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EXISTENCE OF MONOTONE SOLUTIONS TO SOME SINGULAR BOUNDARY AND INITIAL VALUE PROBLEMS

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(Submitted by: Jean Mawhin)

Abstract. We establish conditions sufficient to guarantee existence of nondecreasing solutions on $[0,1]$ of the differential equation $y'' + f(t,y,y') = 0$ subject to the boundary conditions $y(0) = 0$, $y(1) = a > 0$ or the initial conditions $y(0) = 0$, $y'(0) = a > 0$. Here f is a nonnegative function which may be singular as $y \downarrow 0$.

Introduction. We consider the existence of nondecreasing solutions of the problem

$$y''(t) + f(t, y(t), y'(t)) = 0, \qquad y(0) = 0, \quad y(1) = a > 0; \tag{1}$$

here f is allowed, but not required, to be singular at $y = 0$ and $f \geq 0$ is continuous for $y > 0$. The necessity of some further restriction on f or a is clearly seen by considering the elementary problem $y'' + k^2 y = 0$, $y(0) = 0$, $y(1) = a$, which has a monotone solution precisely when $k \in [0, \pi/2]$ (our Theorem 1 below with $g(u) \equiv a$ yields the 10% stronger condition $k \in [0, \sqrt{2}]$).

Two-point boundary value problems for differential equations of the form

$$\big(p(x)u'(x)\big)' + f(x, u(x), p(x)u'(x)) = 0,$$

where $p(x) > 0$ for $x > 0$ and $1/p \in L_1(0,1)$, can be put in the form (1) by setting $t = \int_0^x 1/p(s)\, ds$. Thus certain problems singular at $t = 0$ as well as at $y = 0$ can be included under the present formulation.

In view of the fact that, under our hypotheses, a solution of (1) turns out to have a continuous derivative at $t = 0$, we investigate also the initial value problem for singular equations of the form (1).

Existence of positive solutions to singular two-point boundary value problems similar to (1) has been established by many authors [1–8, 11–21]. In many of these papers a singular problem like (1) is replaced with the family of nonsingular problems $y_n'' + f(t, y_n, y_n') = 0$, $y_n(0) = \frac{1}{n}$, $y_n(1) = a$ $(n = 1, 2, \dots)$. Existence of solutions for these problems is established along with suitable bounds on the family $\{y_n\}$. Then the Ascoli–Arzela theorem is invoked to show that a suitable subsequence $\{y_{n_k}\}$ converges to a solution of (1). Here we contrive to apply the topological transversality theorem [9–10] directly to the singular problem (1). We use two somewhat different

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approaches. The first, suggested by J. Mawhin, turns around showing that all the fixed points of a certain integral map are in fact monotone. The second approach, used in the final two theorems, depends on deriving a priori bounds on monotone solutions and so selecting the range and domain spaces that the Niemytzki map $(Fu)(t) = f(t, u(t), u'(t))$ is continuous and completely continuous even though the function $f(t, u(t), u'(t))$ is discontinuous.

The topological transversality theorem or, equivalently, Schauder theory, seems especially well adapted to establishing existence of solutions with some additional property, such as global existence, positivity, or, as here, monotonicity.

The Dirichlet problem. By a solution of the differential equation

$$y''(t) + f(t, y(t), y'(t)) = 0 \tag{2}$$

we shall mean a function y continuous on $[0,1]$ and continuously differentiable on $(0,1)$ and such that the differential equation is satisfied on $(0,1)$. Our interest lies in the existence of *monotone* solutions to the possibly singular problem composed of (2) and the boundary conditions

$$y(0) = 0, \quad y(1) = a > 0. \tag{3}$$

The first theorem deals with the case of f independent of y'.

Theorem 1. *Let f be continuous on $[0,1] \times (0,a]$; assume there exists a continuous nonincreasing function g on $(0,a]$ with $g \in L_1(0,a)$ such that*

$$0 \le f(t,y) \le g(y) \quad on \quad [0,1] \times (0,a]. \tag{4}$$

Suppose further that $a^3 \ge \int_0^a sg(s)\,ds$. Then a monotone solution of (2)–(3) *exists; furthermore, this solution is continuously differentiable on* $[0,1]$.

Proof. We consider the one-parameter family of problems

$$y''(t) + \lambda f(t, y(t)) = 0, \qquad y(0) = 0, \quad y(1) = a, \tag*{$(5)_\lambda$}$$

where $\lambda \in [0,1]$. Existence of a solution to $(5)_\lambda$ is clearly equivalent to existence of a fixed point in $\mathrm{C}[0,1]$ of the integral operator

$$T(y,\lambda)(t) \equiv at + \lambda \int_0^t s(1-t) f(s, y(s))\,ds + \lambda \int_t^1 t(1-s) f(s,y(s))\,ds. \tag*{$(6)_\lambda$}$$

Let $\big(\mathrm{C}_0[0,1], |\cdot|_0\big)$ be the Banach space of continuous functions on $[0,1]$ that vanish at 0, with the norm $|y|_0 = \max_{x\in[0,1]} |y(x)|$. Let K be the subset of $\mathrm{C}_0[0,1]$ consisting of those functions y that are concave downward on $[0,1]$ and satisfy the boundary condition $y(1) = a$; K is convex. Any y in K satisfies $y(t) \ge at$ for $0 \le t \le 1$. From this and (4) we have that

$$0 \le \int_0^1 f(t, y(t))\,dt \le \int_0^1 g(y(t))\,dt \le \int_0^1 g(at)\,dt = \frac{1}{a}\int_0^a g(s)\,ds < \infty,$$

so the operator T is defined on $K \times [0,1]$. Since also $T(y,\lambda)(0) = 0$, $T(y,\lambda)(1) = a$, and $T(y,\lambda) \in \mathrm{C}^2(0,1)$ with $T(y,\lambda)''(t) = -\lambda f(t, y(t)) \le 0$, we see that $T(\cdot,\lambda)$ maps K into K.

To see that $T : K \times [0,1] \to K$ is continuous, let $(y_i, \lambda_i) \in K \times [0,1]$ for $i = 1,2$. Then

$$\begin{aligned}
&|T(y_1,\lambda_1)(t) - T(y_2,\lambda_2)(t)| \le |\lambda_1 - \lambda_2| \int_0^t s(1-t) f(s, y_1(s))\,ds \\
&+\lambda_2 \int_0^t s(1-t)|f(s,y_1(s)) - f(s,y_2(s))|ds + |\lambda_1 - \lambda_2| \int_t^1 t(1-s) f(s,y_1(s))ds \\
&+ \lambda_2 \int_t^1 t(1-s)|f(s,y_1(s)) - f(s,y_2(s))|\,ds \\
&\le |\lambda_1 - \lambda_2| \int_0^1 g(as)\,ds + \int_0^1 |f(s,y_1(s)) - f(s,y_2(s))|\,ds;
\end{aligned}$$

the final expression is seen to tend to zero as $(y_2,\lambda_2) \to (y_1,\lambda_1)$ if we use the Lebesgue dominated convergence theorem on the last integral. Thus T is continuous. Let S be a subset of K satisfying $|z|_0 \le c$ for some constant c and let $y_\lambda = T(z,\lambda)$; then

$$|y_\lambda(t)| \le a + 2\int_0^1 f(s,z(s))\,ds \le a + 2\int_0^1 g(as)\,ds < \infty,$$

so $\{T(z,\lambda) : z \in S, \lambda \in [0,1]\}$ is uniformly bounded. From

$$\begin{aligned}
&\big|y_\lambda(t) - y_\lambda(v)\big| \\
&\le a|t-v| + \lambda \left| \int_v^t s(1-t) f(s,z(s))\,ds \right| + \lambda |t-v| \int_0^v s f(s,z(s))\,ds \\
&\quad + \left| \int_t^v t(1-s) f(s,z(s))\,ds \right| + \lambda|t-v| \int_v^1 (1-s) f(s,z(s))\,ds \\
&\le a|t-v| + 2\int_v^t g(as)\,ds + |t-v| \int_0^1 g(as)\,ds
\end{aligned}$$

and the integrability of g we get that $\{T(z,\lambda) : z \in S, \lambda \in [0,1]\}$ is also equicontinuous. The Ascoli-Arzela theorem gives us that T is completely continuous.

If y in K is a fixed point of $T(\cdot,\lambda)$ for some $\lambda \in [0,1]$, then from $(6)_\lambda$

$$\begin{aligned}
y'(t) &= a - \lambda \int_0^t s f(s,y(s))\,ds + \lambda \int_t^1 (1-s) f(s,y(s))\,ds \\
&\ge a - \lambda \int_0^t s f(s,y(s))\,ds \ge a - \lambda \int_0^t s g(y(s))\,ds \\
&\ge a - \int_0^1 s g(as)\,ds = \frac{1}{a^2}\left(a^3 - \int_0^a s g(s)\,ds\right) \ge 0.
\end{aligned}$$

Thus y is nondecreasing over $[0,1]$, which immediately implies that

$$0 \le y(t) \le a \tag{7}$$

for all $t \in [0,1]$. If we set

$$U = \{y \in K : |y|_0 < a + \epsilon\}$$

for an arbitrary $\epsilon > 0$, we see that T is a compact homotopy of $\overline{U}$ into K; moreover, no fixed point of T lies on the boundary of U in view of (7). Since $T(y,0)(t) = at$ is a constant map to the interior of U, it is essential [9, 10]. It follows from the topological transversality theorem [9, 10] that $T(y,1)$ is also essential and therefore has a fixed point in $\overline{U}$; this fixed point is the required monotone solution of (2)–(3).

Example. Using $g(u) = u^{-1/2}$, the problem $y'' + (y+t)^{-1/2} = 0$, $y(0) = 0$, $y(1) = a$ is seen to have a monotone solution if $a \ge (2/3)^{2/3}$.

The preceding analysis requires modification when dependence of f on y' is allowed, as the proof of the following theorem shows.

Theorem 2. *Let f be continuous on $[0,1] \times (0,a] \times (-\infty,\infty)$. Assume there exists a continuous nonincreasing function g on $(0,a]$ with $g \in L_1(0,a)$ and a positive constant A such that*

$$0 \le f(t,y,z) \le (1 + Az^2)g(y)$$

on $[0,1] \times (0,a] \times (-\infty,\infty)$. Suppose also that

$$\tfrac{\sqrt{A}}{a} \int_0^a g(s)\,ds + \arctan(a\sqrt{A}) < \frac{\pi}{2}$$

and that

$$a^3 > (1 + AM_1^2) \int_0^a sg(s)\,ds,$$

where

$$M_1 \equiv \tfrac{1}{\sqrt{A}} \tan\Big(\arctan(a\sqrt{A}) + \tfrac{\sqrt{A}}{a} \int_0^a g(s)\,ds\Big).$$

Then a monotone solution of (2)–(3) exists.

Proof. We consider the family of problems

$$y''(t) + \lambda f(t, y(t), y'(t)) = 0, \qquad y(0) = 0, \qquad y(1) = a \tag{8$_\lambda$}$$

indexed by $\lambda \in [0,1]$ and the corresponding integral operator T defined by

$$T(y,\lambda)(t) \equiv at + \lambda \int_0^t s(1-t) f(s, y(s), y'(s))ds + \lambda \int_t^1 t(1-s) f(s, y(s), y'(s))ds. \tag{9$_\lambda$}$$

Clearly, y is a fixed point of $(9)_\lambda$ if and only if it is a solution of $(8)_\lambda$. Define $|\cdot|_1$ by

$$|y|_1 = \max\bigl(\max_{t\in[0,1]} |y(t)|, \max_{t\in[0,1]} |y'(t)|\bigr)$$

on $C_0^1[0,1] = \{y \in C^1[0,1] : y(0) = 0\}$; $\bigl(C_0^1[0,1], |\cdot|_1\bigr)$ is a Banach space. Let now

$$K = \{y \in \mathrm{C}_0^1[0,1] : y \text{ is concave down and } y(1) = a\};$$

K is convex and $y \in K$ implies that $at \le y(t)$. From the calculation

$$\begin{aligned} 0 &\le \int_0^1 f(s, y(s), y'(s))\, ds \le \int_0^1 \bigl(1 + Ay'(s)^2\bigr) g(y(s))\, ds \\ &\le \Bigl(1 + A \max_{s\in[0,1]} (y'(s))^2\Bigr) \int_0^1 g(as)\, ds < \infty \end{aligned}$$

for $y \in K$, we conclude that $T(\cdot, \lambda)$ is defined on $\mathrm{C}_0^1[0,1]$. Moreover, from

$$T(y,\lambda)'(t) = a - \lambda \int_0^t s f(s, y(s), y'(s))\, ds + \lambda \int_t^1 (1-s) f(s, y(s), y'(s))\, ds \qquad (10)_\lambda$$

and

$$T(y,\lambda)''(t) = -\lambda f(s, y(s), y'(s)) \le 0$$

we see that $T : K \times [0,1] \to K$. Continuity of T follows from routine arguments much as before. Let S be a subset of K satisfying $|z|_1 \le c$, so that $|z| \le c$, $|z'| \le c$. For $y_\lambda = T(z, \lambda)$ we have

$$|y_\lambda(t)| \le a + \int_0^1 f(s, z(s), z'(s))\, ds \le a + (1 + Ac^2) \int_0^1 g(as)\, ds < \infty;$$

$|y'_\lambda(t)|$ satisfies the same bound. Thus $\{T(z,\lambda) : z \in S, \lambda \in [0,1]\}$ is uniformly bounded. A similar adaptation of the argument of Theorem 1 shows that $\{T(z,\lambda) : z \in S, \lambda \in [0,1]\}$ is also equicontinuous. Complete continuity of $T : S \times [0,1] \to K$ follows from the Ascoli-Arzela theorem.

We claim that any fixed point $y \in K$ of $T(\cdot,\lambda)$ is monotone. Suppose to the contrary that y is a fixed point that is not monotone; then there exists $x \in (0,1)$ such that $y'(x) = 0$ and $y' \le 0$ on $(x, 1]$. Since $y(t) \ge at$, we have

$$y''(t) \ge -\lambda\bigl[1 + Ay'(t)^2\bigr] g(y(t)) \ge -\bigl[1 + Ay'(t)^2\bigr] g(at);$$

dividing by $1 + Ay'(t)^2$ and integrating over $[x, 1]$ yields

$$\begin{aligned} \sqrt{A} \int_x^1 \frac{y''(t)\, dt}{1 + Ay'(t)^2} &= \arctan[\sqrt{A} y'(1)] \ge -\sqrt{A} \int_x^1 g(at) dt \\ &\ge -\frac{\sqrt{A}}{a} \int_0^a g(s) ds > -\frac{\pi}{2} \end{aligned}$$

by hypothesis. We thus have $-M_1 \le y'(1) \le y'(t)$ for any $t \in [0,1]$. A similar argument shows that $y'(t) \le M_1$ also holds. We then have

$$y'(t) \ge a - \int_0^t s f(s, y(s), y'(s))\, ds \ge a - \int_0^1 s\big(1 + Ay'(s)^2\big) g(y(s))\, ds$$
$$\ge a - \big(1 + AM_1^2\big) \int_0^1 s g(as)\, ds = \frac{1}{a^2}\Big(a^3 - \big(1 + AM_1^2\big)\int_0^a s g(s)\, ds\Big) > 0,$$

a contradiction. We conclude that any fixed point in K for any $\lambda \in [0,1]$ is monotone.

Let y_λ be a (monotone) fixed point of $T(\cdot, \lambda)$ for some $\lambda \in [0,1]$. There exists a value $\hat{t} \in (0,1)$ such that $y'(\hat{t}) = a$, by the mean value theorem. Using a previous argument on the interval $(0, \hat{t})$ we conclude that

$$\arctan[\sqrt{A} y'(0)] \le \arctan[a\sqrt{A}] + \frac{\sqrt{A}}{a}\int_0^a g(s)\, ds;$$

hence

$$0 \le y'(t) \le y'(0) \le M_1.$$

We also have $0 \le y(t) \le a$; thus any fixed point $y_\lambda \in K$ of $T(\cdot, \lambda)$ must satisfy $|y_\lambda|_1 \le \max(a, M_1)$. Let

$$U = \{y \in K : |y|_1 < \max(a, M_1) + \epsilon\}$$

for any fixed $\epsilon > 0$; we have shown above that $T(\cdot, \lambda)$ has no fixed points on ∂U for $\lambda \in [0,1]$. The remainder of the argument is identical with that of Theorem 1. $\square$

The result just obtained can be improved if the function g lies in some L_p space for $p > 1$. For the remainder of this paper we understand by a solution of the differential equation $y''(t) + f(t, y(t), y'(t)) = 0$ a function y continuous on $[0,1]$ with y and y' absolutely continuous on $(0,1)$ and such that the differential equation is satisfied almost everywhere on $(0,1)$. In fact, solutions in this weaker sense will also turn out to be classical $C^2(0,1) \cap C^1[0,1]$ solutions.

Theorem 3. *Let f be continuous on $[0,1] \times (0,a] \times (-\infty, \infty)$; assume there exists a continuous nonincreasing function g on $(0,a|$ with $g \in L_p(0,a)$ for some $p \in (1,\infty)$ and a positive constant A such that*

$$0 \le f(t,y,z) \le (1 + Az^2) g(y) \quad \text{on} \quad [0,1] \times (0,a] \times (-\infty, \infty).$$

Suppose that

$$a^3 > (1 + Aa^2) e^{2A\int_0^a g(u)\, du} \int_0^a s g(s)\, ds.$$

Then a monotone solution of (2)–(3) exists.

Proof. We consider from the start monotone solutions of the family

$$y''(t) + \lambda f(t, y(t), y'(t)) = 0, \qquad y(0) = 0, \quad y(1) = a \qquad (11)_\lambda$$

for $\lambda \in [0,1]$; we have that $0 \le y(t) \le a$, $y'(t) \ge 0$, and $y''(t) \le 0$. Since $A \ne 0$, integration of

$$\frac{y'(t)y''(t)}{1+Ay'(t)^2} \ge -g(y(t))y'(t)$$

over $(\alpha, x) \subset (0,1)$ yields

$$\ln(1+Ay'(\alpha)^2) \le \ln(1+Ay'(x)^2) + 2A\int_0^{y(x)} g(u)\,du.$$

Since $y'(\alpha)$ cannot decrease as $\alpha \downarrow 0$, $\lim_{\alpha\downarrow 0} y'(\alpha) \equiv y'(0)$ exists. Moreover, we have the bound

$$\ln(1+Ay'(t)^2) \le \ln(1+Ay'(0)^2) \le \ln(1+Aa^2) + 2A\int_0^a g(u)\,du$$

since for some x we must have $y'(x) = a$ by the mean value theorem. It follows that $0 \le y'(x) \le M_1$, where M_1 is defined by

$$M_1 = \Big(\exp\Big\{\ln(1+Aa^2) + 2A\int_0^a g(u)\,du\Big\} - 1\Big)^{1/2} A^{-1/2}.$$

It is obvious that $y'(0) \ge a$ and that $y(t) \ge at$. It also follows that

$$\begin{aligned}\|y''\|_{L_p}^p &\le \int_0^1 f(t,y(t),y'(t))^p\,dt \le (1+AM_1^2)^p \int_0^1 g(y(t))^p\,dt \\ &\le (1+AM_1^2)^p \int_0^1 g(at)^p\,dt \le \frac{1}{a}(1+AM_1^2)^p \int_0^a g(s)^p\,ds \equiv M_2^p < \infty.\end{aligned}$$

Let B be the set of functions $\big\{z(t) \in \mathrm{C}[0,1] : z, z'$ are absolutely continuous on $(0,1)$, $z(0)=0$, and $\|z\| < \infty\big\}$, where we define $|z|_0 = \sup_{0<t\le 1}|z(t)|$ and

$$\|z\| = \max\big\{|z|_0/a, |z'|_0/M_1, \|z''\|_{L_p}/M_2\big\};$$

then $(B, \|\cdot\|)$ is a Banach space. Any solution y of $(11)_\lambda$ satisfies $\|y\| \le 1$, by our estimates. Let

$$K = \big\{z \in B : z(t) \ge at, z'(t) \ge 0, z''(t) \le 0 \text{ a.e.}, z(1) = a\big\};$$

K is convex. Let, for some fixed $\delta > 0$,

$$U = \big\{z \in K : \|z\| < 1+\delta\big\}.$$

Then U is an open subset of K. By our a priori estimates we have that any solution y of $(11)_\lambda$ satisfies $\|y\| \le 1$ and so does not lie on ∂U.

Consider the Niemytski map F defined on $\overline{U}$ by $(Fz)(t) = f(t, z(t), z'(t))$. Since $z \in \overline{U}$ implies that $at \le z(t) \le a$,

$$|f(t, z(t), z'(t))| \le (1 + Az'(t)^2)g(z(t)) \le (1 + AM_1^2(1+\delta)^2)g(at) \in L_p. \tag{12}$$

If $\|z_n - z\| \to 0$ in $\overline{U}$, then $z_n \to z$, $z_n' \to z'$ on $(0,1]$, and the dominated convergence theorem with the estimate (12) shows that $Fz_n \to Fz$ in $L_p(0,1)$. Thus $F : \overline{U} \to L_p$ is continuous.

To see that F is compact, let $\overline{U}' = \{z' : z \in \overline{U}\}$; we have at once that $\overline{U}$ and $\overline{U}'$ are uniformly bounded and $\overline{U}$ is equicontinuous. For $z \in \overline{U}$ and $t, s \in [0,1]$ we have from Hölder's inequality that

$$|z'(t) - z'(s)| \le \left| \int_s^t |z''(u)|\, du \right| \le |t-s|^{1/p'} \|z''\|_{L_p} \le M_2(1+\delta)|t-s|^{1/p'}$$

(where $1/p + 1/p' = 1$), so $\overline{U}'$ is also equicontinuous. From the Ascoli–Arzela theorem there is a sequence $\{z_n\} \subset \overline{U}$ and a function $z \in \mathrm{C}^1[0,1]$ such that $z_n \to z$ and $z_n' \to z'$ uniformly on $[0,1]$. From (12) we have

$$\|Fz_n\|_{L_p}^p \le (1 + AM_1^2(1+\delta)^2)^p \int_0^1 g(at)^p\, dt,$$

so from the dominated convergence theorem it follows that $Fz \in L_p$ and that $Fz_n \to Fz$ in L_p. Thus F is a compact map of $\overline{U}$ into the set

$$Q = \{q \in L_p : 0 \le q(t) \le (1 + AM_1^2(1+\delta)^2)g(at) \text{ a.e. } (0,1)\}$$

equipped with the L_p-norm; it is easy to see that $\overline{F\overline{U}} \subset Q$.

Let N be defined on Q by

$$(Nq)(t) = \Big[a + \int_0^1 (1-s)q(s)\, ds\Big]t - \int_0^t (t-s)q(s)\, ds$$

for $q \in Q$. Then Nq is differentiable and

$$\begin{aligned}(Nq)'(t) = a + \int_0^1 (1-s)q(s)\, ds - \int_0^t q(s)\, ds \ge (Nq)'(1) = a - \int_0^1 sq(s)\, ds \\ \ge \frac{1}{a^2}\Big[a^3 - (1 + AM_1^2(1+\delta)^2)\int_0^a sg(s)\, ds\Big] > 0\end{aligned}$$

provided $\delta > 0$ is chosen small enough. $(Nq)'$ is also differentiable and $(Nq)''(t) = -q(t) \le 0$ almost everywhere. Set

$$\zeta(t) = t\int_0^1 (1-s)q(s)\, ds - \int_0^t (t-s)q(s)\, ds.$$

Then $\zeta(0) = \zeta(1) = 0$; but $\zeta''(t) \leq 0$ almost everywhere implies that the minimum of ζ must occur at an endpoint of $[0,1]$. Thus $(Nq)(t) \geq at$. It follows that N maps Q into K.

Continuity of N follows from

$$|Nq - N\overline{q}|_0 \leq \sup_{0<t\leq 1} \Big\{ t \int_0^1 (1-s)|q(s) - \overline{q}(s)|\, ds + \int_0^t (t-s)|q(s) - \overline{q}(s)|\, ds \Big\}$$
$$\leq 2\|q - \overline{q}\|_{L_p},$$

$$|(Nq)' - (N\overline{q})'|_0 \leq 2\|q - \overline{q}\|_{L_p}, \quad \|(Nq)'' - (N\overline{q})''\|_{L_p} = \|q - \overline{q}\|_{L_p}.$$

It follows that the map $N \circ F : K \to K$ is completely continuous and that $H_\lambda : \overline{U} \to K$ defined by $(H_\lambda u)(t) = \lambda N \circ F + (1-\lambda)at$ is a compact homotopy. $y \in \overline{U}$ is a fixed point of H_λ if and only if y is a solution of $(11)_\lambda$; in particular, H_λ is fixed-point free on the boundary of U. Since the constant map H_0 is essential, the map H_1 has a fixed point $y \in U$; y is the required solution to (2)–(3). $\square$

We remark that this proof is easily modified to cover the case $A = 0$, that is, the case of bounded dependence on y'. Note also that we actually have $y \in \mathrm{C}^1[0,1]$.

Example. For the differential equation

$$y'' = \big(1 + \tfrac{1}{10}{y'}^2\big)(y+t)^{-1/2} = 0$$

we may take $g(y) = y^{-1/2}$. Numerical calculation shows that, for this equation plus the boundary conditions (3), the hypothesis of the theorem is met provided $1.085 < a < 12.145$, approximately. The hypotheses of Theorem 2 are met only for a in the approximate range $1.64 < a < 6.66$.

The initial value problem. We turn now to the semilinear initial value problem

$$y''(t) + f(t, y(t)) = 0, \qquad y(0) = 0, \quad y'(0) = a > 0 \tag{13}$$

and establish conditions sufficient to guarantee the existence of a nondecreasing solution on the interval $[0,1]$.

Theorem 4. *Let f be continuous and nonnegative on $[0,1] \times (0,a]$ and assume there exists a continuous, nonincreasing, $L_1(0,a)$ function g satisfying $f(t,y) \leq g(y)$ on $[0,1] \times (0,a]$. Define*

$$G(z) = \int_0^z \frac{du}{\sqrt{a^2 - 2\int_0^u g(s)\, ds}}$$

for $0 \leq z \leq a$, and suppose that

$$a^2 > 2\int_0^a g(s)\, ds, \qquad a > \int_0^1 g(G^{-1}(s))\, ds. \tag{14}$$

Then (13) *has a monotone solution on* $[0,1]$.

Proof. Clearly we have $0 \le y(t) \le at \le a$ for any monotone solution of (13). By standard arguments we see that $G(y(t)) \ge t$, i.e., $y(t) \ge G^{-1}(t)$ (this exists since $G(a) \ge 1$). Note that $G^{-1}(t)$ itself is the unique solution of the problem

$$z''(t) + g(z(t)) = 0, \qquad z(0) = 0, \quad z'(0) = a. \tag{15}$$

Set

$$B = \{z \in \mathrm{C}^1[0,1] : z(0) = 0 \text{ and } \|z\| < \infty\},$$

where we define

$$\|z\| = \tfrac{1}{a} \max\{|z|_0, |z'|_0\}.$$

Let K denote the convex subset

$$K = \{z \in B : G^{-1}(t) \le z(t) \le a, 0 \le z'(t) \le a, z'(0) = a\}$$

and let U denote, for some $\delta > 0$, the open subset of K,

$$U = \{z \in K : \|z\| < 1 + \delta\}.$$

Now $j : \overline{U} \to (\mathrm{C}[0,1], |\cdot|_0)$ is, by a standard application of the Arzela–Ascoli theorem, compact; moreover, $z \in \overline{j\overline{U}}$ satisfies $G^{-1}(t) \le z(t) \le a$. The Niemytzki map $F : \overline{j\overline{U}} \to L_1(0,1)$ is easily seen to be a continuous operator: for $z_1, z_2 \in \overline{j\overline{U}}$ the Lebesgue dominated convergence theorem yields

$$\|Fz_1 - Fz_2\|_{L_1} = \int_0^1 |f(t, z_1(t)) - f(t, z_2(t))| \, dt \to 0$$

as $z_1 \to z_2$ uniformly, since $|f(t, z_i(t))| \le g(G^{-1}(t)) \in L_1(0,1)$. Define the map $N : L_1(0,1) \to B$ by

$$(Nw)(t) = at - \int_0^t (t-s) w(s) \, ds;$$

in fact, $Nw \in \mathrm{C}^1[0,1]$, $(Nw)'$ is absolutely continuous and $(Nw)''(t) = -w(t)$ almost everywhere, and N is a continuous operator into B.

For $z \in \overline{j\overline{U}}$ we have, using (15),

$$\big(NFz - G^{-1}\big)''(t) = -f(t, z(t)) + g(G^{-1}(t)) \ge -g(z(t)) + g(G^{-1}(t)) \ge 0,$$

and $(NFz - G^{-1})(0) = 0$, $(NFz - G^{-1})'(0) = 0$. It follows that $(NFz)(t) \ge G^{-1}(t)$ on $(0,1]$. From

$$(NFz)'(t) = a - \int_0^t f(s, z(s)) \, ds \ge a - \int_0^1 g(G^{-1}(s)) \, ds > 0$$

and $f(t,z(t)) \geq 0$ it follows that $NF : \overline{j\overline{U}} \to K$. Hence $NFj : \overline{U} \to K$ is a compact map.

Let y_λ denote a solution of the one-parameter family of problems

$$y_\lambda''(t) + \lambda f(t, y_\lambda(t)) + (1-\lambda) g(y_\lambda(t)) = 0, \quad y_\lambda(0) = 0, \quad y_\lambda'(0) = a \tag{16$_\lambda$}$$

for $\lambda \in [0,1]$. Noting that $\lambda f(t,y) + (1-\lambda)g(y) \leq g(y)$, we see that the preceding consideration for the solution y of (11) apply equally *and uniformly in* $\lambda \in [0,1]$ to the solutions y_λ of $(16)_\lambda$. We accordingly define the homotopy $H_\lambda : \overline{U} \to K$ by

$$H_\lambda z = \lambda NFjz + (1-\lambda) NQjz,$$

where Q is the Niemytzki operator $(Qz)(t) = g(z(t))$. H is compact. Clearly, y is a fixed point of H_λ if and only if y is a solution of $(16)_\lambda$. It follows from the definition of U that H_λ is fixed-point free on the boundary of U. Moreover, H_0 is the constant map $(H_0 z)(t) = G^{-1}(t)$. Therefore by the topological transversality theorem H_1 is essential and so has a fixed point in U.

Remark. It follows from (15) that $G^{-1}(t) \geq G^{-1}(1)t$, and so

$$\int_0^1 g(G^{-1}(s))\,ds \leq \int_0^1 g(G^{-1}(1)s)\,ds = \frac{1}{G^{-1}(1)} \int_0^{G^{-1}(1)} g(u)\,du;$$

thus the second condition of (14) will be satisfied if

$$aG^{-1}(1) > \int_0^{G^{-1}(1)} g(u)\,du$$

or if the still stronger condition

$$aG^{-1}(1) > \int_0^a g(u)\,du$$

holds.

Example. The problem

$$y''(t) + \tfrac{1}{2}\big(y(t)+t\big)^{-1/2} = 0, \quad y(0) = 0, \quad y'(0) = 3$$

has a solution monotonic on $[0,1]$. Indeed, with $g(y) = y^{-1/2}/2$ and $\xi \equiv G^{-1}(1)$ we have

$$1 = \int_0^\xi \frac{dz}{\sqrt{9 - 2\sqrt{z}}} \leq \frac{\xi}{\sqrt{9-2\sqrt{\xi}}} \;\Leftrightarrow\; 4\xi \geq (9-\xi^2)^2;$$

numerical calculation shows that $\xi > 2.42$. It is now straightforward to see that (14) holds; in particular, the last condition of the remark is satisfied.

We remark that the initial value problem for the general quasilinear equation can be handled by blending the arguments used here in the proofs of Theorems 3 and 4. The details are omitted.

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