^2_H + |w_\varepsilon(\tau)|^2_\varepsilon + \varphi(0)\}d\tau + |\dot w_\varepsilon(t_0)|^2_\varepsilon + (A_\varepsilon w_\varepsilon(t_0), w_\varepsilon(t_0))_\varepsilon.\end{aligned}$$

Thus, by the Gronwall inequality and 7^0,

$$|\dot w_\varepsilon(t)|^2_\varepsilon + \omega\|w_\varepsilon(t)\|^2_\varepsilon \le k\{1 + \varphi(0) + |\xi^{h,\varepsilon}_0|^2_\varepsilon + \|\xi^{h,\varepsilon}_{10}\|^2_\varepsilon, \quad \forall t \in T. \tag{3.1}$$

Rewrite the inequality (2.4) in the form

$$\ddot w_\varepsilon(t) + A_\varepsilon w_\varepsilon(t) + \Phi_\varepsilon(\dot w_\varepsilon(t)) = B_\varepsilon v^h_\varepsilon(t) + f_\varepsilon(t), \tag{3.2}$$

where $\Phi_\varepsilon(\dot w_\varepsilon(t)) \in \partial\varphi_\varepsilon(\dot w_\varepsilon(t))$ almost every $t \in T$. From (3.2) we have, for $v_\varepsilon \in V_\varepsilon$, $\|v_\varepsilon\|_\varepsilon \le 1$,

$$(\ddot w_\varepsilon(t) + A_\varepsilon w_\varepsilon(t) + \Phi_\varepsilon(\dot w_\varepsilon(t)), v_\varepsilon)_\varepsilon = (B_\varepsilon v^h_\varepsilon(t) + f_\varepsilon(t),\, v_\varepsilon)_\varepsilon.$$

This, (2.1), 8^0, the inclusion $f(\,\cdot\,) \in L_2(T;H)$ and (3.1) imply

$$\int_{t_0}^t |p_\varepsilon \ddot w_\varepsilon(\tau)|^2_{v^*}\, d\tau \le \nu_0 < +\infty, \quad \forall t \in T. \tag{3.3}$$

The lemma follows from (3.1), (3.3).

Lemma 2. *There exists a* $\nu_1 \in (0,+\infty)$ *such that the bounds*

$$\int_{t_0}^{t} |\ddot{x}(\tau)|^2_{V^*}\, d\tau + |\dot{x}(t)|_H + |x(t)|_V + \varphi(\dot{x}(t)) \le \nu_1$$

hold, uniformly with respect to all $t \in T$ *and* $x(\cdot) \in X_T$.

The proof of this Lemma is analogous to the proof of Lemma 1.

Introduce the Lyapunov type functional

$$\begin{aligned}\lambda(t) \equiv &\lambda(t; x(\cdot), w_\varepsilon(\cdot), u_*(\cdot), v^h_\varepsilon(\cdot)) = |\dot{x}(t) - p_\varepsilon \dot{w}_\varepsilon(t)|^2_H \\ &+ \langle A p_\varepsilon(r_\varepsilon x(t) - w_\varepsilon(t)), p_\varepsilon(r_\varepsilon x(t) - w_\varepsilon(t))\rangle \\ &+ 2\alpha(h)\int_{t_0}^{t} \{|\,\|v^h_\varepsilon(\tau)\|\,|^2_\varepsilon - |\,\|s_\varepsilon u_*(\tau)\|\,|^2_\varepsilon\} d\tau,\ (u_*(\cdot) = u_*(\cdot\,; x(\cdot))).\end{aligned}$$

Lemma 3. *The bounds*

$$\lambda(t) \le |x_0 - p_\varepsilon \xi^{h,\varepsilon}_0|^2_H + \langle A p_\varepsilon(r_\varepsilon x_{10} - \xi^{h,\varepsilon}_{10}), p_\varepsilon(r_\varepsilon x_{10} - \xi^{h,\varepsilon}_{10})\rangle + \mu(t; h, \delta, \varepsilon, x(\cdot), u_*(\cdot))$$

hold, uniformly with respect to all $h \in (0,1)$, $\varepsilon \in H_0$, $\varepsilon \ne 0$, Δ_h *with diameters* $\delta = \delta(h)$, $\xi^{h,\varepsilon}(\cdot) \in \Xi(x(\cdot), \varepsilon, h)$, $\xi^{h,\varepsilon}_{10} \in V_\varepsilon$, $|x_{10} - p_\varepsilon \xi^{h,\varepsilon}_{10}|_V \le h$ *and* $t \in T$.

Proof. We rewrite the inequalities (1.1) and (2.4) in the forms

$$\ddot{x}(t) + Ax(t) + \Phi(\dot{x}(t)) = Bu_*(t) + f(t) \quad \text{a.e.} \quad t \in T,$$

and (3.2), where $t \to \Phi(\dot{x}(t)) \in L_2(T; H)$, $\Phi(\dot{x}(t)) \in \partial\phi(\dot{x}(t))$ for almost every $t \in T$. It is easy to establish the relation

$$\sum_{j=1}^{3} J^{(j)}_{t,\varepsilon} = \sum_{j=4}^{5} J^{(j)}_{t,\varepsilon} \quad \text{a.e.} \quad t \in T. \tag{3.4}$$

Here

$$\begin{aligned} J^{(1)}_{t,\varepsilon} &= (p^*_\varepsilon \ddot{x}(t) - \ddot{w}_\varepsilon(t),\ r_\varepsilon \dot{x}(t) - \dot{w}_\varepsilon(t))_\varepsilon, \\ J^{(2)}_{t,\varepsilon} &= (p^*_\varepsilon \Phi(\dot{x}(t)) - \Phi_\varepsilon(\dot{w}_\varepsilon(t)),\ r_\varepsilon \dot{x}(t) - \dot{w}_\varepsilon(t))_\varepsilon, \\ J^{(3)}_{t,\varepsilon} &= (p^*_\varepsilon Ax(t) - A_\varepsilon w_\varepsilon(t),\ r_\varepsilon \dot{x}(t) - \dot{w}_\varepsilon(t))_\varepsilon, \\ J^{(4)}_{t,\varepsilon} &= (p^*_\varepsilon Bu_*(t) - B_\varepsilon v^h_\varepsilon(t),\ r_\varepsilon \dot{x}(t) - \dot{w}_\varepsilon(t))_\varepsilon, \\ J^{(5)}_{t,\varepsilon} &= (p^*_\varepsilon f(t) - f_\varepsilon(t),\ r_\varepsilon \dot{x}(t) - \dot{w}_\varepsilon(t))_\varepsilon. \end{aligned}$$

Note that the following relations are true:

$$J^{(1)}_{t,\varepsilon} \ge (\ddot{x}(t) - p_\varepsilon \ddot{w}_\varepsilon(t), \dot{x}(t) - p_\varepsilon \dot{w}_\varepsilon(t))_H - |\ddot{x}(t) - p_\varepsilon \ddot{w}_\varepsilon(t)|_{V^*} |(I - p_\varepsilon r_\varepsilon)\dot{x}(t)|_V, \tag{3.5}$$

$$\begin{aligned} J^{(3)}_{t,\varepsilon} &= \langle A p_\varepsilon(r_\varepsilon x(t) - w_\varepsilon(t)), p_\varepsilon(r_\varepsilon \dot{x}(t) - \dot{w}_\varepsilon(t))\rangle + I_{t,\varepsilon}, \\ I_{t,\varepsilon} &= \langle A(I - p_\varepsilon r_\varepsilon)x(t),\ p_\varepsilon(r_\varepsilon \dot{x}(t) - \dot{w}_\varepsilon(t))\rangle. \end{aligned} \tag{3.6}$$

From the definition of the function $f_\varepsilon(\,\cdot\,)$ it follows that

$$J^{(5)}_{t,\varepsilon} = 0, \quad \forall t \in T. \tag{3.7}$$

Let us observe that for $t \in \delta_i$,

$$\begin{aligned} &|p_\varepsilon(r_\varepsilon \dot{x}(t) - \dot{w}_\varepsilon(t))|_{V^*} = |\dot{x}(t) - (I - p_\varepsilon r_\varepsilon)\dot{x}(t) - p_\varepsilon \dot{w}_\varepsilon(t))|_{V^*} \\ &\le |(I - p_\varepsilon r_\varepsilon)\dot{x}(t)|_{V^*} + |\dot{x}(\tau_i) - p_\varepsilon \dot{w}_\varepsilon(\tau_i)|_{V^*} + \int_{\tau_i}^t \{|\ddot{x}(\tau)|_{V^*} + |p_\varepsilon \ddot{w}_\varepsilon(\tau)|_{V^*}\}\, d\tau. \end{aligned} \tag{3.8}$$

Using (2.2), (3.8), 5^0, 7^0 and Lemmas 1, 2 we conclude, for $t \in \delta_i$,

$$\begin{aligned} &J^{(4)}_{t,\varepsilon} = (B(u_*(t) - q_\varepsilon v^h_\varepsilon(t)), p_\varepsilon(r_\varepsilon \dot{x}(t) - \dot{w}_\varepsilon(t)))_H \\ &\le (B_\varepsilon(s_\varepsilon u_*(t) - v^h_\varepsilon(t)), \dot{x}(\tau_i) - p_\varepsilon \dot{w}_\varepsilon(\tau_i))_\varepsilon + |B|_{L(U;H)}|(I - q_\varepsilon s_\varepsilon)u_*(t)|_U \\ &|\dot{x}(\tau_i) - p_\varepsilon \dot{w}_\varepsilon(\tau_i)|_H + |B|_{L(U;V)}\{|u_*(t)|_U + c_2\}|(I - p_\varepsilon r_\varepsilon)\dot{x}(t)|^*_V \\ &+ \int_{\tau_i}^t \{|\ddot{x}(\tau)|_{V^*} + |p_\varepsilon \ddot{w}_\varepsilon(\tau)|_{V^*}\}\, d\tau \le -(B_\varepsilon(s_\varepsilon u_*(t) - v^h_\varepsilon(t)), s^{(i)}_\varepsilon)_\varepsilon \\ &+ k_1|(I - q_\varepsilon s_\varepsilon)u_*(t)|_U + k_2(\delta^{1/2} + h) + k_3|(I - p_\varepsilon r_\varepsilon)\dot{x}(t)|_{V^*}. \end{aligned} \tag{3.9}$$

Also, due to the equality $\varphi_\varepsilon(\dot{w}_\varepsilon(t)) = \varphi(p_\varepsilon \dot{w}_\varepsilon(t))$, the inclusion $p_\varepsilon \in L(V_\varepsilon; V)$ and condition 8^0 we have

$$\begin{aligned} &J^{(2)}_{t,\varepsilon} = (p^*_\varepsilon \Phi(\dot{x}(t)) - p^*_\varepsilon F(p_\varepsilon \dot{w}_\varepsilon(t)), r_\varepsilon \dot{x}(t) - \dot{w}_\varepsilon(t))_\varepsilon \\ &\ge (\Phi(\dot{x}(t)) - F(p_\varepsilon \dot{w}_\varepsilon(t)), \dot{x}(t) - p_\varepsilon \dot{w}_\varepsilon(t))_H - |\Phi(\dot{x}(t)) - F(p_\varepsilon \dot{w}_\varepsilon(t))|_{V^*} \\ &|(I - p_\varepsilon r_\varepsilon)\dot{x}(t)|_V \ge -|\Phi(\dot{x}(t)) - F(p_\varepsilon \dot{w}_\varepsilon(t))|_{V^*}|(I - p_\varepsilon r_\varepsilon)\dot{x}(t)|_V, \end{aligned} \tag{3.10}$$

where $F(p_\varepsilon \dot{w}_\varepsilon(t)) \in \partial\varphi(p_\varepsilon \dot{w}_\varepsilon(t))$ almost every $t \in T$, $t \to F(p_\varepsilon \dot{w}_\varepsilon(t) \in L_\infty(T; H)$. Taking into account (3.4)–(3.7), (3.9), (3.10) one can get the estimate

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\{|\dot{x}(t) - p_\varepsilon \dot{w}_\varepsilon(t)|^2_H + \langle Ap_\varepsilon(r_\varepsilon x(t) - w_\varepsilon(t)), p_\varepsilon(r_\varepsilon x(t) - w_\varepsilon(t))\rangle \\ &\le I_{t,\varepsilon} + F_{x,w_\varepsilon}(t)|(I - p_\varepsilon r_\varepsilon)\dot{x}(t)|_V - (B_\varepsilon(s_\varepsilon u_*(t) - v^h_\varepsilon(t)), s^{(i)}_\varepsilon)_\varepsilon \\ &+ k_1|(I - q_\varepsilon s_\varepsilon)u_*(t)|_U + k_2(\delta^{1/2} + h) + k_3|(I - p_\varepsilon r_\varepsilon)\dot{x}(t)|_{V^*}. \end{aligned} \tag{3.11}$$

Here

$$F_{x,w_\varepsilon}(t) = |\Phi(\dot{x}(t)) - F(p_\varepsilon \dot{w}_\varepsilon(t))|_{V^*} + |\ddot{x}(t) - p_\varepsilon \ddot{w}_\varepsilon(t)|_{V^*},$$

$$\sup\{\int_{t_0}^\theta F^2_{x,w_\varepsilon}(t)\, dt : x(\cdot) \in X_T, w_\varepsilon(\cdot) \in W_\varepsilon, \varepsilon \ne 0\} < +\infty. \tag{3.12}$$

It is also evident that

$$\begin{aligned} &I_{t,\varepsilon} = \langle A(I - p_\varepsilon r_\varepsilon)x_{10}, p_\varepsilon(r_\varepsilon x_{10} - w_\varepsilon(t_0))\rangle - \langle A(I - p_\varepsilon r_\varepsilon)x(t), \\ &p_\varepsilon(r_\varepsilon x(t) - w_\varepsilon(t))\rangle + \int_{t_0}^t \langle A(I - p_\varepsilon r_\varepsilon)\dot{x}(\tau), p_\varepsilon(r_\varepsilon x(\tau) - w_\varepsilon(\tau))\rangle\, d\tau. \end{aligned}$$

Consequently,

$$\int_{t_0}^{t} I_{\tau,\varepsilon} d\tau \le k_* \nu(t; \varepsilon, x(\cdot)), \quad \forall t \in T. \tag{3.13}$$

After some manipulations we obtain from (2.5), (3.11)–(3.13) and Lemma 1.2 that

$$\lambda(t) \le \lambda(t_0) + \mu(t; h, \delta, \varepsilon, x(\cdot), u_*(\cdot)).$$

The Lemma is proved.

Lemma 4. *Let* $u^{h,\varepsilon(h)}(\cdot) \to u_0(\cdot)$ *weakly in* $L_2(T;U)$ *as* $h \to 0$. *Then* $u_0(\cdot) \in U_*(x(\cdot))$.

Proof. It is sufficient to show that

$$x_*(\cdot) = x(\cdot). \tag{3.14}$$

Here $x_*(\cdot) = x(\cdot\,; t_0, x_0, x_{10}, u_0(\cdot))$, $x(\cdot) = x(\cdot\,; t_0, x_0, x_{10}, u_*(\cdot))$. Following the proof of (3.11) we come to the estimate (for almost every $t \in T$)

$$\begin{aligned} &d/dt\{|\dot{x}_*(t) - p_\varepsilon \dot{w}_\varepsilon(t)|_H^2 + \langle A p_\varepsilon (r_\varepsilon x_*(t) - w_\varepsilon(t)), p_\varepsilon(r_\varepsilon x_*(t) - w_\varepsilon(t))\rangle\} \\ &\le \mu_1(t; \varepsilon, \delta, h, x_*(\cdot)) + 2(B(q_\varepsilon s_\varepsilon u_0(t) - q_\varepsilon v_\varepsilon^h(t)), p_\varepsilon(r_\varepsilon \dot{x}_*(t) - \dot{w}_\varepsilon(t)))_H, \end{aligned}$$

where

$$\int_{t_0}^{\theta} \mu_1(t; \varepsilon, \delta, h, x_*(\cdot))\, dt \to 0 \quad \text{as } h, \varepsilon, \delta \to 0. \tag{3.15}$$

Further, we have

$$\begin{aligned} &(B(q_\varepsilon s_\varepsilon u_0(t) - q_\varepsilon v_\varepsilon^h(t)), p_\varepsilon(r_\varepsilon \dot{x}_*(t) - \dot{w}_\varepsilon(t)))_H \le k_1|(I - p_\varepsilon r_\varepsilon)\dot{x}_*(t)|_{V^*} \\ &+ (B q_\varepsilon s_\varepsilon u_0(t) - B u^{h,\varepsilon}(t)), \dot{x}_*(t) - p_\varepsilon \dot{w}_\varepsilon(t)))_H \le k_1|(I - p_\varepsilon r_\varepsilon)\dot{x}_*(t)|_{V^*} \\ &+ k_2|(I - q_\varepsilon s_\varepsilon)u_0(t)|_U + (B u_0(t) - B u^{h,\varepsilon}(t)), \dot{x}_*(t) - p_\varepsilon \dot{w}_\varepsilon(t)))_H \\ &\le k_1|(I - p_\varepsilon r_\varepsilon)\dot{x}_*(t)|_{V^*} + k_2|(I - q_\varepsilon s_\varepsilon)u_0(t)|_U + k_3|\dot{x}(t) - p_\varepsilon \dot{w}_\varepsilon(t)|_{V^*} \\ &+ (B u_0(t) - B u^{h,\varepsilon}(t)), \dot{x}_*(t) - \dot{x}(t))_H. \end{aligned} \tag{3.16}$$

Taking into account (3.15), (3.16), Lemma 3 and the weak convergence of $u^{h,\varepsilon(h)}(\cdot)$ to $u_0(\cdot)$, we come to the equality (3.14). This completes the proof of Lemma 4.

4. Proof of Theorem. The proof if similar to those of the corresponding assertions from the papers [13, 18, 21]. We give it to complete the narration. Assume that the Theorem is not true. Then there exists a subsequence of the sequence $\{u_j(\cdot)\}$, $u_j(\cdot) = u^{h_j,\varepsilon(h_j)}(\cdot)$ (denote it for simplicity by the same symbol $\{u_j(\cdot)\}$), such that

$$h_j \to 0, \quad u_j(\cdot) \to u_0(\cdot) \text{ weakly in } L_2(T;U), \quad u_0(\cdot) \ne u_*(\cdot\,; x(\cdot)). \tag{4.1}$$

Note that Lemma 4 implies the inclusion $u_0(\cdot) \in U_*(x(\cdot))$. Also, due to the property of the weak limit,

$$\varliminf_{j\to\infty} \|u_j(\cdot)\|_{L_2(T;U)} \ge \|u_0(\cdot)\|_{L_2(T;U)} \ge \|u_*(\cdot\,, x(\cdot))\|_{L_2(T;U)}. \tag{4.2}$$

Thanks to Lemma 3 and 4^0,

$$\overline{\lim_{j\to\infty}} \|u_j(\cdot)\|_{L_2(T;U)} \leq \|u_*(\cdot\,, x(\cdot))\|_{L_2(T;U)}. \tag{4.3}$$

Now the Theorem follows from (4.1)–(4.3).

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