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EIKONAL EQUATIONS WITH DISCONTINUITIES

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Abstract. This paper is concerned with the Hamilton-Jacobi equation of eikonal type

$$H(Du) = n(x) \qquad x \in \Omega \subset \mathbb{R}^N, \tag{E}$$

where H is convex, Du represents the gradient of u with respect to x, and $n(x)$ is lower semi-continuous. In this work, a new notion of generalized solution for (E) is developed which is appropriate for this class of discontinuous right-hand sides $n(x)$. Such solutions we term *Monge* solutions. The Monge notion arises in a natural way from the variational formulation of (E) and is consistent with the well-known viscosity notion when $n(x)$ is continuous. In the class of lower semi-continuous $n(x)$, we establish the comparison principle for Monge subsolutions and supersolutions, existence and uniqueness results for (E) with Dirichlet boundary conditions, and a stability result. Moreover, we show that the Monge solution can be smaller than the maximal Lipschitz subsolution.

1. Introduction. Consider the Hamilton-Jacobi equation of eikonal type

$$H(Du) = n(x) \qquad x \in \Omega \subset \mathbb{R}^N, \tag{E}$$

where Du is the gradient of the unknown u and H is assumed to be convex. Such equations arise in geometric optics, for example, where $H(p) = |p|$ and $n(x)$ is the index of refraction ([1]). In case $n(x)$ is continuous, the notion of viscosity solution has lead to a complete comparison and stability theory for (E) and uniqueness and existence theory for the Dirichlet problem ([3], [5], [6], [8]). Here we seek to develop a new theory for (E) that allows the right-hand side $n(x)$ to be lower semi-continuous and which is consistent with the known notion of viscosity solution when $n(x)$ is continuous. Such a theory is desirable in view of the occurrence of optical problems with discontinuous media.

When $n(x)$ is only lower semi-continuous, the direct application of known notions of generalized solution of (E) does not yield the 'natural' solution, in the following sense. Recall that the *optical length function* $L : \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$ relative to the triple (H, n, Ω) is given by

$$L(x,y) = \inf_{\gamma \in P} \Big\{ \int_0^T N(n(\gamma(t)), \gamma'(t))\, dt \Big\}, \tag{1.1}$$

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where $N(r,\zeta) = \sup\{-\zeta\cdot p : H(p) = r\}$ and P denotes the set of absolutely continuous paths $\gamma : [0,T] \to \bar{\Omega}$ such that $\gamma(0) = x$ and $\gamma(T) = y$. When $n(x)$ is continuous, Lions in [8] has shown that if the Dirichlet problem

$$\begin{aligned} H(Du) &= n(x) \qquad &&\text{in } \Omega \\ u &= \varphi &&\text{on } \partial\Omega \end{aligned} \tag{1.2}$$

has a viscosity solution u, then it is given by the formula

$$u(x) = \inf_{y\in\partial\Omega} \{L(x,y) + \varphi(y)\}\,. \tag{1.3}$$

However, when $n(x)$ is lower semi-continuous, this formula does not necessarily yield a classical viscosity solution. For example, if we let $H(p) = |p|$, $\Omega = (-1, \frac{1}{2})$, put $n(x) \equiv 1$ on $[-1, 0]$ and $n(x) \equiv 2$ on $(0, \frac{1}{2}]$, and let $\varphi(-1) = 2$, $\varphi(\frac{1}{2}) = 0$, then formula (1.3) yields

$$u(x) = \begin{cases} 1-x & \text{if } \ -1 \le x \le 0; \\ 1-2x & \text{if } \ 0 \le x \le \frac{1}{2}. \end{cases}$$

One easily checks that u is not a viscosity solution of $|Du| = n(x)$. We refer the reader to the User's Guide [4] and its references for the basic definitions and theory of viscosity solutions. We remark that using upper semi-continuous right-hand sides does not eliminate this difficulty in general, as a similar example shows. Moreover, the same difficulty arises for the notion of solution proposed by Kruzkov in [7], and for the notion of *maximal Lipschitz subsolution*. For the latter, we cite the example considered at the end of Section 5 below.

We introduce now the notion of *Monge* solution for Hamilton-Jacobi equations of eikonal type. Solutions of Monge type are consistent with formula (1.3) and yield a satisfactory uniqueness and existence theory for the Dirichlet problem when $n(x)$ is lower semi-continuous. At the same time some of the useful properties of viscosity solutions carry over. For example, subsolutions and supersolutions of Monge type can be compared and uniform limits of Monge solutions of equations converging in a natural way remain Monge solutions (stability).

Definition. Let $u \in C(\Omega)$ and suppose for each $x_0 \in \Omega$ that

$$\liminf_{x\to x_0} \frac{u(x) - \big(u(x_0) - L(x_0,x)\big)}{|x-x_0|} = 0 \qquad (\text{resp. } \ge, \le). \tag{1.4}$$

Then u is a *Monge solution* (resp. *subsolution*, *supersolution*) of (E) on Ω.

Our terminology derives from the fact that, for sufficiently smooth $H(p)$ and $n(x)$, the graph of $C(x) \equiv u(x_0) - L(x_0, x)$ is the classical lower Monge cone for the Hamilton-Jacobi equation (E). Using the following set of standard assumptions (SA) on the Hamiltonian H, the 'index of refraction' n, and the domain Ω:

(H1) H is convex on $\mathbb{R}^N$

(H2) $H(0) = 0$ and $H(p) > 0$ if $p \neq 0$
(H3) $H(p) \to +\infty$ as $|p| \to +\infty$
(N1) $n : \bar{\Omega} \to \mathbb{R}$ is lower semi-continuous
(N2) $0 < n_- \equiv \inf\{n(x) : x \in \bar{\Omega}\} \leq n_+ \equiv \sup\{n(x) : x \in \bar{\Omega}\} < +\infty$
(Ω1) $\Omega \subset \mathbb{R}^N$ is open, connected, and bounded
(Ω2) $\partial\Omega$ is Lipschitz

we state the following main results:

Theorem 1.1. (Comparison Theorem) *Let* (H, n, Ω) *satisfy* (SA)*. Suppose that* $u \in C(\bar{\Omega})$ *is a Monge subsolution of* (E) *and that* $v \in C(\bar{\Omega})$ *is a Monge supersolution of* (E)*. If* $u \leq v$ *on* $\partial\Omega$ *then* $u \leq v$ *in* Ω.

Theorem 1.2. (Solvability of the Dirichlet Problem) *Let* (H, n, Ω) *satisfy* (SA)*. Suppose the boundary data* φ *satisfies the compatibility condition*

$$\varphi(x) - \varphi(y) \leq L(x, y) \qquad \forall x, y \in \partial\Omega, \tag{1.5}$$

where $L(x, y)$ *is the optical length for the triple* (H, n, Ω)*. Then the Dirichlet problem* (1.2) *has a unique Monge solution* u *in* $C(\bar{\Omega})$*, given by formula* (1.3).

Theorem 1.3. (Stability Theorem) *Let* $n_k : \bar{\Omega} \to \mathbb{R}$ *be given for* $k \in \mathbb{N}$ *with positive constants* c *and* C *such that*

$$c \leq n_k(x) \leq n_{k+1}(x) \leq C \qquad \forall k \in \mathbb{N}\ \forall x \in \bar{\Omega}. \tag{1.6}$$

Suppose also that (SA) *holds for* (H, n_k, Ω)*, that each* $u_k \in C(\bar{\Omega})$ *is a Monge solution of* $H(Du) = n_k$*, and that the sequence* $\{u_k\}$ *converges uniformly on* $\bar{\Omega}$ *to* u*. Let* $n \equiv \lim_{k\to\infty} n_k$*. Then* (SA) *holds for* (H, n, Ω) *and* u *is a Monge solution of* $H(Du) = n$.

In Section 2 we prove some preliminary results concerning the optical length function and the equivalence of viscosity and Monge notions when $n(x)$ is continuous, needed in the sequel. Section 3 contains the proof of Theorem 1.1. In Section 4 we prove both Theorem 1.2 and Theorem 1.3. We conclude in Section 5 by comparing and contrasting Monge solutions and maximal Lipschitz subsolutions.

2. Preliminaries. In the following two propositions we obtain some useful inequalities for the optical length function and prove that optimal paths exist.

Proposition 2.1. *Under* (SA)*, the optical length function* $L(x, y)$ *is well-defined (finite-valued) on* $\bar{\Omega} \times \bar{\Omega}$ *and satisfies the following properties:*

(L1) $0 \leq L(x, z) \leq L(x, y) + L(y, z) \quad \forall x, y, z \in \bar{\Omega}$.
(L2) *Let* $\alpha = \min\{|p| : H(p) = n_-\}$, $\beta = \max\{|p| : H(p) = n_+\}$*, and* Γ *be a Lipschitz constant for* $\partial\Omega$*. Then we have that* $\alpha|x - y| \leq L(x, y) \leq \beta(\Gamma + 1)|x - y|$ *for all* $x, y \in \bar{\Omega}$.
(L3) *The optical length function is Lipschitz on* $\bar{\Omega} \times \bar{\Omega}$ *with a Lipschitz constant equal to* $2\beta(\Gamma + 1)$.

Proof. For each path $\gamma \in P$, the mapping defined by

$$t \mapsto N\big(n(\gamma(t)), \gamma'(t)\big) \tag{2.1}$$

is measurable since the map N is continuous and the map n is Borel measurable. The mapping of (2.1) is also integrable since it is bounded above in norm by $\beta|\gamma'(t)|$. The existence of paths in P follows from properties (Ω1) and (Ω2). In fact, given any pair of points (x, y) in Ω, these conditions imply the existence of a path $\gamma \in P$ with length bounded above by the quantity $(\Gamma+1)|x-y|$. This establishes (L2) and the fact that the optical length is well-defined. The proof of (L1) is clear. Now choose (x_1, y_1) and (x_2, y_2) in $\bar{\Omega} \times \bar{\Omega}$. From (L1) it follows that

$$L(x_1, y_1) - L(x_2, y_2) \leq L(x_1, x_2) + L(y_2, y_1).$$

This inequality together with (L2) yields property (L3).

Proposition 2.2. *Under* (SA)*, given x and y in $\bar{\Omega}$, there exists a path $\xi \in P$ so that*

$$L(x, y) = \int_0^T N(n(\xi(t)), \xi'(t))\, dt.$$

Proof. Since n is lower semi-continuous on $\bar{\Omega}$ there exists an increasing sequence $\{n_k\}$ of positive functions in $C(\bar{\Omega})$ that converge pointwise to n on $\bar{\Omega}$. Let L_k be the optical length associated to the triple (H, n_k, Ω). In view of Theorem 14.1 of [2] the functional $F_k : P \to \mathbb{R}$ given by

$$F_k(\gamma) \equiv \int_0^T N(n_k(\gamma(t)), \gamma'(t))\, dt$$

is lower semi-continuous with respect to the usual Fréchet topology on P. An application of the Lebesgue dominated convergence theorem yields

$$\lim_{k\to\infty} F_k(\gamma) = F(\gamma) \equiv \int_0^T N(n(\gamma(t)), \gamma'(t))\, dt. \tag{2.2}$$

We claim that F is lower semi-continuous as well. Let $\gamma_k \to \gamma$ in the Fréchet topology. Fix $l \in \mathbb{N}$. From $n \geq n_l$ we have $F(\gamma_k) \geq F_l(\gamma_k)$. Let $k \to \infty$ to obtain

$$\liminf_{k\to\infty} F(\gamma_k) \geq \liminf_{k\to\infty} F_l(\gamma_k) \geq F_l(\gamma). \tag{2.3}$$

Letting $l \to \infty$ in (2.3) and using (2.2) proves the claim. The remainder of the proof follows in the standard way via the Hilbert compactness theorem, as Ω is bounded and the lengths of the paths in any minimizing sequence can be bounded above by $\frac{\beta}{\alpha}(\Gamma+1)|x-y|$ in view of (L2).

Remark. As the map $\gamma \mapsto F(\gamma)$ is independent of re-parameterizations that do not affect orientation, we assume throughout the remainder that all paths appear parameterized by arc-length.

Corollary 2.3. *Let $x, y \in \bar{\Omega}$ with ξ an optimal path for $L(x,y)$. Then for $0 \le t \le s \le T$ we have*

$$\frac{\alpha}{\beta(\Gamma+1)}(s-t) \le |\xi(s) - \xi(t)| \le s - t. \tag{2.4}$$

Proof. The second inequality follows from the arc-length parameterization of ξ. As ξ is optimal from x to y it is likewise optimal from $\xi(t)$ to $\xi(s)$. By virtue of (L2) we have

$$\alpha(s-t) \le \int_t^s N(n(\xi(\sigma), \xi'(\sigma))\,d\sigma = L(\xi(t), \xi(s)) \le \beta(\Gamma+1)|\xi(s) - \xi(t)|,$$

which completes the proof of the corollary. □

If the right-hand side $n(x)$ of (E) is continuous on $\bar{\Omega}$, it turns out that Monge subsolutions and supersolutions are precisely viscosity subsolutions and supersolutions, respectively. This useful fact will be employed in the sequel. We give a demonstration for the supersolution case only, since the subsolution case is similar. We require the following simple approximation lemma, whose proof we omit:

Lemma 2.4. *Let (H, n, Ω) satisfy* (SA). *Fix $x_0 \in \Omega$. If $n(x)$ is continuous in a neighborhood of x_0 then the optical length function can be locally approximated as follows:*

$$L(x_0, x) = N(n(x_0), x - x_0) + o(|x - x_0|), \tag{2.5}$$

$$L(x_0, x) = L_r(x_0, x) + o(r) \qquad \forall x \in B_r(x_0), \tag{2.6}$$

where L_r is the optical length function relative to $B_r(x_0)$.

Proposition 2.5. *Let (H, n, Ω) satisfy* (SA). *In addition, assume that $n(x)$ is continuous on $\bar{\Omega}$. Then the set of Monge supersolutions of* (E) *coincides with the set of viscosity supersolutions of* (E).

Proof. From (1.1) and (2.5) it follows that v is a Monge supersolution if and only if for each $x_0 \in \Omega$ we have

$$\liminf_{x \to x_0} \max_{p \in C_0} \Big(\frac{v(x) - (v(x_0) + p \cdot (x - x_0))}{|x - x_0|} \Big) \le 0, \tag{2.7}$$

where $C_0 = \{p : H(p) \le n(x_0)\}$. Let v be a Monge supersolution of (E). Choose any $q \in D^- v(x_0) \equiv \{p \in \mathbb{R}^n : v(x) \ge v(x_0) + p \cdot (x - x_0) + o(|x - x_0|)\}$. From (2.7) and the compactness of the sets C_0 and $S^{N-1} \equiv \{x \in \mathbb{R}^N : |x| = 1\}$, we have the existence of a direction $\xi \in S^{N-1}$ such that $(q - p) \cdot \xi \le 0$ for every $p \in C_0$. By the separation theorem for convex sets, it follows that $q \notin \operatorname{int} C_0$. Hence, v is a viscosity supersolution of (E).

Let v be a viscosity supersolution of (E) on Ω. If v is not a Monge supersolution, there exists an $x_0 \in \Omega$ and a pair of positive constants r and δ such that

$$v(x) - (v(x_0) - L(x_0, x)) \ge \delta |x - x_0| \tag{2.8}$$

for $|x - x_0| \le r$. Without loss of generality we may suppose that $v(x_0) = 0$. Let $\varphi(x) = -L(x_0, x) + \delta r$. Then, if L_r is the optical length for the triple $(H, n, B_r(x_0))$, it follows that

$$\varphi(x) - \varphi(y) = L(x_0, y) - L(x_0, x) \le L(x, y) \le L_r(x, y) \tag{2.9}$$

because (L1) holds and L is monotone decreasing in Ω if we use set inclusion as a partial ordering. Let

$$w(x) = \inf_{y \in \partial B_r(x_0)} \{L_r(x, y) + \varphi(y)\}.$$

Since the consistency condition (2.9) holds, one easily checks (see Chapter 5 of [8]) that w is continuous on Ω and is the unique viscosity solution of

$$\begin{aligned} H(Dw) &= n \qquad && \text{in } B_r(x_0) \\ w &= \varphi && \text{on } \partial B_r(x_0). \end{aligned}$$

Moreover, by standard comparison results (see [5] or [6]) we have $w \le v$ in $B_r(x_0)$ since $\varphi \le v$ on $\partial B_r(x_0)$ by (2.8). However, this is impossible as

$$w(x_0) = \inf_{y \in \partial B_r(x_0)} \{L_r(x_0, y) - L(x_0, y) + \delta r\} \ge \delta r > 0 = v(x_0).$$

Hence, v is a Monge supersolution.

Remark. We note that for the subsolution case we can use (2.6) in place of the estimate $L \le L_r$. Also, we note that when $n(x)$ is lower semi-continuous, Monge supersolutions *still* coincide with viscosity supersolutions, as will be shown in Section 4 below. However, subsolutions may differ. This will be clear from the example presented in Section 5.

3. Proof of Theorem 1.1.

Before proving the comparison theorem we establish two lemmas.

Lemma 3.1. *Assume* (SA) *holds. If u is a Monge subsolution of (E) in Ω then the graph of u lies locally on or above its lower Monge cones.*

Proof. Fix $x_0 \in \Omega$. We need to show that there is an $r > 0$ so that

$$u(x) \ge u(x_0) - L(x_0, x) \qquad x \in B_r(x_0). \tag{3.1}$$

Choose $r > 0$ sufficiently small so that all optimal paths for $L(x_0, x)$ with $x \in B_r(x_0)$ lie inside a ball $B \subset\subset \Omega$. Put $\Lambda(t) = L(x_0, \gamma(t))$, where γ is optimal for $L(x_0, x)$. By (L2), Λ is Lipschitz. Also, note that u is Lipschitz on B since u is a viscosity subsolution of $H(Du) = n_+$ on B. We can then calculate that

$$\lim_{s \downarrow t} \frac{u \circ \gamma(s) - u \circ \gamma(t) + \Lambda(s) - \Lambda(t)}{s - t} = (u \circ \gamma)'(t) + \Lambda'(t) \tag{3.2}$$

for almost every t in $[0,\, T]$. Thus,

$$\begin{aligned}(u\circ\gamma+\Lambda)'(t) &= \lim_{s\downarrow t}\frac{u(\gamma(s))-(u(\gamma(t))-L(\gamma(t),\gamma(s)))}{s-t}\\ &\geq \frac{\beta(\Gamma+1)}{\alpha}\lim_{s\downarrow t}\frac{u(\gamma(s))-(u(\gamma(t))-L(\gamma(t),\gamma(s)))}{|\gamma(s)-\gamma(t)|}\geq 0 \quad \text{a.e.} \quad t\in[0,T],\end{aligned}$$

where we have used Corollary 2.3 and the definition of Monge subsolution. Integration of this last inequality yields (3.1).

Lemma 3.2. *Let* (SA) *hold and let* $m:\bar{\Omega}\to\mathbb{R}$ *satisfy* (N1) *and* (N2) *with* $m\leq\delta n$*, for some* $0<\delta<1$*. If* $u\in C(\bar{\Omega})$ *is a Monge subsolution of* $H(Du)=m$ *and* $v\in C(\bar{\Omega})$ *is a Monge supersolution of* $H(Dv)=n$*, then* $u\leq v$ *on* $\partial\Omega$ *implies that* $u\leq v$ *in* Ω*.*

Proof. Consider the auxiliary function

$$\varphi(x,y)=u(x)-v(y)-\frac{L^2(x,y)}{\varepsilon}\,,$$

where $\varepsilon>0$ and L is the optimal length for (H,n,Ω). By (L3), $\varphi\in C(\bar{\Omega}\times\bar{\Omega})$. Suppose our conclusion fails. For all $\varepsilon>0$ sufficiently small, the existence of a point $(x_\varepsilon,y_\varepsilon)\in\Omega\times\Omega$ where φ has a local maximum is then guaranteed. Fix such an $\varepsilon>0$ and then fix an optimal path γ for $L(x_\varepsilon,y_\varepsilon)$. Define $g(t)\equiv[L(x_\varepsilon,y_\varepsilon)+L(\gamma(t),y_\varepsilon)]\varepsilon^{-1}$. We claim that $g(0)<1$. This is trivial if $x_\varepsilon=y_\varepsilon$. Suppose then that $x_\varepsilon\neq y_\varepsilon$. Since $\varphi(\cdot,y_\varepsilon)$ has a local maximum at $x=x_\varepsilon$ we have

$$u(x_\varepsilon)-u(\gamma(t))\geq g(t)\big(L(x_\varepsilon,y_\varepsilon)-L(\gamma(t),y_\varepsilon)\big)=g(t)L(x_\varepsilon,\gamma(t)). \tag{3.3}$$

Let L_m be the optical length associated to (H,m,Ω). In view of (3.3) and Lemma 3.1 we obtain

$$L_m(x_\varepsilon,\gamma(t))\geq g(t)L(x_\varepsilon,\gamma(t))$$

and now since $m\leq\delta n$ and γ is optimal it follows that

$$\int_0^t N\left(\delta n(\gamma(s)),\gamma'(s)\right)\,ds\geq g(t)L(x_\varepsilon,\gamma(t))=g(t)\int_0^t N\left(n(\gamma(s)),\gamma'(s)\right)\,ds. \tag{3.4}$$

Let us define the function

$$f(r)\equiv\max\Big\{\frac{N(\delta r,\zeta)}{N(r,\zeta)}L:|\zeta|=1\Big\}.$$

From (H1) and (H2) it follows that $f\in C(0,\infty)$ and that $f(r)<1$ for $0<r<\infty$. Hence, there exists a constant c satisfying $f(r)\leq 1-c$ for $r\in[n_-,n_+]$ and $0<c<1$. This implies, letting $t\downarrow 0$ in (3.4), that $g(0)\leq 1-c$. This establishes our claim.

Now since $\varphi(x_\varepsilon, \cdot)$ has a local maximum at y_ε, we have the first inequality in

$$\begin{aligned} v(y_\varepsilon) - v(y) &\le \left(\frac{L(x_\varepsilon, y_\varepsilon) + L(x_\varepsilon, y)}{\varepsilon}\right) \big(L(x_\varepsilon, y) - L(x_\varepsilon, y_\varepsilon) \big) \\ &\le \left(\frac{L(x_\varepsilon, y_\varepsilon) + L(x_\varepsilon, y)}{\varepsilon}\right) L(y_\varepsilon, y) \le \left(1 - \frac{c}{2}\right) L(y_\varepsilon, y), \end{aligned}$$

where the last inequality holds for y close enough to y_ε. Here we also used (L1).

We rewrite this last inequality, using (L2):

$$\frac{v(y) - \big(v(y_\varepsilon) - L(y_\varepsilon, y) \big)}{|y - y_\varepsilon|} \ge \frac{cL(y_\varepsilon, y)}{2|y_\varepsilon - y|} \ge \frac{c\alpha}{2} > 0.$$

Yet this contradicts the fact that v is a Monge supersolution. Thus, we must have had $u \le v$ in Ω to begin with. $\square$

Proof of Theorem 1.1. By replacing u by $u + C$, and v by $v + C$ for a constant $C > 0$ large enough, we can assume that u is non-negative in Ω. Fix $0 < \delta < 1$ and put $u_\delta(x) \equiv \delta u(x)$. The convexity of H and the normalization $H(0) = 0$ imply that $H(\delta p) \le \delta H(p)$. Hence, u_δ solves $H(Du_\delta) \le \delta n(x)$. By Lemma 3.2, $u_\delta \le v$ in Ω. Now let $\delta \uparrow 1$.

4. Proofs of Theorems 1.2 and 1.3. We first establish the unique solvability of the Dirichlet problem (1.2).

Proof of Theorem 1.2. Uniqueness in the class $C(\bar\Omega)$ is an immediate consequence of the comparison theorem. We claim that

$$u(x) = \inf_{y \in \partial\Omega} \{L(x, y) + \varphi(y)\} \qquad x \in \bar\Omega \tag{1.3}$$

is the desired Monge solution.

In view of (L2) and the compatibility condition (1.5) we see that $\varphi \in C(\partial\Omega)$. In particular, φ is bounded on $\partial\Omega$, so that in the formula (1.3), u is expressed as an infimum of uniformly Lipschitz functions (in x) that are also uniformly bounded (in x and y). It follows that u is Lipschitz on $\bar\Omega$, so *a fortiori* u is continuous there.

Furthermore, since $\partial\Omega$ is compact, the infimum in (1.3) is always achieved. Fix $x_0 \in \Omega$ and fix an arbitrary sequence $\{x_k\}$ converging to x_0. Choose $y_k \in \partial\Omega$ so that $u(x_k) = L(x_k, y_k) + \varphi(y_k)$. Then we have

$$u(x_k) \ge L(x_0, y_k) - L(x_0, x_k) + \varphi(y_k) \ge u(x_0) - L(x_0, x_k),$$

which implies that u is a Monge subsolution of (E) in Ω.

Now choose $y_0 \in \partial\Omega$ so that $u(x_0) = L(x_0, y_0) + \varphi(y_0)$. Let γ be a minimizing path for $L(x_0, y_0)$. For large k, put $z_k = \gamma(\frac{1}{k})$. We claim that then

$$u(z_k) = L(z_k, y_0) + \varphi(y_0) \tag{4.1}$$

holds for all large k. If not, there exists a $w_k \in \partial\Omega$ so that

$$L(z_k, w_k) + \varphi(w_k) < L(z_k, y_0) + \varphi(y_0). \tag{4.2}$$

Adding $L(x_0, z_k)$ to both sides of (4.2) we obtain from our choice of y_0 and the fact that γ is minimizing for $L(x_0, y_0)$ the inequality

$$L(x_0, w_k) + \varphi(w_k) < u(x_0)$$

which contradicts (1.3) and establishes (4.1). Hence

$$u(z_k) - (u(x_0) - L(x_0, z_k)) = L(z_k, y_0) - L(x_0, y_0) + L(x_0, z_k) = 0. \tag{4.3}$$

Letting $k \to \infty$ in (4.3) shows that u is a Monge supersolution on Ω.

Now fix $y_0 \in \partial\Omega$. From (4.1), choosing $x = y_0 = y$, we have $u(y_0) \leq \varphi(y_0)$. Choose $y_1 \in \partial\Omega$ so $u(y_0) = L(y_0, y_1) + \varphi(y_1)$. From (1.5) we immediately see that $u(y_0) \geq \varphi(y_0)$. Hence, $u = \varphi$ on $\partial\Omega$, which establishes that u is a Monge solution of the Dirichlet problem (1.2). □

As promised above, we now strengthen Proposition 2.5.

Proposition 4.1. *Let* (SA) *hold, and let* $v \in C(\bar{\Omega})$. *The function* v *is a Monge supersolution of* (E) *if and only if* v *is a viscosity supersolution of* (E).

Proof. Let v be a Monge supersolution of (E). The first part of the proof of Proposition 2.5 is still valid since, under our assumptions here, the approximation (2.5) holds, but with '$\leq$' in place of '$=$'; and it is only this direction of the approximation that is required. Hence, v is a viscosity supersolution as well. Let v be a viscosity supersolution of (E). Since we now have comparison and existence results allowing $n(x)$ to be lower semi-continuous, the second half of the proof of Proposition 2.5 is valid here without further change. Hence, v is also a Monge supersolution of (E). □

We are now in a position to prove the last of our main theorems.

Proof of Theorem 1.3. As the supremum of lower semi-continuous functions remains lower semi-continuous, and (1.6) holds, then both (N1) and (N2) hold for the limit function n. So (SA) is valid for the triple (H, n, Ω). Of course, we also have $u \in C(\bar{\Omega})$.

Let $L_k(x, y)$ be the optical length for (H, n_k, Ω), and $L(x, y)$ the optical length for (H, n, Ω). As the n_k are uniformly bounded above and away from zero, the proof of Lemma 3.1 applies uniformly in k and so for fixed $x_0 \in \Omega$ we obtain an $r > 0$ independent of k so that

$$u_k(x_0) - u_k(y) \leq L_k(x_0, y) \quad \text{for } y \in B_r(x_0). \tag{4.4}$$

From $n \geq n_k$ it follows that $L \geq L_k$. Letting $k \to \infty$ in (4.4) we see that u is a Monge subsolution of $H(Du) = n$ in Ω.

By Proposition 4.1 each u_k is a viscosity supersolution of $H(Du_k) \geq n_k$. We claim that $H(Du) \geq n$ holds in the viscosity sense. To show this let $\varphi \in C^1(\Omega)$ with $u - \varphi$

possessing a strict local minimum at $x_0 \in \Omega$. For large k, $u_k - \varphi$ has a local minimum at $x_k \in \Omega$ with $\lim_{k\to\infty} x_k = x_0$. From the lower semi-continuity of the n_k and their monotonicity ($n_k \le n_{k+1}$) we obtain

$$H(D\varphi(x_0)) = \lim_{k\to\infty} H(D\varphi(x_k)) \ge \liminf_{k\to\infty} n_k(x_k) \ge n(x_0).$$

By applying Proposition 4.1 again we see that u is a Monge supersolution of $H(Du) = n$. This establishes the theorem.

5. Maximal Lipschitz subsolutions. For the Dirichlet problem (1.2), the map $w : \bar{\Omega} \to \mathbb{R}$ is a *Lipschitz subsolution* if w is Lipschitz on $\bar{\Omega}$, $H(Du) \le n$ almost everywhere (with respect to N dimensional Lebesgue measure) in Ω, and $u \le \varphi$ on $\partial\Omega$. The proof of the following result can be found in Proposition 7.3 of [8].

Proposition 5.1. *Let* (SA) *hold and in* (1.2) *put* $\varphi \equiv 0$. *Then there exists a maximal Lipschitz subsolution* w *of* (1.2) *so that*

$$\begin{cases} H(Dw) = n(x) & a.e.\ x \in \Omega \\ H(Dw) \ge n(x) & \text{in the viscosity sense.} \end{cases} \tag{5.1}$$

It would seem reasonable then to define a weak solution of (1.2) as this maximal Lipschitz subsolution. By Proposition 5.1 such solutions exist and are clearly unique. Several questions naturally arise. Is the maximal Lipschitz subsolution identical to the Monge solution? Does condition (5.1) along with the boundary condition characterize maximal Lipschitz subsolutions? We will answer these negatively, but first we show that Monge solutions also satisfy condition (5.1).

Theorem 5.2. *Let* (SA) *hold and let* φ *satisfy the compatibility condition* (1.5). *Then the Monge solution* u *of* (1.2) *is also a Lipschitz subsolution of* (1.2).

Proof. We begin by choosing $n_k \in C(\bar{\Omega})$ so that $n_k \le n_{k+1} \le n$ and $n_k \to n$ pointwise on $\bar{\Omega}$. Let $L_k(x, y)$ be the optical length function for (H, n_k, Ω) and put

$$\varphi_k(x) \equiv \inf_{z\in\partial\Omega} \{\varphi(z) + L_k(x, z)\} \qquad x \in \partial\Omega.$$

We leave it to the reader to check that

$$\varphi_k(x) - \varphi_k(y) \le L_k(x, y) \qquad \forall x, y \in \partial\Omega. \tag{5.2}$$

We claim that $L_k \to L$ pointwise on $\bar{\Omega} \times \bar{\Omega}$. Let $x, y \in \bar{\Omega}$ with $x \ne y$. Let γ_k be an optimal path for $L_k(x, y)$. Since the lengths of the paths γ_k are uniformly bounded, and $\bar{\Omega}$ is compact, the Hilbert compactness theorem demonstrates the existence of a subsequence $\{\gamma_{k_j}\}$ converging to a limit path γ in the Fréchet topology. With F_k defined as in the proof of Proposition 2.2, we have for $j \ge l$

$$L(x, y) \ge L_{k_j}(x, y) = F_{k_j}(\gamma_{k_j}) \ge F_{k_l}(\gamma_{k_j}).$$

Then for each l fixed we have, on letting $j \to \infty$,

$$L(x,y) \geq \lim_{j\to\infty} L_{k_j}(x,y) \geq F_{k_l}(\gamma),$$

which follows in light of the lower semi-continuity of F_{k_l}. Now let $l \to \infty$, to obtain

$$L(x,y) \geq \lim_{j\to\infty} L_{k_j}(x,y) \geq L(x,y).$$

As $L_k \leq L_{k+1}$, this shows not only that $L_k \to L$ pointwise, but also uniformly (Dini). From the definition of φ_k we conclude that $\{\varphi_k\}$ converges to a function $\bar{\varphi}$ given by

$$\bar{\varphi}(x) = \inf_{z\in\partial\Omega}\{\varphi(z) + L(x,z)\}. \tag{5.3}$$

We claim that $\bar{\varphi} = \varphi$. On substitution of $z = x$ in (5.3) we have $\bar{\varphi}(x) \leq \varphi(x)$ while from (1.5) written as $\varphi(z) + L(x,z) \geq \varphi(x)$, we have $\bar{\varphi}(x) \geq \varphi(x)$.

By virtue of (5.2), let u_k be the Monge solution of

$$\begin{aligned} H(Dw) &= n_k \qquad \text{in } \Omega \\ w &= \varphi_k \qquad \text{on } \partial\Omega \end{aligned}$$

given by

$$u_k(x) = \inf_{y\in\partial\Omega}\{\varphi_k(y) + L_k(x,y)\}.$$

As $\varphi_k \to \varphi$ and $L_k \to L$ both converge uniformly, it is clear that $u_k \to u$ uniformly on $\bar{\Omega}$. Now the sequence $\{u_k\}$ is uniformly Lipschitz on $\bar{\Omega}$; it follows that, as Ω is bounded, $\{u_k\}$ is a bounded sequence in $W^{1,2}(\Omega)$. Without loss of generality we can assume that

$$Du_k \rightharpoonup Du \quad \text{in} \quad \big(L^2(\Omega)\big)^N.$$

By Mazur's theorem we can find $0 \leq \lambda_k^i \leq 1$, $i = 1, \ldots, k$, so that $\sum_{i=1}^{k} \lambda_k^i = 1$ and

$$\sum_{i=1}^{k} \lambda_k^i Du_i \to Du \quad \text{in} \quad \big(L^2(\Omega)\big)^N. \tag{5.4}$$

Passing to a subsequence (again indexed just by k) we can assume that the convergence in (5.4) is pointwise almost everywhere on Ω. The convexity of H implies that the following inequalities hold almost everywhere on Ω:

$$H\Big(\sum_{i=1}^{k} \lambda_k^i Du_i\Big) \leq \sum_{i=1}^{k} \lambda_k^i H(Du_i) \leq \sum_{i=1}^{k} \lambda_k^i n_i \leq n_k.$$

In the second inequality we used $H(Du_i) \leq n_i$ which holds almost everywhere on Ω since $n_i \in C(\bar{\Omega})$. The third inequality follows from $n_i \leq n_k, i = 1, \ldots, k$. Now (5.4) yields

$$H(Du) \leq \sup_k n_k = n \quad \text{a.e. in} \quad \Omega.$$

Hence, u is itself a Lipschitz subsolution of (1.2).

Corollary 5.3. *Under the same assumptions as in Theorem* 5.2*, the Monge solution* u *of* (1.2) *also satisfies condition* (5.1)*.*

Proof. The second part of condition (5.1) is a consequence of Proposition 4.1. Thus $H(Du) \geq n$ holds almost everywhere in Ω. Theorem 5.2 establishes the reverse inequality, and thus all of (5.1) holds. □

When $n \in C(\bar{\Omega})$, then it is known that the viscosity solution of (1.2) is the maximal Lipschitz subsolution of (1.2); likewise so with the Monge solution. In certain particular cases when $n(x)$ is 'mildly' discontinuous, Monge solutions will remain the same as the maximal Lipschitz subsolution.

Theorem 5.4. *Let* $\Omega \subset \mathbb{R}^2$ *satisfy* (Ω1) *and* (Ω2)*. Section* Ω *by a finite number of non-intersecting lines* $l_1, \dots, l_k$ *into open subdomains* $\Omega_1, \dots, \Omega_{k+1}$*. Let* $n : \bar{\Omega} \to \mathbb{R}$ *satisfy* (N1) *and* (N2)*. In addition, suppose that* $n \in C(\Omega_i)$ *for* $i = 1, \dots, k+1$ *and that on a neighborhood of each side of the line* l_i *the index of refraction* n *is equal to a distinct constant, and that, on the line* l_i *itself,* n *equals the smaller of these constants. If* $w : \bar{\Omega} \to \mathbb{R}$ *is a Lipschitz subsolution of*

$$\begin{aligned} |Du| &= n \qquad \text{in } \Omega \\ u &= \varphi \qquad \text{on } \partial\Omega, \end{aligned} \tag{5.5}$$

then w *is a Monge subsolution of* (5.5)*.*

Proof. Suppose there exists a Lipschitz subsolution w which is not a Monge subsolution. A proof similar to that of Proposition 2.5 shows that w is a Monge subsolution on each Ω_i. Then there is a point $(x_0, y_0) \in \Omega$ so that, for instance, $(x_0, y_0) \in \partial\Omega_1 \cap \partial\Omega_2$ and

$$\liminf_{(x,y)\to(x_0,y_0)} \frac{w(x,y) - \Big(w(x_0,y_0) - L((x_0,y_0),(x,y))\Big)}{|(x - x_0, y - y_0)|} < 0. \tag{5.6}$$

For convenience we may assume that $\partial\Omega_1 \cap \partial\Omega_2$ lies on the x-axis, with Ω_1 (resp. Ω_2) in the lower (resp. upper) half-plane. Also, let $n \equiv n_1$ on $\bar{\Omega}_1$ and $n \equiv n_2$ on Ω_2, where n_1 and n_2 are constants, with $0 < n_1 < n_2$; and finally put $(x_0, y_0) = (0, 0)$. For (x, y) near (0,0) a straight-forward calculation yields

$$L((0,0),(x,y)) = \begin{cases} n_1(x^2+y^2)^{\frac{1}{2}} & \text{if } (x,y) \in S_1; \\ n_2(x^2+y^2)^{\frac{1}{2}} & \text{if } (x,y) \in S_2; \\ n_1|x| + (n_2^2 - n_1^2)^{\frac{1}{2}} y & \text{otherwise,} \end{cases} \tag{5.7}$$

where $S_1 = \{(x,y) | y \leq 0\}$ and $S_2 = \{(x,y) : |x| \leq n_1(n_2^2 - n_1^2)^{-\frac{1}{2}} y\}$. For each (x, y) near $(0, 0)$ there is only one optimal path for $L((0,0),(x,y))$. If $(x,y) \in S_1$ (resp. S_2), the optimal path γ_1 (resp. γ_2) is the straight line joining $(0,0)$ to (x,y). If $(x,y) \notin S_1 \cup S_2$ then the optimal path γ_3 (traced backward) follows a line of slope $\pm n_1^{-1}(n_2^2 - n_1^2)^{\frac{1}{2}}$ from (x,y) until intersecting the x-axis and then follows the x-axis to $(0,0)$.

We now verify that for each of these optimal paths γ_i there exists a unit vector $\nu_i \in \mathbb{R}^2$ and a positive constant a_0 such that for $0 \le a \le a_0$ the translates $\gamma_i + a\nu_i$ sweep out a positive area, are disjoint, and satisfy

$$\lim_{a \downarrow 0} n(\gamma_i(t) + a\nu_i) = n(\gamma_i(t)) \qquad \forall t \in [0,T]. \tag{5.8}$$

We remark that condition (5.8) is not necessarily trivial as n is discontinuous. If $(x,y) \in S_1$, choose ν_1 normal to γ_1 so that the second component of ν_1 is negative. Since each translate stays in S_1, (5.8) follows. If $(x,y) \in S_2$, choose ν_2 normal to γ_2 so that the second component of ν_2 is positive. If $(x,y) \notin S_1 \cup S_2$ choose $\nu_3 = (0,-1)$. Since $n = n_1$ on the x-axis, (5.8) holds here as well.

Fix (x,y) near $(0,0)$. For convenience, we now drop the subscripts on the optimal paths γ_i and unit vectors ν_i. Since the classical gradient Dw exists almost everywhere on Ω, Fubini's Theorem implies the existence of a null sequence $\{a_k\}$ such that Dw exists on $\gamma + a_k\nu : [0,T] \to \Omega$ and satisfies $H(Dw(\gamma(t) + a_k)) \le n(\gamma(t) + a_k)$ for almost every $t \in [0,T]$ and for each $k \in \mathbb{N}$. This gives

$$\begin{aligned} w((x,y) + a_k\nu) - w(a_k\nu) &= \int_0^T Dw(\gamma(t) + a_k) \cdot \gamma'(t)\, dt \\ &\ge -\int_0^T N(n(\gamma(t) + a_k), \gamma'(t))\, dt. \end{aligned}$$

In view of (5.8), letting $a_k \to 0$ we have

$$w(x,y) - w(0,0) \ge -\int_0^T N(n(\gamma(t)), \gamma'(t))\, dt = -L((0,0),(x,y)),$$

which contradicts (5.6).

Corollary 5.5. *Under the assumptions of Theorem* 5.4 *the Monge solution of* (5.5) *is also the maximal Lipschitz subsolution of* (5.5).

Proof. Let u be the Monge solution and w be the maximal Lipschitz subsolution. By Theorem 5.4, w is a Monge subsolution with $w \le \varphi$ on $\partial\Omega$. Hence by our comparison theorem, $w \le u$ on $\bar{\Omega}$. However, by Theorem 5.2, u is a Lipschitz subsolution, so that $u \le w$ on $\bar{\Omega}$. □

We now answer negatively the questions posed earlier with respect to the relation between the Monge solution and the maximal Lipschitz solution by considering the following example. Let $\Omega = (-1,1) \times (-1,1) \subset \mathbb{R}^2$ and for $k \in \mathbb{N}$ let Ω_k be the closed subset of $\bar{\Omega}$ lying between the lines $y = \frac{1}{k}$ and $y = -\frac{1}{k}$. Define

$$n_k(x,y) = \begin{cases} 1 & \text{if } (x,y) \in \Omega_k; \\ 2 & \text{otherwise,} \end{cases}$$

and let u_k be the Monge solution of

$$\begin{aligned} |Du_k| &= n_k && \text{in } \Omega \\ u_k &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{5.9}$$

By Corollary 5.5, u_k is also the maximal Lipschitz subsolution of (5.9). In view of the formula (1.3), we have $u_k \geq u_j$ for $k > j$. Applying the theorem of Arzela-Ascoli, there exists $u \in C(\bar{\Omega})$ so that $u_k \to u$ uniformly on $\bar{\Omega}$. From the stability theorem we see that u is the Monge solution of

$$\begin{aligned} |Du| &= n && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{5.10}$$

where

$$n(x,y) = \begin{cases} 1 & \text{if } y = 0; \\ 2 & \text{otherwise.} \end{cases}$$

Since $n = 2$ almost everywhere in Ω, the maximal Lipschitz subsolution of (5.10) is given by

$$w(x,y) = 2\ \mathrm{dist}((x,y), \partial\Omega).$$

Since $u_k(x,0) = 1 - |x|$ for each k, then $u(x,0) = 1 - |x|$. Hence $u \neq w$.

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