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ASYMPTOTIC BEHAVIOR OF THE QUANTUM DEFECT AS A FUNCTION OF AN ANGULAR MOMENTUM AND ALMOST NORMALIZATION OF EIGENFUNCTIONS FOR A THREE-DIMENSIONAL SCHRÖDINGER OPERATOR WITH NEARLY COULOMB POTENTIAL

Marianna A. Shubov[1]

Department of Mathematics, Texas Tech University, Lubbock, Texas 79409

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Abstract. We study two questions connected to the discrete spectrum of a three-dimensional Schrödinger operator with Coulomb potential perturbed by a spherically symmetric compactly supported function $q(r)$. It is known that the eigenvalues of this operator can be described by the formula $\lambda_n^\ell = -(n+\ell+\mu_n^\ell)^{-2}$, where $n = 1, 2, \dots$ is the principal quantum number, $\ell = 0, 1, 2, \dots$ is the azimuthal quantum number (or angular momentum) and μ_n^ℓ is the so-called quantum defect. It is, also, known that for each ℓ there exists a limit $\mu_\infty^\ell = \lim_{n\to\infty} \mu_n^\ell$. We assume that $q(r) \geq 0$ and prove the estimate $|\mu_\infty^\ell| \leq C(ae^2/2)^{2\ell+1}(\ell+0.5)^{-(4\ell+3)}$, where a is the radius of the support of $q(r)$ and C is $0.25\sqrt{\pi}$ times the first moment of $q(r)$ (which is assumed to be finite). It follows from this estimate that μ_∞^ℓ tends to zero as $\ell \to \infty$ faster than any negative power of ℓ. Our second result deals with the eigenfunctions $\{\Phi_n^\ell(r)\}_{n=1}^\infty$ of the radial Schrödinger operator corresponding to a fixed value of ℓ. The natural definition of $\Phi_n^\ell(r)$ which allows one to construct them explicitly, is based on the condition $\lim_{r\to 0} r^{-\ell-1}\Phi_n^\ell(r) = 1$. Such defined eigenfunctions are not normalized in $L^2(0,\infty)$ and the normalization constant C_n^ℓ is known only for the pure Coulomb potential. We prove that in the case of a perturbed Coulomb potential the constants C_n^ℓ behave as $n^{-3/2}$ when $n \to \infty$ for any fixed ℓ and, therefore, for any ℓ the system $\{n^{-3/2}\Phi_n^\ell(r)\}_{n=1}^\infty$ forms an orthogonal Riesz basis in its closed linear hull.

1. Introduction. An electron in the field of a positively charged atomic ion can be approximately described by the Hamiltonian operator

$$H = -(1/2)\Delta - Z/r + q(x) \tag{1.1}$$

acting on $L^2(\mathbb{R}^3)$. Here $Z > 0$ is the charge of the ion, $x = (x_1, x_2, x_3) \in \mathbb{R}^3, r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ and q represents the perturbation of the nucleus by an electron cloud. In this paper we assume that q is compactly supported and spherically symmetric: $q = q(r)$. More precise conditions on q will be given later.

The operator (1.1) is very well studied. However, there are nontrivial questions connected to its discrete spectrum which are of importance in quantum physics and which have not yet been clarified. To describe these questions and to formulate the results of the paper we have to recall some well-known facts and introduce some notation.

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Due to the spherical symmetry of the potential the Hamiltonian (1.1) is separable in spherical coordinates. The radial Schrödinger equation corresponding to the unperturbed Hamiltonian $H_0 = -(1/2)\Delta + Z/r$ has the form

$$\Big(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} + \frac{2Z}{r} + E\Big)\psi = 0, \tag{1.2}$$

where $\ell = 0, 1, 2, \ldots$ is the azimuthal quantum number or the angular momentum. Setting $\rho = Zr, E = -Z^2/k^2, E < 0$, we rewrite (1.2) in the form

$$\Big(\frac{d^2}{d\rho^2} - \frac{\ell(\ell+1)}{\rho^2} + \frac{2}{\rho} - \frac{1}{k^2}\Big)\psi = 0. \tag{1.3}$$

So, the radial Schrödinger equation corresponding to Hamiltonian (1.1) can be written in the form

$$\Big(\frac{d^2}{d\rho^2} - \frac{\ell(\ell+1)}{\rho^2} + \frac{2}{\rho} - q(\rho) - \frac{1}{k^2}\Big)\psi = 0, \tag{1.4}$$

where we renamed $2q(r)/Z^2$ to be $q(\rho)$.

The discrete spectrum of (1.1) splits into infinite sequence of infinite series $\{\lambda_n^\ell\}_{n=1}^{\infty}$ corresponding to different values of the angular momentum $\ell = 0, 1, 2, \ldots$. It can be described by means of the so-called quantum defect method, the idea of which goes back to the works of Rydberg on the analysis of alkali spectra (see [1, 4]). Namely, the eigenvalues can be represented in the form $\lambda_n^\ell = -1/(k_n^\ell)^2$, where

$$k_n^\ell = n + \ell + \mu_n^\ell. \tag{1.5}$$

Here n is the principal quantum number and the quantity μ_n^ℓ is called the quantum defect. The quantum defect characterizes the influence on the spectrum of the non-Coulomb part of the potential.

There exists an extensive physical literature, both experimental and theoretical, devoted to the quantum defect. A systematic theoretical study of the quantum defect theory was initiated in the works of M. Seaton and his collaborators. (The bibliography can be traced through the book [1].) It was suggested in these works that for each value of ℓ there must exist a limit $\mu_\infty^\ell = \lim_{n\to\infty} \mu_n^\ell$, which is of a particular importance for physicists. The existence of the above limit was rigorously proved in [11, 12] for a special class of analytic potentials $q(r)$. Under more natural restrictions on the perturbation potential $q(r)$ a similar result was obtained in [5]. In papers [13, 14] devoted to the problem of quantum defect we derived an explicit (and very complicated) formula for the limiting quantum defect μ_∞^ℓ in terms of the perturbation $q(r)$. It was also shown that $\mu_n^\ell = \mu_\infty^\ell + O(n^{-1})$. In [13, 14] $q(r)$ was assumed to be compactly supported.

It was predicted by physicists that the quantum defect μ_n^ℓ varies slowly with n but very rapidly goes to zero when $\ell \to \infty$. This is due to the fact that the greater ℓ, the less amount of time the electron spends in the vicinity of the nucleus, and, therefore, the energy levels have to approach the hydrogen levels. Experimental observations of the spectra of the alkali elements (see [1], [6]) had shown the absence of the limiting shift μ_∞^ℓ for high values of the angular momentum ℓ. Practically, this shift disappears starting from $\ell = 5, 6, \dots$.

The fact that $\mu_\infty^\ell \to \infty$ as $\ell \to \infty$ and, moreover, $\sum_{\ell=0}^\infty \ell\mu_\infty^\ell < \infty$ for rapidly decreasing potentials is well known (see, e.g. [5]). It is shown rigorously in [5], that for $q \in L^2(0,\infty) \cap L^1(0,\infty)$ the limiting quantum defect μ_∞^ℓ is equal to the spectral shift function ξ_ℓ at zero energy for the pair of operators (1.3), (1.4). For the pair of three-dimensional Schrödinger operators H and H_0 (see (1.1), (1.2)) the spectral shift function Ξ is also well defined. Separation of variables leads to

$$\Xi = \sum_{\ell=0}^\infty (2\ell+1)\xi_\ell.$$

The latter suggests that the series $\sum(2\ell+1)\mu_\infty^\ell$ is convergent.

However, the above rate of convergence of μ_∞^ℓ to zero (which is slightly faster than ℓ^{-2}) does not explain completely the experimental results. According to these results (see e.g. [1], [6]), μ_∞^ℓ tends to zero much faster than it follows from the convergence of the above series.

In this paper we consider a compactly supported nonnegative potential $q(r)$ with finite first moment and under these assumptions show that a significantly stronger estimate for μ_∞^ℓ takes place. It follows, in particular, from our result that $\sum_{\ell=0}^\infty (2\ell+1)^p \mu_\infty^\ell < \infty$ for any $p \ge 1$. Before we formulate this result precisely, we would like to emphasize that in its proof we do not use the above-mentioned explicit formula for μ_∞^ℓ derived in [14]. It is very difficult to study the dependence of μ_∞^ℓ on ℓ using this formula, because the formula is very complicated. Instead we use a method which will be briefly outlined at the end of the Introduction.

Now we turn to the precise formulation of the first main result of the present paper.

Theorem 1.1. *Assume, that $q = q(\rho)$ is a measurable nonnegative function, which is equal to zero outside a finite interval $[0, a]$, and its first moment is finite:*

$$\int_0^a tq(t)\,dt < \infty. \tag{1.6}$$

Then for the sequence of limiting values $\{\mu_\infty^\ell\}_{\ell=0}^\infty$ the following estimate holds:

$$|\mu_\infty^\ell| \le C(a\ e^2/2)^{2\ell+1}(\ell+0.5)^{-(4\ell+3)}, \tag{1.7}$$

where

$$C = 0.25\sqrt{\pi}\int_0^a tq(t)\,dt. \tag{1.8}$$

Remark 1.1. The quantum defect theory is, also, concerned with the so-called quantum defect of the continuous spectrum. This defect is the increment over the interval $[0,\infty)$ of the limit phase $\delta_\ell(s)$ which is defined in the scattering theory (see, for example, [9]).

It was predicted by physicists ([1,6]) and rigorously shown in [19] that $\delta_\ell(\infty) - \delta_\ell(0) = \pi\mu_\infty^\ell$. In [5] for a wide class of potentials it was also shown that $\delta_\ell(\infty) = 0$ and, therefore, $-\delta_\ell(0) = \pi\mu_\infty^\ell$. We can conclude from Theorem 1.1 that the same estimate (1.7) holds for $\delta_\ell(0)$.

The second result of this paper deals with the problem of the expansion of a given function $f \in L^2(0,\infty)$ with respect to the system of eigenfunctions of the radial operator $H_\ell = -\frac{d^2}{d\rho^2} + \frac{\ell(\ell+1)}{\rho^2} - q(\rho)$. This problem is important in quantum mechanics (see [5, 6]). To formulate this result we have to recall some well-known facts. The perturbed radial Schrödinger equation (1.4) has two linearly independent solutions $\Phi_\ell(k,\rho)$ and $F_\ell(k,\rho)$. The regular solution $\Phi_\ell(k,\rho)$ is naturally defined by the condition

$$\lim_{\rho\to 0} \rho^{-\ell-1}\Phi_\ell(k,\rho) = 1; \tag{1.9}$$

i.e., it has a zero of the order $(\ell+1)$ at $\rho = 0$. The singular solution $F_\ell(k,\rho)$ has a singularity of the type $\rho^{-\ell}$ at $\rho = 0$. At a discrete set of points $\{k_n^\ell\}_{n=1}^\infty$ these two solutions become linearly dependent; i.e., their Wronskian $\{\Phi_\ell, F_\ell\} = 0$. These points define the eigenvalues by $\lambda_n^\ell = -(k_n^\ell)^{-2}$. The corresponding eigenfunctions are simply $\Phi_\ell(k_n^\ell,\rho)$. This definition of eigenfunctions based on the condition (1.9) is natural. It allows one to obtain them as solutions of certain Volterra integral equations (see [9] or Section 2). However, thus defined eigenfunctions are not normalized. The normalization constant $\mathcal{C}_n^\ell$ is known only in the case of pure Coulomb potential (1.3), when $\Phi_\ell(k_n^\ell,\rho)$ can be expressed in terms of the Whittaker functions. This constant behaves as $n^{-3/2}$ when $n\to\infty$. In spite of the fact that the problem is very well known the behavior of the normalization constant for the case of a perturbed Coulomb potential was never clarified.

For our second result the condition $q(\rho) \ge 0$ is not required. This result consists of the following.

Theorem 1.2. *Assume that $q(\rho)$ has a compact support and satisfies $\int_0^a t|q(t)|\,dt < \infty$. Then for each $\ell = 0,1,2,\dots$ there exist two positive constants C_1^ℓ and C_2^ℓ such that for the eigenfunctions $\{\Phi_\ell(k_n^\ell,\rho)\}_{n=1}^\infty$ of the problem* (1.4) *with $q(\rho)$, satisfying* (1.6)*, the following estimates hold*:

$$C_1^\ell n^{3/2} \le \|\Phi_\ell(k_n^\ell,\cdot)\| \le C_2^\ell n^{3/2}. \tag{1.10}$$

Both constants may grow with ℓ. (*Here and below $\|f\|$ is the $L^2(0,\infty)$-norm of f.*)

As an application of Theorem 1.2 we consider the estimates for the mean values of the different powers of ρ that is of a special interest for physicists. By the definition (see [15]), the mean values of successive powers of ρ in a given quantum state are computed by the formula

$$\overline{\rho^m} = \int_0^\infty \rho^m |\Phi_\ell(k_n^\ell,\rho)|^2 \|\Phi_\ell(k_n^\ell,\rho)\|^{-2}\,d\rho,$$

where $\overline{\rho^m}$ is the mean value of ρ^m. From Theorem 1.2, it follows that

$$\int_0^\infty |\Phi_\ell(k_n^\ell,\rho)|^2 \|\Phi_\ell(k_n^\ell,\rho)\|^{-2}\,d\rho \asymp \int_0^\infty |\mathring{\Phi}_n(\rho)|^2 \|\mathring{\Phi}_n(\rho)\|^{-2}\,d\rho. \tag{1.11}$$

Here and below we write $\psi(x) \asymp \varphi(x)$ for two positive functions $\psi(x)$ and $\varphi(x)$, defined on one and the same domain, if there exist two positive constants $\mathcal{C}_1$ and $\mathcal{C}_2$ such that $\mathcal{C}_1\psi(x) \leq \varphi(x) \leq \mathcal{C}_2\psi(x)$.

It is known ([1]) that in the pure Coulomb case $\overline{\rho^s} \asymp n^{2s}$. Due to (1.11) the same is valid in the presence of the perturbation q.

Now we describe briefly the organization of the paper and the method used in the proof of the main results. Section 2 is devoted to the proof of Theorem 1.1 and to the formulation of all auxiliary results required for this proof. The proof of Theorem 1.1 is based on the direct analysis of the condition $\{\Phi_\ell, F_\ell\}(k) = 0$, which defines eigenvalues k_n^ℓ (or, more precisely, $\lambda_n^\ell = -(k_n^\ell)^{-2}$) of the problem. This analysis in turn is based on special uniform asymptotic representations of the so-called Liouville-Green type for the solutions of equation (1.4). These representations are given by Theorem 2.1 which we prove in Section 3 together with another auxiliary statement—Lemma 2.2. Section 4 is devoted to the proof of Theorem 1.2, the result of which is, in fact, a combination of three propositions: Lemmas 4.1, 4.2 and 4.3 proved in this section. Lemmas 4.2 and 4.3 related to the behavior of the irregular Whittaker function might be of interest in themselves.

We would like to mention that similar (but more complicated) techniques of Liouville-Green approximations are used in our recent paper [15], in which we study the resonances of the same operator (1.1). Based on the aforementioned techniques it is shown in [15] that, in contrast with the non-Coulomb case, the full three-dimensional operator (1.1) has an infinite series of low energy resonances accumulating to zero on the second Riemann sheet of the spectral parameter. These resonances correspond to different values of ℓ.

Remark 1.2. 1. We believe that (1.10) must be correct for sufficiently rapidly decreasing (but not compactly supported) potentials. The proof of such a result would require more involved asymptotic techniques than those we use in Section 4.

2. The estimate (1.7) contains the radius a of the support $q(r)$. So, it cannot be extended directly to noncompactly supported potentials. However, based on our proof of Theorem 1.1 (see (2.21)–(2.26)) we conjecture that for rapidly decreasing $q(r)$ (faster than any negative power of r) the following estimate is valid:

$$|\mu_\infty^\ell| \leq 0.25\sqrt{\pi}(e^2/2)^{2\ell+1} \int_0^\infty t^{2\ell+2} q(t)\, dt (\ell + 0.5)^{-(4\ell+3)}.$$

The proof of such an estimate would require an extension of our Liouville-Green arguments to rapidly decreasing potentials.

2. Formulation of auxiliary propositions. Proof of Theorem 1.1.

This section is devoted to the proof of Theorem 1.1. We derive it from several auxiliary propositions. To formulate them, we have to recall some basic facts about the solutions of the radial equations (1.3) and (1.4).

Equation (1.3) has two linearly independent solutions. As the first one we take a solution that behaves as $\rho^{\ell+1}$ near $\rho = 0$. As the second solution we choose the one that decreases exponentially as $\rho \to \infty$; at $\rho = 0$ this solution behaves as $\rho^{-\ell}$. We call the first solution a regular solution and denote it by $\Phi_{0\ell}(k,\rho)$, while the second solution is called a singular solution and denoted by $F_{0\ell}(k,\rho)$. These solutions are proportional to the regular and irregular Whittaker functions, the standard notation for which are $M_{k,\ell+1/2}(2\rho/k)$ and $W_{k,\ell+1/2}(2\rho/k)$ (see, for example, [7, 16]). The solution $\Phi_{0\ell}(k,\rho)$ is given by the formula

$$\Phi_{0\ell}(k,\rho) = (k/2)^{\ell+1} M_{k,\ell+1/2}(2\rho/k) = \rho^{\ell+1}(1 + \rho(\ell+1)^{-1} + O(\rho^2)). \tag{2.1}$$

The estimate $O(\rho^2)$ when $\rho \to 0$ is uniform with respect to $k \in [s,\infty)$ for any $s > 0$. The regular Whittaker function is given by the formula

$$M_{k,\ell+1/2}(z) = z^{\ell+1} e^{-z/2} M(\ell+1-k, 2\ell+2, z),$$

where $M(\ell+1-k, 2\ell+2, z)$ is a confluent hypergeometric function

$$M(\ell+1-k, 2\ell+2, z) = \sum_{n=0}^{\infty} \frac{(\ell+1-k)_n}{(2\ell+2)_n n!} z^n, \tag{2.2}$$

and $(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$ is the Pohhammer symbol. The function $F_{0\ell}(k,\rho)$ is given by the following formula:

$$\begin{aligned} F_{0\ell}(k,\rho) &= -2^{\ell} k^{\ell+1} \Gamma(-\ell-k)[(2\ell+1)!]^{-1} W_{k,\ell+1/2}(2\rho/k) \\ &= \rho^{-\ell}(1 + \rho\ell^{-1} + O(\rho^2 k^{-1})), \end{aligned} \tag{2.3}$$

where $W_{k,\ell+1/2}(z) = z^{\ell+1} e^{-z/2} U(\ell+1-k, 2\ell+2, z)$. Let $\alpha = \ell+1-k$; then

$$\begin{aligned} U(\alpha, 2\ell+2, z) = {} & \frac{(2\ell+1)!}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha-2\ell-1)_n}{(-2\ell-1)_n n!} z^{n-2\ell-1} \\ & - \frac{1}{(2\ell+1)!\Gamma(\alpha-2\ell-1)} \big\{ M(\alpha, 2\ell+2, z) \ell n z \\ & + \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(2\ell+2)_n n!} z^n \big[\psi(\alpha+n) - \psi(1+n) - \psi(2\ell+2+n) \big] \big\}. \end{aligned} \tag{2.4}$$

Here $\psi(z)$ is the digamma function $\psi(z) = \Gamma'(z)/\Gamma(z)$. In (2.3) the estimate $O(\rho^2 k^{-1})$ when $k \to \infty$ is uniform with respect to ρ from any interval $[0,b]$, the same estimate when $\rho \to 0$ is uniform with respect to $k \in [s,\infty)$ for any $s > 0$. Denoting the Wronskian of two functions by curly brackets, we have

$$W_{0\ell}(k) = \{\Phi_{0\ell}, F_{0\ell}\} = -(2\ell+1) k^{2\ell+1} \Gamma(-\ell-k)/\Gamma(\ell+1-k). \tag{2.5}$$

By analogy with the unperturbed equation, we call the solution of equation (1.4), that near $\rho = 0$ has a zero of order $(\ell+1)$, the regular solution. The solution that is proportional to the irregular Whittaker function $W_{k,\ell+1/2}(2\rho/k)$ when $\rho \geq a$ is called the singular solution of the same equation. It has a singularity of the type $\rho^{-\ell}$ near $\rho = 0$. The regular solution, denoted by $\Phi_\ell(k,\rho)$, and the singular solution of equation (1.4), denoted by $F_\ell(k,\rho)$, satisfy the following Volterra integral equations:

$$\Phi_\ell(k,\rho) = \Phi_{0\ell}(k,\rho) + W_{0\ell}(k)^{-1} \int_0^\rho \mathfrak{K}(k,\rho,t) q(t) \Phi_\ell(k,t)\, dt, \tag{2.6}$$

$$F_\ell(k,\rho) = F_{0\ell}(k,\rho) - W_{0\ell}(k)^{-1} \int_\rho^\infty \mathfrak{K}(k,\rho,t) q(t) F_\ell(k,t)\, dt, \tag{2.7}$$

where the kernel is given by

$$\mathfrak{K}(k,\rho,t) = \Phi_{0\ell}(k,\rho) F_{0\ell}(k,t) - \Phi_{0\ell}(k,t) F_{0\ell}(k,\rho). \tag{2.8}$$

Equations (2.6) and (2.7) were studied in [14] by the method of successive approximations and estimates for the solutions were obtained. It follows from these estimates, that the solutions $\Phi_\ell(k,\rho)$ and $F_\ell(k,\rho)$ preserve the behavior of the functions $\Phi_{0\ell}(k,\rho)$ and $F_{0\ell}(k,\rho)$ at the origin; namely,

$$|\Phi_\ell(k,\rho)| \leq \mathcal{C}(k,\ell)\rho^{\ell+1}, \quad |F_\ell(k,\rho)| \leq \mathcal{D}(k,\ell)\rho^{-\ell}, \tag{2.9}$$

where $\mathcal{C}(k,\ell)$ and $\mathcal{D}(k,\ell)$ may grow with k and ℓ. We also mention, that the function $F_{0\ell}(k,\rho)$ is "double singular" with respect to both its arguments: it has a singularity with respect to ρ of the type $\rho^{-\ell}$ at $\rho = 0$ and a singularity with respect to k of the type $\cot(\pi k)$ for $k \geq 0$. The same fact holds for the singular solution $F_\ell(k,\rho)$.

By eigenfunctions of the problem, defined by (1.4) and (1.5), we understand the solutions of equation (1.4) that behave as $\rho^{\ell+1}$ at the neighborhood of $\rho = 0$ and are square summable with respect to $\rho \in [0,\infty)$. In the general case, a solution $\Phi_\ell(k,\rho)$ that is bounded near $\rho = 0$, does not belong to $L_2(0,\infty)$, whereas the solution $F_\ell(k,\rho)$ belongs to $L_2(R,\infty), R > 0$, but behaves as $\rho^{-\ell}$ at $\rho = 0$. However, there exists a countable set of points $\{k_n^\ell\}_{n=1}^\infty$ at which the solutions $\Phi_\ell(k_n^\ell,\rho)$ and $F_\ell(k_n^\ell,\rho)$ become linearly dependent, being transformed into eigenfunctions of the problem (1.4), (1.5). The corresponding eigenvalues are $\lambda_n^\ell = -(k_n^\ell)^{-2}$.

In the next statement we collect all necessary asymptotic facts related to the special functions occurring in the paper.

Lemma 2.1. a) *The asymptotic behavior of the irregular Whittaker function for* $|z| \leq 1$ *when* $k \to \infty$ *is given by the formula* (see [17, 18])

$$W_{k,\ell+1/2}(z) = (-1)^{\ell+1} k^k e^{-k} \sqrt{2\pi z} \big[\sin(\pi k) Y_{2\ell+1}(2\sqrt{kz}) + \cos(\pi k) \mathcal{J}_{2\ell+1}(2\sqrt{kz})\big] (1 + \tilde{w}(k,\ell,z)), \tag{2.10}$$

where $Y_{2\ell+1}(t)$ *is the Neumann function of the order* $(2\ell+1)$ *and* $\mathcal{J}_{2\ell+1}(t)$ *is the Bessel function of the order* $(2\ell+1)$*;* $\tilde{w}(k,\ell,z)$ *is the function, such that* $\tilde{w}(k,\ell,z) \to 0$ *when*

$k \to \infty$ uniformly with respect to $z, |z| \le 1$, and, possibly, nonuniformly with respect to ℓ. The latter means, that there exists a positive continuous function $C_\ell(k)$ such that $|\tilde{w}(k,\ell,z)| \le C_\ell(k)$. This dominant function $C_\ell(k) \to 0$ when $k \to \infty$; it does not depend on $z, |z| \le 1$, and $\sup_{k\in[0,\infty)} C_\ell(k)$ may grow when $\ell \to \infty$.

b) *The irregular solution $F_{0\ell}(k,\rho)$ of* (1.3) *can be approximated by*

$$\begin{aligned} &F_{0\ell}(k,\rho) \\ &= -\pi 2^\ell [(2\ell)!]^{-1}\sqrt{2\rho}\big[Y_{2\ell+1}(2\sqrt{2\rho}) + \cot(\pi k)\mathcal{J}_{2\ell+1}(2\sqrt{2\rho})\big](1+w(k,\ell,z)), \end{aligned} \tag{2.11}$$

where the behavior of the function $w(k,\ell,z)$ is similar to that of the function $\tilde{w}(k,\ell,z)$ from (2.10).

c) *The asymptotic behavior of the Neumann and Bessel functions with respect to the order for bounded argument can be described by the formulae* (see [3, 7])

$$Y_\nu(z) = -(0.5\pi\nu)^{-1/2}(0.5ez/\nu)^{-\nu}(1+\epsilon_1(\nu,z)), \quad |z| \le 1, \tag{2.12}$$

$$\mathcal{J}_\nu(z) = (2\pi\nu)^{-1/2}(0.5ez/\nu)^{\nu}(1+\epsilon_2(\nu,z)), \quad |z| \le 1, \tag{2.13}$$

where $\epsilon_i(\nu,z) \to 0$ when $\nu \to \infty$ uniformly with respect to z, $|z| \le 1$, $i = 1,2$.

d) *The following asymptotic representation for the gamma function in the region $|\arg z| \le \pi - \epsilon$ is valid:*

$$\Gamma(z) = e^{-z}z^{z-0.5}(1+O(z^{-1})). \tag{2.14}$$

The following identity holds:

$$\Gamma(z)\Gamma(1-z) = \pi csc(\pi z). \tag{2.15}$$

Now we formulate two auxiliary propositions—Theorem 2.1 and Lemma 2.2. Theorem 2.1 will be obtained in Section 3 as a corollary of Theorem 3.1 from Section 3. The latter theorem deals with the special (uniform with respect to ℓ) approximations for two linearly independent solutions of equation (1.4). These approximations are known as Liouville-Green (LG) approximations (see [10]). In the derivation of these approximations it will be convenient to use a new parameter $m = \ell + 0.5$ instead of ℓ. We will use m in the rest of this section.

Theorem 2.1. *For $k \in [m^2,\infty)$ and $\rho \in [0,a]$, the regular solution $\Phi_\ell(k,\rho)$ of* (1.4) *can be approximated by the following function:*

$$\Phi_\ell(k,\rho) = \rho^{m+1/2}(1+O(m^{-1})), \quad m = \ell + 1/2, \tag{2.16}$$

where the estimate $O(m^{-1})$ is uniform with respect to k and ρ from the above intervals. In other words there exist two constants $\mathcal{C}_1$ and $\mathcal{C}_2$ such that

$$0 < \mathcal{C}_1 \le \big|(\Phi_\ell(k,\rho)\rho^{-m-0.5} - 1)m\big| \le \mathcal{C}_2 < \infty$$

for any $\rho \in [0,a]$, any $m = \ell + 1/2, \ell = 0,1,2\ldots$ and $k \in [m^2,\infty)$.

Lemma 2.2. *For $k \in [m^2, \infty)$ the Wronskian $W_{0\ell}(k)$ given by (2.5) can be represented in the form*

$$W_{0\ell}(k) = -2m(1 + O(m^{-1})), \quad m = \ell + 1/2, \tag{2.17}$$

where the estimate $O(m^{-1})$ is uniform with respect to k. In other words, there exist two constants $\mathcal{D}_1$ and $\mathcal{D}_2$ such that

$$0 < \mathcal{D}_1 \leq |(W_{0\ell}(k)(2m)^{-1} + 1)m| \leq \mathcal{D}_2 < \infty$$

for any $m = \ell + 1/2, \ell = 0, 1, 2, \ldots$ and $k \in [m^2, \infty)$.

We would like to emphasize that the uniformity of the estimates in Theorem 2.1 and Lemma 2.2 takes place only when ρ belongs to a compact set of a real axis and $k \geq m^2$.

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. As was mentioned above the eigenvalues of the problem (1.4), (1.5) are those and only those points of parameter k, where the solutions $\Phi_\ell(k, \rho)$ and $F_\ell(k, \rho)$ become linearly dependent; i.e., their Wronskian $W_\ell(k) = \{\Phi_\ell, F_\ell\} = 0$. Since $W_\ell(k)$ does not depend on ρ, we can evaluate it at any ρ, in particular at $\rho \geq a$. Based on (2.7) and the fact that $q(\rho) = 0$ for $\rho > a$ we can conclude that $F_\ell(k, \rho) = F_{0\ell}(k, \rho)$ for $\rho \geq a$. Using (2.6) and (2.7) for $\rho \geq a$ and (2.8), we obtain the following formula for $W_\ell(k)$:

$$W_\ell(k) = W_{0\ell}(k) + \int_0^a F_{0\ell}(k, t) q(t) \Phi_\ell(k, t)\, dt. \tag{2.18}$$

The equation for the spectrum, which we call "the eigenvalue equation," can be written in the form

$$1 + W_{0\ell}(k)^{-1} \int_0^a F_{0\ell}(k, t) q(t) \Phi_\ell(k, t)\, dt = 0. \tag{2.19}$$

Now we are going to transform the eigenvalue equation (2.19) using the above auxiliary proposition.

Substituting (2.11), (2.16) and (2.17) into (2.19) we can transform the latter equation to the form

$$\begin{aligned} 1 + (2m)^{-1} \pi 2^{m-0.5} \big[(2m-1)!\big]^{-1} (1 + O(m^{-1})) \int_0^a t^{m+0.5} q(t) \\ \times \big[Y_{2\ell+1}\big(2\sqrt{2t}\big) + \cot(\pi k) \mathcal{J}_{2\ell+1}\big(2\sqrt{2t}\big)\big](1 + w(k, \ell, t/k))\, dt = 0. \end{aligned} \tag{2.20}$$

Substituting the asymptotics for $Y_{2\ell+1}(z)$ and $\mathcal{J}_{2\ell+1}(z)$ into (2.20) (see (2.12), (2.13)) and taking into account that $(2m-1)! = \Gamma(2m) = (2m)^{2m-0.5} e^{-2m}(1 + O(m^{-1}))$, we transform (2.20) to

$$\begin{aligned} 1 - \frac{\sqrt{\pi} e^{2m}}{2m(\sqrt{2}m)^{2m}} (1 + O(m^{-1})) \int_0^a t^{m+1} q(t) \Big[\Big(\frac{e\sqrt{t}}{\sqrt{2}m}\Big)^{-2m} (1 + \epsilon_2(m, t)) \\ - \frac{\cot(\pi k)}{2} \Big(\frac{e\sqrt{t}}{\sqrt{2}m}\Big)^{2m} \big(1 + \epsilon_1(m, t)\big)\Big](1 + w(k, \ell, t/k))\, dt = 0. \end{aligned} \tag{2.21}$$

Let us denote $\mathcal{A}(k,m) = \sqrt{\pi}2^{-(2m+2)}e^{4m}m^{-(4m+1)}$, $Q_m = \int_0^a t^{2m+1}q(t)\,dt$ and $Q_0 = \int_0^a tq(t)\,dt$. After elementary transformations we can rewrite (2.21) in the form

$$\tan(\pi k) = \mathcal{A}(k,m)Q_m(1+\eta_1(k,m))\big(\sqrt{\pi}(2m)^{-1}Q_0(1+\eta_2(k,m))-1\big)^{-1}, \qquad (2.22)$$

where

$$\begin{aligned}&\eta_1(k,m)\\ &= Q_m^{-1}(1+O(m^{-1}))\int_0^a t^{2m+1}q(t)\big(\epsilon_1(m,t)+\epsilon_1(m,t)w(k,\ell,\tfrac{t}{k})+w(k,\ell,\tfrac{t}{k})\big)\,dt\\ &\eta_2(k,m)\\ &= Q_0^{-1}(1+O(m^{-1}))\int_0^a tq(t)(1+w(k,\ell,\tfrac{t}{k})(\epsilon_2(m,t)+(1+\epsilon_2(m,t))O(m^{-1}))\,dt.\end{aligned}$$

If we introduce the notation

$$1+\Theta(k,m) = (1+\eta_1(k,m))\big(1-\sqrt{\pi}(2m)^{-1}Q_0(1+\eta_2(k,m))\big)^{-1}, \qquad (2.23)$$

then (2.22) can be rewritten in the form

$$\tan(\pi k) = -\mathcal{A}(k,m)Q_m(1+\Theta(k,m)). \qquad (2.24)$$

We notice now, that $\Theta(k,m)\to 0$, when $k\to\infty$ along the positive real semiaxis, but the convergence is not uniform with respect to m. Indeed, it is easy to see, that $\eta_1(k,m)\to 0$ and $\eta_2(k,m)\to 0$ when $k\to\infty$, and the convergence is not uniform with respect to m. We also notice, that $Q_m \le a^{2m}Q_0$ and, therefore, $|\mathcal{A}(k,m)Q_m| \le \mathcal{C}_0 Q_0(a/2)^{2m}e^{4m}m^{-(4m+1)} \to 0$ when $m\to\infty$. (Here $\mathcal{C}_0$ is an absolute constant.) Taking into account (2.23) and these remarks we arrive at the above conclusion about the behavior of $\Theta(k,m)$.

Let us come back to the eigenvalue equation (2.24). We know already that for each m there exists an infinite sequence of real roots $\{k_n^\ell\}_{n=1}^\infty(\ell = m-0.5)$ of this equation. Notice, that $\Theta(k,m)$ is a complex-valued function of real k. The fact that the roots of (2.24) are real means that $Im\,\Theta(k_n^\ell,m)=0$. Therefore, (2.24) can be replaced by a real equation

$$\tan(\pi k) = -\mathcal{A}(k,m)Q_m(1+Re\ \Theta(k,m)), \qquad (2.25)$$

where $\lim_{k\to\infty} Re\Theta(k,m) = 0$ nonuniformly with respect to m. It follows immediately from (2.25) and the properties of the function $\tan x$ that for each m there exists $k_0(m)$ such that

$$|k_n^\ell - n - \ell| \le \mathcal{A}(k,m)Q_m(1+|Re\ \Theta(k_n^\ell,m)|) \quad \text{for} \ \ n > k_0(m), \qquad (2.26)$$

where $\{k_n^\ell\}_{n=1}^\infty$ are the above-mentioned roots of (2.25) and $\ell = m-0.5$. Using an explicit expression for $\mathcal{A}(k,m)$ the estimate (2.26) can be given in the following form:

$$|k_n^\ell - n - \ell| \le C(ae^2/2)^{2m}m^{-(4m+1)}(1+|Re\Theta(k_n^\ell,m)|) \quad \text{for} \ \ n \ge k_0(m) \qquad (2.27)$$

with C as given by (1.8).

Therefore, in terms of ℓ the estimate (2.27) obtains the following form:

$$|k_n^\ell - n - \ell| \le C(ae^2/2)^{2\ell+1}(\ell+0.5)^{-(4\ell+3)}\big(1+|Re\Theta(k_n^\ell,m)|\big). \qquad (2.28)$$

Recall now that the quantum defect $\mu_n^\ell = k_n^\ell - n - \ell$. We know (see [11, 14, 19]) that there exists $\lim_{n\to\infty}\mu_n^\ell = \mu_\infty^\ell$ and $k_n^\ell \to \infty$ when $n \to \infty$. Passing to the limit $n \to \infty$ in (2.26) and taking into account that $\lim_{n\to\infty} Re\ \Theta(k_n^\ell, m) = 0$ we arrive exactly at the inequality (1.7) with C given by (1.8). The proof is completed.

3. Proofs of Theorem 2.1 and Lemma 2.2. To prove Theorem 2.1 we construct LG approximations for the solutions of (1.4). First of all, by changing the variable and introducing a new function in (1.4) we transfer it to a certain standard form (see [10]). Having (1.4) in this new form, we can clarify the question about the absence of transition points. So, we have to reduce (1.4) to the form

$$(d^2/dx^2)S = (1+\sigma)S + gf^{-1}S. \qquad (3.1)$$

To do this we introduce new variables and new parameters by the formulae

$$m = \ell + 0.5,\ x = 2\rho(km)^{-1},\ \lambda = mk^{-1}. \qquad (3.2)$$

Then (1.4) takes the form

$$(d^2/dx^2)\Psi = (k^2\lambda f(\lambda,x) + g(\lambda,x))\Psi \qquad (3.3)$$

with

$$f(\lambda,x) = (\lambda x^2 - 4x + 4\lambda)(2x)^{-2}, \quad g(\lambda,x) = q\lambda^2k^4/4 - (2x)^{-2}. \qquad (3.4)$$

Next we carry out the change of variable $x \mapsto \xi(x)$ assuming that the inverse function is also uniquely defined. Differentiation with respect to ξ will be denoted by $\dot\Psi$ in contrast with the differentiation with respect to x, denoted by Ψ'. We also introduce a new function S instead of Ψ by the formula $S = \sqrt{\xi'}\Psi$. The following formulae can be easily verified:

$$\Psi'' = (\ddot\Psi\dot x - \dot\Psi\ddot x)(\dot x)^{-3}, \quad \ddot\Psi = \ddot S\sqrt{\dot x} + \dot S\ddot x(\dot x)^{-1/2} + S(4\dot x\sqrt{\dot x})^{-1}(2\dddot x\dot x - (\ddot x)^2). \qquad (3.5)$$

Setting $u^2 = k^2\lambda$ and inserting (3.5) into (3.3) we have

$$\ddot S = \{\dot x^2u^2f(x(\xi)) + 0.75(\ddot x/\dot x)^2 - 0.5\dddot x/\dot x\}S + (\dot x)^2gS. \qquad (3.6)$$

Now we choose $\xi(x)$ such that $(\dot x)^2f(x) = 1$, which gives $\xi(x) = \int_\alpha^x \sqrt{f(t)}\,dt$ where $\alpha = 2a/(km)$. (3.6) becomes

$$\ddot S = (u^2+\sigma)S + gf^{-1}S, \qquad (3.7)$$

where

$$\sigma = \frac{4f(x)f''(x) - 5(f'(x))^2}{16f^3(x)} = -\frac{1}{f^{3/4}}\frac{d^2}{dx^2}\Big(\frac{1}{f^{1/4}}\Big). \qquad (3.8)$$

The transformation from (3.3) to (3.7) cannot be carried out in domains containing zeros of f. In such case σ becomes infinite and the approximation fails at these points called transition points of (3.6). Now we show that all regions under consideration in this paper are free of transition points. The following statement is valid.

Lemma 3.1. *If $k \geq m^2$ and q is a function described in Theorem* 1.1, *then for $\rho \in [0, a]$ the function f (see* (3.4)) *is bounded from below; i.e.,* (3.7) *has no transition points.*

Proof. The function $f(x)$ given by (3.4) has two different roots $x_{1,2} = 2\lambda^{-1}(1 \pm \sqrt{1-\lambda^2})$. Formally, (3.7) has two transition points, but, in fact, we consider such values of x, that are less than $x_1 = 2\lambda^{-1}(1-\sqrt{1-\lambda^2})$. Indeed, for large m and $k \geq m^2$ we have $\lambda = mk^{-1} \to 0$ when $m \to \infty$ and, therefore, $x_1 \cong 2\lambda^{-1}(1-(1-0.5\lambda^2)) \cong \lambda$. Since $x_1 = 2\rho_1 k^{-1} m^{-1}$, then $\rho_1 \cong m^2$. It is clear that for $\rho \in [0, a]$ and large m: $\rho \leq \rho_1 - \delta, \delta > 0$. Therefore, if $\rho \in [0, a]$ and $k \geq m^2$, then $f(\lambda, x) \geq \gamma > 0$. The lemma is shown.

In the next statement we prove the existence and formulae for two linearly independent solutions of (3.7).

Theorem 3.1. *Equation* (3.7) *has two linearly independent solutions given by the following formulae:*

$$S_1(u, \lambda, x) = e^{-u\xi}(1 + h_1(u, \lambda, x)), \tag{3.9}$$

$$S_2(u, \lambda, x) = e^{u\xi}(1 + h_2(u, \lambda, x)), \tag{3.10}$$

where the functions $h_i(u, \lambda, x)$, $i = 1, 2$ are estimated by $|h_i(u, \lambda, x)| \leq \tilde{\mathcal{C}} m^{-1}$; the constant $\tilde{\mathcal{C}}$ does not depend on u or x when $k \in [m^2, \infty)$ and $x = 2\rho k^{-1} m^{-1}$, $\rho \in [0, a]$.

Proof. We seek for a solution in the form

$$S_1 = e^{-u\xi}(1 + h_1(u, \lambda, x)), \tag{3.11}$$

where the function $h_1(u, \lambda, x)$ satisfies the following integral equation:

$$h_1 = (2u)^{-1} \int_0^{\xi} [e^{2u(\xi - v)} - 1] \mathcal{X}(v)(1 + h_1)\, dv, \tag{3.12}$$

with $\mathcal{X} = \sigma + gf^{-1}$.

To proceed with the proof we need two statements which are formulated now and will be proved after the theorem.

Lemma 3.2. *The function $exp\{2u(\xi - v)\}$ is estimated by*

$$|e^{2u(\xi - v)}| \leq 1. \tag{3.13}$$

Lemma 3.3. *The following estimate holds:*

$$u^{-1} \int_0^{2a/(km)} |\mathcal{X}(\xi)|\, d\xi = O(m^{-1}), \tag{3.14}$$

where the estimate $O(m^{-1})$ is uniform with respect to $k \in [m^2, \infty)$.

Lemmas 3.2 and 3.3 guarantee the convergence of successive approximation series for (3.12). Indeed, let K be the Volterra integral operator with the kernel

$$\mathcal{K}(\xi, v) = (2u)^{-1}[e^{2u(\xi-v)} - 1]\mathcal{X}(v) \quad \text{and let} \quad h_0(\xi) = \int_0^\xi \mathcal{K}(\xi, v)\, dv;$$

then the Neumann series for (3.12) has the form $h_1(\lambda, \xi) = \sum_{n=0}^{\infty}(K^n h_0)(\xi)$, where

$$(K^n h_0)(\xi) = \int_0^\xi \mathcal{K}(\xi, v)\, dv \int_0^v \mathcal{K}(v, t)\, dt \cdots d\tau \int_0^\tau \mathcal{K}(\tau, w) h_0(w)\, dw.$$

Due to Lemmas 3.2 and 3.3 the latter term is dominated by

$$|(K^n h_0)(\xi)| \le \tilde{\mathcal{C}}^n m^{-(n+1)}[n!]^{-1},$$

which leads to the convergence of successive approximations and the estimate

$$|h_1(\lambda, \xi)| \le \tilde{\mathcal{C}} m^{-1}.$$

The approximation (3.10) can be shown analogously. The theorem is shown.

Now we prove both lemmas.

Proof of Lemma 3.2. By the definition of $\xi(x)$ we have

$$\xi - v = \int_{2a/(km)}^{2\rho/(km)} \sqrt{f(\tau)}\, d\tau - \int_{2a/(km)}^{2t/(km)} \sqrt{f(\tau)}\, d\tau.$$

Since $v \in [0, \xi]$ then t is necessarily greater than or equal to ρ. Thus,

$$\xi - v = -\int_{2\rho/(km)}^{2t/(km)} \sqrt{f(\tau)}\, d\tau \le 0.$$

The lemma is proved.

Proof of Lemma 3.3. The integral in (3.14) can be represented in the form

$$I = \int_{2a/(km)}^{0} |\mathcal{X}(\xi(x))\sqrt{f(x)}|\, dx.$$

First, we consider the integrand. We have (see (3.4), (3.8))

$$\begin{aligned} \mathcal{X}\sqrt{f(x)} = -\Big\{ &\frac{2\sqrt{x}}{(\lambda x^2 - 4x + 4\lambda)^{1/4}} \frac{d^2}{dx^2} \frac{\sqrt{x}}{(\lambda x^2 - 4x + 4\lambda)^{1/4}} \\ &+ \frac{1}{2x\sqrt{\lambda x^2 - 4x + 4\lambda}} - \frac{2xq\lambda^2 k^4}{(\lambda x^2 - 4x + 4\lambda)^{1/4}} \Big\}. \end{aligned} \tag{3.15}$$

Setting $\eta = (\lambda x^2 - 4x + 4\lambda)^{1/4}$ and $\tau = \sqrt{x}$ we rewrite (3.15) as

$$\mathcal{X}\sqrt{f(x)} = -\Big\{\frac{2\tau}{\eta}\frac{d^2}{dx^2}\Big(\frac{\tau}{\eta}\Big)+\frac{1}{2\tau^2\eta^2}-\frac{2xq\lambda^2k^4}{\eta^2}\Big\} = -\Big\{\frac{2\eta'}{\eta^3}+\frac{2x}{\eta}\Big(\frac{\eta'}{\eta^2}\Big)'-\frac{2xq\lambda^2k^4}{\eta^2}\Big\}. \quad (3.16)$$

Now we take advantage of the explicit form of η. If $x = 2\rho k^{-1}m^{-1}$, then $\lambda x^2 - 4x + 4\lambda = 4\lambda(1 - 2\rho m^{-2} + O(k^{-2}m^{-2}))$ for $\rho \in [0, a]$. For $\eta, \eta', \eta'/\eta^2$ the following asymptotic representations can be easily verified:

$$\begin{aligned}\eta &= (4\lambda)^{1/4}(1 - x\lambda^{-1} + 0.25x^2)^{1/4}\\ &= (4\lambda)^{1/4}(1 + ax\lambda^{-1} + bx^2\lambda^{-2} + O(x^3\lambda^{-3})) = C_1\lambda^{1/4}(1 + O(m^{-2})),\end{aligned} \quad (3.17)$$

$$\eta' = (4\lambda)^{1/4}(a\lambda^{-1} + 2bx\lambda^{-1} + O(x^2\lambda^{-3})) = C_2\lambda^{-3/4}(1 + O(m^{-1}k^{-1})), \quad (3.18)$$

$$\eta'\eta^{-3} = C_3\lambda^{-3/2}(1 + O(m^{-2})); \quad (\eta'\eta^{-2})' = C_4\lambda^{-9/4}(1 + O(m^{-2})). \quad (3.19)$$

Here C_i, $i = 1, 2, 3, 4$ are absolute constants, exact values of which are immaterial for us. Substituting (3.17)–(3.19) into (3.16) and integrating we get

$$\begin{aligned}u^{-1}\int_0^{2a/(km)} \mathcal{X}(\xi(x))\sqrt{f(x)}\,dx &= u^{-1}\int_0^{2a/(km)}[2C_3\lambda^{-3/2}(1 + O(m^{-2}))\\ &+ 2xC_1^{-1}\lambda^{-1/4}C_4\lambda^{-9/4}(1 + O(m^{-2})) - 2qx\lambda^2k^4C_1^{-2}\lambda^{-1/2}(+O(m^{-2}))]\,dx.\end{aligned} \quad (3.20)$$

Evaluating all integrals in (3.20) we obtain

$$1)\quad u^{-1}\int_0^{2ak^{-1}m^{-1}} \lambda^{-3/2}\,dx = (km)^{-1/2}(k/m)^{3/2}2a/(km) = 2am^{-3}, \quad (3.21)$$

$$2)\quad u^{-1}\int_0^{2ak^{-1}m^{-1}} x\lambda^{-5/2}\,dx = 0.5(km)^{-1/2}(k/m)^{5/2}(2a/(km))^2 = 2a^2m^{-5}, \quad (3.22)$$

$$3)\quad u^{-1}\int_0^{2ak^{-1}m^{-1}} xq\lambda^{3/2}k^4\,dx = 4m^{-1}\int_0^a \rho q(\rho)\,d\rho. \quad (3.23)$$

From (3.20)–(3.23) Lemma 3.3 follows immediately.

Proof of Theorem 2.1. First we show, that the LG approximation for the regular solution of (1.4) can be given by the formula

$$\Phi_\ell(k,\rho) = \sqrt{0.5}a^m m^{3/4}k^{1/4}f(x)^{-1/4}\exp\Big(k\sqrt{\lambda}\int_\alpha^x \sqrt{f(t)}\,dt\Big)(1 + O(m^{-1})\Big), \quad (3.24)$$

where the estimate $O(m^{-1})$ is uniform with respect to $k \in [m^2, \infty)$ and $\rho = 0.5kmx \in [0, a]$. To identify $\Phi_\ell(k,\rho)$ with the approximation (3.10) we have to show, that at the neighborhood of $\rho = 0$ the following fact holds: $\exp\{k\sqrt{\lambda}\int_\alpha^x \sqrt{f}(t)\,dt\} \sim \rho^m$. To do this we evaluate all integrals involving in (3.10). In what follows any function $f(x)$ considered at the point x will be denoted as f^x instead of $f(x)$. Letting $R^t = \lambda t^2 - 4t + 4\lambda$, we have (see [2])

$$\int_\alpha^x \sqrt{f^t}\,dt = \int_\alpha^x \sqrt{R^t}/t\,dt = \sqrt{R^t}\Big|_\alpha^x \tag{3.25}$$
$$= \frac{2}{\sqrt{\lambda}}\ell n\Big|\frac{\sqrt{\lambda R^x} + \lambda x - 2}{\sqrt{\lambda R^\alpha} + \lambda\alpha - 2}\Big| - 2\sqrt{\lambda}\ell n\Big|\frac{\sqrt{\lambda R^x}/x + 2\lambda/x - 1}{\sqrt{\lambda R^\alpha}/\alpha + 2\lambda/\alpha - 1}\Big|.$$

So, for the function in (3.10) we have the following approximation:

$$\exp\big(0.5k\sqrt{\lambda}\int_\alpha^x \sqrt{R^t}/t\,dt\big) \tag{3.26}$$
$$= \exp\big(0.5k\sqrt{\lambda}[\sqrt{R^x} - \sqrt{R^\alpha}]\big)\Big|\frac{\sqrt{\lambda R^x} + \lambda x - 2}{\sqrt{\lambda R^\alpha} + \lambda\alpha - 2}\Big|^{-k}\Big|\frac{\sqrt{\lambda R^x}/x + 2\lambda/x - 1}{\sqrt{\lambda R^\alpha}/\alpha + 2\lambda/\alpha - 1}\Big|^{-m}.$$

Now we study the function (3.26) for $\rho \in [0, a]$, large m and $k \in [m^2, \infty)$. By straightforward calculation we obtain, that $\sqrt{R^x} = 2\sqrt{m/k}(1 - \rho m^{-2} + O(\rho^2 m^{-4}))$, $\rho \in [0, a], k \geq m^2$. Based on the latter representation we derive

$$\exp\big(0.5k\sqrt{\lambda}[\sqrt{R^x} - \sqrt{R^\alpha}]\big) = e^{m((a-\rho)m^{-2} + O(m^{-4}))} = 1 + (a-\rho)m^{-1} + O(m^{-2}). \tag{3.27}$$

Then we have $\sqrt{\lambda R^x} = 2\lambda(1 - \rho m^{-2} + O(\rho^2 m^{-4}))$, which leads to $(\sqrt{\lambda R^x} + \lambda x - 2)(\sqrt{\lambda R^\alpha} + \lambda\alpha - 2)^{-1} = (1 - (a-\rho)k^{-1}m^{-1})(1 + O(k^{-2}) + O(k^{-1}m^{-3}))$. Therefore,

$$\big|(\sqrt{\lambda R^x} + \lambda x - 2)/(\sqrt{\lambda R^\alpha} + \lambda\alpha - 2)\big|^{-k} = 1 + O(m^{-1}). \tag{3.28}$$

Finally, we evaluate the last fraction from (3.26).

$$\frac{\sqrt{\lambda R^x}/x + 2\lambda/x - 1}{\sqrt{\lambda R^\alpha}/\alpha + 2\lambda/\alpha - 1} = \frac{1 - \rho^{-1}m^2 + O(\rho m^{-2})}{1 - a^{-1}m^2 + O(\rho m^{-2})} = \frac{a}{\rho}\big(1 - \frac{a-\rho}{m^2}\big)(1 + O(m^{-4})).$$

From the latter expression we have

$$|(\sqrt{\lambda R^x}/x + 2\lambda/x - 1)/(\sqrt{\lambda R^\alpha}/\alpha + 2\lambda/\alpha - 1)|^{-m} = \rho^m a^{-m}(1 + O(m^{-1})). \tag{3.29}$$

Collect together (3.27)–(3.29) for $\rho \in [0, a]$, $k \in [m^2, \infty)$; we get

$$\exp\big(0.5k\sqrt{\lambda}\int_\alpha^x \sqrt{f(t)}\,dt\big) = \rho^m a^{-m}(1 + O(m^{-1})). \tag{3.30}$$

For the function $f(x)^{-1/4}$ the following asymptotic representation is valid:

$$f(x)^{-1/4} = \sqrt{2x}(\lambda x^2 - 4x + 4\lambda)^{-1/4} = \sqrt{2\rho}\ k^{-1/4}m^{-3/4}(1 + O(k^{-2})). \tag{3.31}$$

Collecting (3.30) and (3.31) we can write for $\rho \in [0,a]$, $k \in [m^2,\infty)$

$$\Phi_\ell(k,\rho) = \sqrt{0.5}a^m m^{3/4}\exp\big(k\sqrt{\lambda}\int_\alpha^x \sqrt{f(t)}\,dt\big)(1+O(m^{-1})) = \rho^m(1+O(m^{-1})).$$

The theorem is completely shown.

Remark 3.1. As is shown, the regular solution of (1.4) can be approximated by one of the LG functions; namely, in terms of (3.10). But, we cannot guarantee that the singular solution $F_\ell(k,\rho)$ can be approximated by the complementary LG function (3.9). The reason for that consists of the following: to approximate the regular solution using LG approximation (3.10) it is sufficient to prove that the approximate function goes to zero as $\rho^{\ell+1}$ when $\rho \to 0$. While to identify the singular solution with the second LG function (3.9) it is insufficient to prove that the approximate function behaves at the vicinity of zero as $\rho^{-\ell}$. The characteristic property of the singular solution is an exponential decrease when $\rho \to \infty$. But we derive LG approximation (3.9) only for $\rho \in [0,a]$, using the technique, that does not involve the transition points. Certainly it is possible to derive an approximation for the singular solution $F_\ell(k,\rho)$ for $\rho \in (0,\infty)$. To do this we have to overcome substantial technical routine connected with the derivation of the approximations for the interval involving two distant turning points (our function $f(\lambda,x)$ (see (3.4)) has two different simple roots). We believe, that the aforementioned approximation related to two turning points, coincides with (3.9) for small x, but this requires nontrivial proof. Fortunately, to study the eigenvalue equation (2.19) we do not need LG approximation for the singular solution, though such approximation for $F_\ell(k,\rho)$ when $\rho \in (0,\infty)$ could be of interest in itself.

Proof of Lemma 2.2. Based on formulae (2.5) and (2.15) we obtain

$$W_{0\ell}(k) = -2mk^{2m}\Gamma(k-m+0.5)[\Gamma(m+k+0.5)]^{-1}. \tag{3.32}$$

Using the asymptotics (2.14) for $\Gamma(z)$ we have

$$\begin{aligned} W_{0\ell}(k) = {} & -2me^{2m}k^{2m}(k-m)^{k-m}(k+m)^{-(k+m)}\big(1+\frac{0.5}{k-m}\big)^{k-m} \\ & \times \big(1+\frac{0.5}{k+m}\big)^{-(k+m)}(1+O(k^{-1})). \end{aligned} \tag{3.33}$$

Based on the fact, that

$$\big(1+\frac{0.5}{k\pm m}\big)^{k\pm m} = e^{0.5}\big(1+O(|k\pm m|^{-1})\big) = e^{0.5}(1+O(k^{-1})),$$

we transfer (3.33) to the form

$$W_{0\ell}(k) = -2mk^{2m}e^{2m}((k-m)/(k+m))^k(k^2-m^2)^{-m}(1+O(k^{-1})). \tag{3.34}$$

Let $S_m(k) = \big[(1-\frac{m}{k})/(1+\frac{m}{k})\big]^k$ and $T_m(k) = (1-m^2/k^2)^{-m}$; then

$$W_{0\ell}(k) = -2me^{2m}S_m(k)T_m(k). \tag{3.35}$$

We estimate $S_m(k)$ and $T_m(k)$ when $k \geq m^2$. We have

$$\ell n S_m(k) = k\big[\ell n(1-m/k) - \ell n(1+m/k)\big] = -2m + O(m^3k^{-2}) = -2m(1+O(m^{-1})). \tag{3.36}$$

From (3.36) it follows that $S_m(k) = e^{-2m}(1+O(m^{-1}))$. By analogy with (3.36), we have

$$\ell n T_m(k) = -m\,\ell n(1-m^2/k^2) = m(m^2/k^2 + O(m^4/k^4)) = m^3k^{-2} + O(m^5/k^4). \tag{3.37}$$

From (3.37) it follows that $T_m(k) = 1 + O(m^{-1})$. Substituting the approximations for $S_m(k)$ and $T_m(k)$ into (3.35) we have

$$W_{0\ell}(k) = -2me^{2m}e^{-2m}(1+O(m^{-1})) = -2m(1+O(m^{-1})).$$

Lemma 2.2 is completely shown.

4. Almost normalization of eigenfunctions. Proof of Theorem 1.2. In this section we need one more singular solution of (1.4), which increases as $\rho \to \infty$. This solution should be proportional to the function $F_{0\ell}(-k,\rho)$ for $\rho \geq a$ (for $F_{0\ell}(k,\rho)$ see (2.3)). It is easily verified that the second singular solution also satisfies a certain Volterra integral equation. From now on this second singular solution is denoted by $F_\ell^-(k,\rho)$, while the singular solution considered in the previous sections is renamed by $F_\ell^+(k,\rho)$. $F_\ell^+(k,\rho)$ satisfies (2.7) and $F_\ell^-(k,\rho)$ satisfies the following one:

$$F_\ell^-(k,\rho) = F_{0\ell}(-k,\rho) - W_{0\ell}(k)^{-1}\int_\rho^\infty \mathfrak{K}(k,\rho,t)q(t)F_\ell^-(k,t)\,dt, \tag{4.1}$$

where $W_{0\ell}(k)$ is given by (2.5).

Equation (4.1) has the same kernel $\mathfrak{K}(k,\rho,t)$ as (2.7). As is shown in [2] (4.1) can be solved by the method of successive approximations. Since the solutions $F_\ell^+(k,\rho)$ and $F_\ell^-(k,\rho)$ form a fundamental system of solutions of (1.4), the regular solution $\Phi_\ell(k,\rho)$ is represented as a linear combination

$$\Phi_\ell(k,\rho) = \mathcal{A}(k)F_\ell^+(k,\rho) + \mathcal{B}(k)F_\ell^-(k,\rho). \tag{4.2}$$

If $\{\varphi,\psi\}$ is the Wronskian of two functions φ and ψ, then $\mathcal{A}(k)$ and $\mathcal{B}(k)$ are given by the formulae

$$\mathcal{A}(k) = \frac{\{\Phi_\ell(k,\rho), F_\ell^-(k,\rho)\}}{\{F_\ell^+(k,\rho), F_\ell^-(k,\rho)\}}, \quad \mathcal{B}(k) = \frac{\{\Phi_\ell(k,\rho), F_\ell^+(k,\rho)\}}{\{F_\ell^-(k,\rho), F_\ell^+(k,\rho)\}}. \tag{4.3}$$

In what follows the $L^2(0,\infty)$-norm of any function ψ will be denoted by $\|\psi\|$. The goal of this section is to prove a two-sided estimate for $\|\Phi_\ell(k_n^\ell,\cdot)\|$. When $k=k_n^\ell$ the regular $\Phi_\ell(k,\rho)$ and singular $F_\ell^+(k,\rho)$ solutions become proportional and $\mathcal{B}(k_n^\ell)=0$. So, the relation (4.2) takes the form

$$\Phi_\ell(k_n^\ell,\rho)=\mathcal{A}(k_n^\ell)F_\ell^+(k_n^\ell,\rho). \tag{4.4}$$

The proof of Theorem 1.2 consists of the proofs of three lemmas. Lemma 4.1 deals with the estimate for the sequence $\mathcal{A}(k_n^\ell)$. Lemmas 4.2 and 4.3 provide the estimates for $\|F_\ell^+(k_n^\ell,\cdot)\chi_{(a,\infty)}\|$ from below and from above respectively. Here $\chi_{(\alpha,\beta)}$ is the characteristic function of the interval (α,β).

The next statement describes the behavior of the function $\mathcal{A}(k)$ in (4.3) at the points $\{k_n^\ell\}_{n=1}^\infty$. (We recall, that the spectrum of the problem, defined by (1.4), (1.5), is given in the form: $\lambda_n=-(k_n^\ell)^{-2}$.) As in the rest of this section we do not need the uniformity with respect to an angular momentum ℓ of all the estimates, we omit the superscript ℓ in the notation for the eigenvalues. From now on we use k_n instead of k_n^ℓ.

Lemma 4.1. *If μ_n^ℓ is the quantum defect, then the following estimates take place:*

$$\left|\sin(\pi\mu_n^\ell)\left\{F_\ell^+(k_n,\rho),F_\ell^-(k_n,\rho)\right\}\right|\asymp\mathcal{C}_\ell, \tag{4.5}$$
$$\left|\sin(\pi\mu_n^\ell)\left\{\Phi_\ell(k_n,\rho),F_\ell^-(k_n,\rho)\right\}\right|\asymp\mathcal{D}_\ell, \tag{4.6}$$

where the constants $\mathcal{C}_\ell$ and $\mathcal{D}_\ell$ may grow with ℓ. (For the notation $\asymp$ see the remark after (1.11).*) It follows from* (4.5), (4.6) *that $|\mathcal{A}(k_n^\ell)|\asymp\mathcal{P}_\ell$, where $\mathcal{P}_\ell$ is a constant.*

Proof. Since the Wronskian of the solutions of (1.4) does not depend on ρ, it can be evaluated at any $\rho\in[0,\infty)$; in particular, at $\rho>a$.

$$\left\{F_\ell^-(k,\rho),F_\ell^+(k,\rho)\right\}=\left\{F_\ell^-(k,\rho),F_\ell^+(k,\rho)\right\}_{\rho>a}=\left\{F_{0\ell}(-k,\rho),F_{0\ell}(k,\rho)\right\}. \tag{4.7}$$

Due to (2.3), (2.4) we can represent (4.7) in the form

$$\begin{aligned}&\left\{F_{0\ell}(-k,\rho),F_{0\ell}(k,\rho)\right\}\\&=2^{2\ell}k^{2\ell+2}(-1)^{\ell+1}\Gamma(-\ell-k)\Gamma(-\ell+k)\left\{W_{-k,\ell+1/2}(-2\rho/k),W_{k,\ell+1/2}(2\rho/k)\right\}.\end{aligned} \tag{4.8}$$

Due to (2.4) and (4.8) we have the following formulae for the Wronskian:

$$\begin{aligned}&\left\{F_{0\ell}(-k,\rho),F_{0\ell}(k,\rho)\right\}=\\&\tilde{\mathcal{C}}_\ell(k)\rho^{2\ell+2}\big[2k^{-1}U(\ell+1+k,2\ell+2,-2\rho/k)U(\ell+1-k,2\ell+2,2\rho/k)\\&\quad+\left\{U(\ell+1+k,2\ell+2,-2\rho/k),U(\ell+1-k,2\ell+2,2\rho/k)\right\}\big],\end{aligned} \tag{4.9}$$

where

$$\tilde{\mathcal{C}}_\ell(k)=-2^{4\ell+1}\Gamma(-\ell-k)\Gamma(-\ell+k)\big[(2\ell+1)!\big]^{-2}. \tag{4.10}$$

Setting $a = \ell + 1 - k$ and $b = 2\ell + 2$ and using the following relation for the Kummer functions (see [11], [14]):

$$\{U(a,b,z), e^z U(b-a,b,-z)\} = e^{-i\pi(b-a)} z^{-b} e^z,$$

we obtain

$$\begin{aligned}\{U(\ell+1+k, 2\ell+2, z), U(\ell+1-k, 2\ell+2, -z)\} &= 2k^{-1}\big[e^{-i\pi(\ell+k+1)} z^{-2\ell-2} \\ &- U(\ell+1+k, 2\ell+2, -2\rho/k) U(\ell+1-k, 2\ell+2, 2\rho/k)\big]. \quad (4.11)\end{aligned}$$

In (4.11) the differentiation with respect to ρ is replaced by the differentiation with respect to $z = 2\rho/k$. Substituting (4.11) into (4.9) we have

$$\{F_{0\ell}(-k,\rho), F_{0\ell}(k,\rho)\} = 2^{2\ell+1} k^{2\ell+1} e^{-i\pi(k+1)} \big[(2\ell+1)!\big]^{-2} \Gamma(-\ell-k)\Gamma(-\ell+k). \quad (4.12)$$

Substituting k_n for k in (4.9) and using the fact that $\Gamma(z)\Gamma(1-z) = csc(\pi z)$ and $\Gamma(z+1) = z\Gamma(z)$, we obtain

$$\begin{aligned}\big|k_n^{2\ell+1}\Gamma(-\ell-k_n)\Gamma(-\ell+k_n)\big| &= \big|csc(\pi k_n) k_n^{2\ell+1}\Gamma(-\ell+k_n)/\Gamma(\ell+1+k_n)\big| \\ &= \big|csc(\pi k_n)\Gamma(-\ell+k_n)/(k_n-\ell)_{2\ell+1}\big|,\end{aligned} \quad (4.13)$$

where $(k_n-\ell)_{2\ell+1} = (k_n-\ell)(k_n-\ell+1)(k_n-\ell+2)\cdots(k_n+\ell)$, which implies (4.5) immediately.

To prove (4.6) we need an estimate for the solution of the integral equation (4.1). In [14] the solvability and an estimate for the solution were obtained for the Volterra integral equation with the same integral operator but with another free term. It is not necessary to repeat step by step the proof of an analogous result from [14], so we formulate the final result. Equation (4.1) can be solved by successive approximations and its solution satisfies the following estimate for $\rho \in (0,a]$:

$$\begin{aligned}&\Big|\frac{\rho^\ell}{1+\rho^\ell}\big[F_\ell^-(k,\rho) - F_{0\ell}(-k,\rho) - \sum_{n=1}^{\infty}(K_0^n F_{0\ell})(-k,\rho)\big]\Big| \quad (4.14) \\ &\le \gamma_2(k)\int_\rho^\infty tq(t)\,dt\, \exp\{(\gamma_1(k)+\gamma_2(k))\int_\rho^\infty tq(t)\,dt\} \max_{\rho\in[0,a]}\Big|\frac{\rho^\ell}{1+\rho^\ell}F_{0\ell}(-k,\rho)\Big|,\end{aligned}$$

where $\gamma_i(k) = \mathcal{C}_0|k|^{-1}|\cot(\pi k)|^{i-1}$, $i = 1,2$, $\mathcal{C}_0$ is an absolute constant. K_0 is an integral operator:

$$(K_0\varphi)(\rho) = \frac{1}{2\ell+1}\int_\rho^\infty \mathfrak{K}_0(\rho,t)q(t)\varphi(t)\,dt,$$

whose kernel is a continuous bounded function for $\rho, t \in [0,a]$. (An explicit expression for it may be seen in [14]).

The following representation for $F_{0\ell}(-k,\rho)$ can be easily derived from (2.3), (2.4):

$$F_{0\ell}(-k,\rho) = e^{\rho/k}\tilde{F}_0(\rho) + O(\rho^{-\ell}/k), \tag{4.15}$$

where $\tilde{F}_0(\rho) \sim \rho^{-\ell}$ when $\rho \to 0$ and the estimate $O(\rho^{-\ell}/k)$ is uniform with respect to ρ from any compact vicinity of zero.

Based on (4.14), (4.15) and (4.1) we immediately obtain the asymptotic representation for $F_\ell^-(k,\rho)$

$$F_\ell^-(k,\rho) = e^{\rho/k}\tilde{F}_\ell(\rho) + \hat{\mathcal{C}}_1 e^{\rho/k}\cot(\pi k)[\hat{F}_\ell(\rho) + O(\rho^{\ell+1}/k)] + O(\rho^{-\ell}/k), \tag{4.16}$$

with $\tilde{F}_\ell(\rho) \cong \rho^{-\ell}$ when $\rho \to 0$.

Using the integral equations (2.6) and (4.1) for the solutions Φ_ℓ and F_ℓ^- we can derive the following formula for the Wronskian of these functions (for the derivation of a similar formula see [14]):

$$\big\{\Phi_\ell(k,\rho), F_\ell^-(k,\rho)\big\} = \lim_{\rho\to 0}\frac{\ell+1}{\rho}\Phi_{0\ell}(k,\rho)F_\ell^-(k,\rho).$$

Recall that $\lim_{\rho\to 0}\Phi_{0\ell}(k,\rho)\rho^{-\ell-1} = 1$. And using (4.16) we have

$$\big\{\Phi_\ell(k_n,\rho), F_\ell^-(k_n,\rho)\big\} = (\ell+1)\big[\lim_{\rho\to 0}\rho^\ell\tilde{F}_\ell(\rho) + \hat{\mathcal{C}}_1\cot(\pi k_n)\lim_{\rho\to 0}\rho^\ell\hat{F}_\ell(\rho)\big] + O(k_n^{-1}). \tag{4.17}$$

The estimate (4.6) follows from (4.17). The lemma is completely shown.

Remark 4.1. We mention that the following estimate holds:

$$\|\Phi_\ell(k_n,\cdot)\chi_{(0,a)}\| \le \tilde{\mathcal{C}}_\ell. \tag{4.18}$$

Indeed, as can be easily derived (see [14]), the solution of (2.6) obtained by the method of successive approximations is represented in the form

$$\Phi_\ell(k_n,\rho) = \Phi_{0\ell}(k_n,\rho) + \sum_{\nu=1}^{\infty}(K_1^\nu\Phi_{0\ell})(k_n,\rho) + \Phi_{1\ell}(k_n,\rho),$$

where $\Phi_{1\ell}(k_n,\rho)$ is estimated by

$$|\Phi_{1\ell}(k_n,\rho)| \le$$

$$\mathcal{C}_1 k_n^{-1}\rho^{\ell+1}(1+\rho^{\ell+1})^{-1}\int_0^\rho tq\,dt\exp(\mathcal{C}_1\int_0^\rho tq\,dt)\max_{\rho\in[0,a]}\big|\big(1+\rho^{\ell+1}\big)\rho^{-\ell-1}\Phi_{0\ell}(k_n,\rho)\big|,$$

and K_1 is a Volterra integral operator $(K_1\varphi)(\rho) = \int_0^\rho \mathfrak{R}_1(\rho,t)q(t)\varphi(t)\,dt$ with a bounded kernel $|\mathfrak{R}_1(\rho,t)q(t)| \le \mathcal{C}_4$. This fact gives us

$$\big\|\sum_{\nu=1}^{\infty}(K_1\Phi_{0\ell})(k_n,\cdot)\chi_{(0,a)}\big\| \le \mathcal{C}_2\|\Phi_{0\ell}(k_n,\cdot)\chi_{(0,a)}\|,$$

and, therefore, we have $\|\Phi_\ell(k_n,\cdot)\chi_{(0,a)}\| \le \mathcal{C}_3\|\Phi_{0\ell}(k_n,\cdot)\chi_{(0,a)}\|$. Now we take into account that when $\rho \in [0,a]$ for the regular hypergeometric function $M(\ell+1-k, 2\ell+2, 2\rho/k)$ the following representation is valid: $M(\ell+1-k, 2\ell+2, 2\rho/k) = \Gamma(2\ell+2)(2\rho)^{-\ell-0.5}\mathcal{J}_{2\ell+1}(2\sqrt{2\rho}) + O(k^{-1})$, where $\mathcal{J}_{2\ell+1}(z)$ is the Bessel function. From the latter formula through (2.2) we arrive at (4.18). Thus, we have to prove Theorem 1.2 only on the interval (a,∞).

Our next statement is

Lemma 4.2. *For each ℓ there exists a constant $\mathcal{N}_\ell$ such that*

$$\|F_\ell^+(k_n,\cdot)\chi_{(a,\infty)}\| \geq \mathcal{N}_\ell\; n^{3/2}, \tag{4.19}$$

where $\mathcal{N}_\ell$ may grow with ℓ.

Proof. If $\mathring{\Phi}_n(\rho)$ is normalized-to-unity n^{th} eigenfunction of the unperturbed operator, then it satisfies (1.3) with $k = n$. Using (1.4) for $F_\ell^+(k_n,\rho)$ and (1.3) for $\mathring{\Phi}_n(\rho)$ we obtain

$$\int_a^\infty \big[\big(F_\ell^+(k_n,\rho)\big)'\mathring{\Phi}_n(\rho) - F_\ell^+(k_n,\rho)(\mathring{\Phi}_n(\rho))''\big]\,d\rho = \frac{n^2-k_n^2}{n^2k_n^2}\int_a^\infty F_\ell^+(k_n,\rho)\mathring{\Phi}_n(\rho)\,d\rho. \tag{4.20}$$

Due to the asymptotics for k_n, $n\to\infty$ and $\|\mathring{\Phi}_n(\cdot)\| = 1$, (4.20) leads to the following inequality:

$$\big|\big(F_\ell^+(k_n,\rho)\big)'\mathring{\Phi}_n(\rho) - F_\ell^+(k_n,\rho)(\mathring{\Phi}_n(\rho))'\big|_{\rho=a} \leq \frac{\tilde{\mathcal{C}}}{n^3}\|F_\ell^+(k_n,\cdot)\chi_{(a,\infty)}\|. \tag{4.21}$$

Our next step is to show that

$$\big|(F_\ell^+(k_n,\rho))'\mathring{\Phi}_n(\rho) - F_\ell^+(k_n,\rho)(\mathring{\Phi}_n(\rho))'\big|_{\rho=a} \asymp n^{-3/2}. \tag{4.22}$$

From (4.21) and (4.22) the estimate (4.19) will follow immediately. Let us prove (4.22). First of all, we show that

$$\big|(F_\ell^+(k_n,\rho))'\mathring{\Phi}_n(\rho) - F_\ell^+(k_n,\rho)(\mathring{\Phi}_n(\rho))'\big|_{\rho=a} \asymp n^{-3/2}\big|\{F_{0\ell}(k,\rho),\Phi_{0\ell}(k,\rho)\}_{k=k_n}\big|. \tag{4.23}$$

Due to (2.5)

$$\big\{F_{0\ell}(k,\rho),\Phi_{0\ell}(k,\rho)\big\}_{k=k_n} = \frac{-(2\ell+1)n^{2\ell+1}\Gamma(-\ell-k_n)}{\Gamma(\ell+1-k_n)} \to (2\ell+1), \quad n\to\infty. \tag{4.24}$$

From (4.24) through (4.23) we achieve (4.22). Therefore, the proof of (4.22) is reduced to the proof of (4.23). We mention, that the normalized eigenfunction $\mathring{\Phi}_n(\rho)$ is connected with the function $\Phi_{0\ell}(k,\rho)$, at $k = n$ by the formula

$$\mathring{\Phi}_n(\rho) = \frac{2^{\ell+1}}{(2\ell+1)!n^{\ell+2}}\sqrt{\frac{(n+\ell)!}{(n-\ell-1)!}}\Phi_{0\ell}(n,\rho) \equiv \mathcal{G}_{n\ell}\Phi_{0\ell}(n,\rho). \tag{4.25}$$

Due to the fact that $F_\ell^+(k_n,\rho) = F_{0\ell}(k_n,\rho)$ for $\rho \geq a$ and (4.23) we have

$$\begin{aligned}
&\big[F_\ell^+(k_n,\rho)'\mathring{\Phi}_n(\rho) - F_\ell^+(k_n,\rho)(\mathring{\Phi}_n(\rho))'\big]_{\rho=a}\\
&= \mathcal{G}_{n\ell}\big[F_{0\ell}'(k_n,\rho)\Phi_{0\ell}(n,\rho) - F_{0\ell}(k_n,\rho)\Phi_{0\ell}'(n,\rho)\big]_{\rho=a}\\
&= \mathcal{G}_{n\ell}\big[S_1 + S_2 + \{F_{0\ell}(k_n,\rho),\Phi_{0\ell}(k_n,\rho)\}\big],
\end{aligned}$$

where

$$S_1 = \{F'_{0\ell}(k_n,\rho), [\Phi_{0\ell}(n,\rho) - \Phi_{0\ell}(k_n,\rho)]\}_{\rho=a},$$
$$S_2 = \{F_{0\ell}(k_n,\rho), [\Phi'_{0\ell}(k_n,\rho) - \Phi'_{0\ell}(n,\rho)]\}_{\rho=a}. \tag{4.26}$$

In order to estimate S_1 and S_2 we use the following relations for the Whittaker and Kummer functions (see [3, 16]):

a) $$\lim_{k\to\infty} \Gamma(-\ell-k)(2\rho/k)^{-\ell-1} W_{k,\ell+1/2}(2\rho/k) = 2(2\rho)^{-\ell-1/2} K_{2\ell+1}(2\sqrt{2\rho}), \tag{4.27}$$

where $K_{2\ell+1}(z)$ is the Macdonald function;

b) $$U(a,b+1,z) = U(a,b,z) - U'(a,b,z), \tag{4.28}$$

c) $$(b-a)M(a,b+1,z) = bM(a,b,z) - bM'(a,b,z). \tag{4.29}$$

Due to b) and c) we obtain

$$F'_{0\ell}(k,\rho) = \frac{\ell+1}{\rho} F_{0\ell}(k,\rho) + \frac{1}{k} F_{0\ell}(k,\rho) - \frac{2}{k} F_{0\ell}(k,\rho) \frac{U(\ell+1-k, 2\ell+3, 2\rho/k)}{U(\ell+1-k, 2\ell+2, 2\rho/k)}, \tag{4.30}$$

$$\begin{aligned}\Phi'_{0\ell}(k,\rho) = &\frac{\ell+1}{\rho}\Phi_{0\ell}(k,\rho) + \frac{1}{k}\Phi_{0\ell}(k,\rho)\\ &- \frac{1}{k}\Phi_{0\ell}(k,\rho)\frac{\ell+1+k}{\ell+1}\frac{M(\ell+1-k,2\ell+3,2\rho/k)}{M(\ell+1-k,2\ell+2,2\rho/k)}.\end{aligned} \tag{4.31}$$

Passing to the limit in (2.2) with respect to k, we obtain that $M(\ell+1-k, 2\ell+2+j, 2a/k) = \mathcal{M}_j(1+O(k^{-1}))$, $j=0,1$, where positive constants $\mathcal{M}_j$ are given by the formulae: $\mathcal{M}_j = \sum_{n=0}^{\infty} \frac{a^n}{(2\ell+2+j)_n n!}$, $j=0,1$. Thus, we have from (4.31)

$$\Phi'_{0\ell}(k_n,a) - \Phi'_{0\ell}(n,a) = (\ell+1)a^\ell [\Phi_{0\ell}(k_n,a) - \Phi_{0\ell}(n,a)] + O(n^{-1}). \tag{4.32}$$

In turn, we also have

$$\begin{aligned}\Phi_{0\ell}(k_n,a) - \Phi_{0\ell}(n,a) &= a^{\ell+1}e^{-a/k_n}(\mathcal{M}_0 + O(k_n^{-1})) - a^{\ell+1}e^{-a/n}(\mathcal{M}_0 + O(n^{-1}))\\ &= O(n^{-1}).\end{aligned} \tag{4.33}$$

Using the asymptotics for $\{k_n\}$, (4.32), (4.33) and relation (4.27) for the Whittaker function, we obtain that $\lim_{n\to\infty} nS_j(n) < \infty$, $j=1,2$. From the latter fact we reduce (4.23) to the form

$$\left|(F_\ell^+(k_n,\rho))'\mathring{\Phi}_n(\rho) - F_\ell^+(k_n,\rho)(\mathring{\Phi}_n(\rho))'\right|_{\rho=a} \asymp \mathcal{G}_{n\ell}\left|\{F_{0\ell}(k_n,\rho), \Phi_{0\ell}(k_n,\rho)\}\right|_{\rho=a},$$

where $\mathcal{G}_{n\ell} = 2^{\ell+1}n^{-\ell-2}[(2\ell+1)!]^{-1}\sqrt{(n-\ell)(n-\ell+1)(n-\ell+2)\cdots(n+\ell)} \asymp n^{-3/2}$. The lemma is proved.

Lemma 4.3. *For each $\ell \geq 0$ there exists a constant $\tilde{\mathcal{N}}_\ell$, such that*

$$\|F_{0\ell}(k_n,\cdot)\chi_{(0,a)}\| \leq \tilde{\mathcal{N}}_\ell n^{3/2}, \tag{4.34}$$

where $\tilde{\mathcal{N}}_\ell$ may grow with ℓ.

Proof. In this proof we use another notation for the solutions $\Phi_{0\ell}(k,\rho)$ and $F_{0\ell}(k,\rho)$; namely, we shall write $\Phi_0(\ell,k,\rho)$ and $F_0(\ell,k,\rho)$ respectively. First we show, that if (4.34) takes place for some definite angular momentum ℓ_0, then this result holds for any other ℓ. More precisely, we prove that

$$\|F_0(\ell,k,\cdot)\chi_{(a,\infty)}\| \asymp \|F_0(\ell_0,k,\cdot)\chi_{(a,\infty)}\|. \tag{4.35}$$

Equation (1.3) for the function $F_0(\ell,k,\rho)$ can be rewritten in the form

$$\psi'' - \frac{\ell_0(\ell_0+1)}{\rho^2}\psi + \frac{2}{\rho}\psi - \frac{1}{k^2}\psi = \frac{(\ell-\ell_0)(\ell+\ell_0+1)}{\rho^2}\psi, \tag{4.36}$$

Equation (4.36) is equivalent to the following Volterra integral equation:

$$\psi(k,\rho) = F_0(\ell_0,k,\rho) - W_{0\ell_0}(k)^{-1}\int_\rho^\infty \mathfrak{K}_{\ell_0}(k,\rho,t)(\ell-\ell_0)(\ell+\ell_0+1)t^{-2}\psi(k,t)\,dt, \tag{4.37}$$

where the kernel $\mathfrak{K}_{\ell_0}(k,\rho,t)$ is given by the formula

$$\mathfrak{K}_{\ell_0}(k,\rho,t) = \Phi_0(\ell_0,k,\rho)F_0(\ell_0,k,t) - \Phi_0(\ell_0,k,t)F_0(\ell_0,k,\rho). \tag{4.38}$$

To apply the method of successive approximations to this equation we have to estimate its kernel. The following formula holds (see [16]):

$$\lim_{k\to\infty} M(\ell_0+1-k, 2\ell+2, 2\rho/k) = \Gamma(2\ell_0+2)(2\rho)^{-\ell_0-0.5}\mathcal{J}_{2\ell_0+1}(2\sqrt{2\rho}), \tag{4.39}$$

where $\mathcal{J}_{2\ell_0+1}(z)$ is the Bessel function. Using (2.1), (4.39) and asymptotics for the Bessel function when $\rho \to \infty$, we obtain

$$|\Phi_0(\ell_0,k,\rho)| = |\rho^{\ell_0+1}e^{-\rho/k}M(\ell_0+1-k, 2\ell_0+2, 2\rho/k)| \leq \mathcal{B}e^{-\rho/k}\rho^{1/4}. \tag{4.40}$$

Using (4.27) and the asymptotics for the Macdonald function $K_{2\ell+1}(z)$, we obtain

$$|F_0(\ell_0,k,\rho)| \leq \mathcal{C}_1|\sqrt{\rho}K_{2\ell_0+1}(2\sqrt{2\rho})| \leq \mathcal{C}_2 e^{-2\sqrt{2\rho}}\rho^{1/4}, \quad \rho \geq a. \tag{4.41}$$

Collect together (4.40) and (4.41) and we obtain the estimate for the kernel

$$|\mathfrak{K}_{\ell_0}(k,\rho,t)| \leq \mathcal{D}(\rho t)^{1/4}e^{-\rho/k-2\sqrt{2\rho}} \leq \tilde{\mathcal{D}}e^{-\rho/k}t^{1/4}. \tag{4.42}$$

Equation (4.37) can be solved by successive approximations and the solution can be represented in the form

$$F_0(\ell,k,\rho)=F_0(\ell_0,k,\rho)+\sum_{n=1}^{\infty}(-1)^n(\ell-\ell_0)^n(\ell+\ell_0+1)^n W_{0\ell_0}(k)^{-n}K_{\ell_0}^n(F_0(\ell_0,k,\rho)),$$

where K_{ℓ_0} is an integral Volterra operator of the form

$$K_{\ell_0}\cdot=\int_\rho^\infty \mathfrak{K}_{\ell_0}(k,\rho,t)t^{-2}\cdot\, dt.$$

The common term of the series is dominated by

$$\begin{aligned}&|(K_{\ell_0}^nF_0)(\ell_0,k,\rho)|=\big|\int_\rho^\infty \mathfrak{K}_{\ell_0}(k,\rho,t)t^{-2}\,dt\int_t^\infty \mathfrak{K}_{\ell_0}(k,t,\tau)\tau^{-2}\,d\tau\\ &\times\int_\tau^\infty \mathfrak{K}_{\ell_0}(k,\tau,w)w^{-2}\,dw\cdots\int_\xi^\infty \mathfrak{K}_{\ell_0}(k,\xi,\eta)F_0(\ell_0,k,\eta)\eta^{-2}d\eta\big|\\ &\le \tilde{\mathcal{D}}^n e^{-\rho/k}\int_\rho^\infty e^{-t/k}t^{-7/4}dt\int_t^\infty e^{-\tau/k}\tau^{-7/4}d\tau\cdots e^{-\xi/k}\,d\xi\int_\xi^\infty \eta^{-7/4}|F_0(\ell_0,k,\eta)|d\eta\\ &\le\frac{\mathcal{C}_1\tilde{\mathcal{D}}^n e^{-2\rho/k}}{(n-1)!}\big(\int_\rho^\infty e^{-t/k}t^{-7/4}\,dt\big)^{n-1}\rho^{-5/4}\|F_0(\ell_0,k,\cdot)\chi_{(a,\infty)}\|.\end{aligned}$$

Based on the latter representation we can write

$$\begin{aligned}\|F_0(\ell,k,\cdot)\chi_{(a,\infty)}\|\le\|F_0(\ell_0,k,\cdot)\chi_{(a,\infty)}\|\big[1+\mathcal{C}_1\big\|e^{-2\rho/k}\rho^{-5/4}\\ \times\sum_{n=1}^{\infty}\tilde{\mathcal{D}}^n\frac{1}{(n-1)!}\big(\int_\rho^\infty e^{-t/k}t^{-7/4}dt\big)^{n-1}\chi_{(a,\infty)}\big\|\big].\end{aligned}\tag{4.43}$$

Evaluating the integral in (4.43) we have

$$\begin{aligned}\int_\rho^\infty e^{-t/k}t^{-7/4}\,dt&=k^{-3/4}\int_{\rho/k}^\infty e^{-z}z^{-7/4}dz\\ &=k^{-3/4}\big[-\frac{4}{3}e^{-z}z^{-3/4}\big|_{\rho/k}^\infty+\frac{4}{3}\int_{\rho/k}^\infty e^{-z}z^{-3/4}dz\big]\\ &\le\frac{4}{3}\big(e^{-\rho/k}\rho^{-3/4}+k^{-3/4}\Gamma(1/4)\big)\le\tilde{\mathcal{C}}.\end{aligned}\tag{4.44}$$

Inserting (4.44) in (4.43) we obtain the dominant for $\|F_0(\ell,k,\cdot)\chi_{(a,\infty)}\|$ of the following form:

$$\|F_0(\ell,k,\cdot)\chi_{(a,\infty)}\|\le\|F_0(\ell_0,k,\cdot)\chi_{(a,\infty)}\|\big[1+\tilde{\mathcal{C}}\|e^{-2\rho/k}\rho^{-5/4}\chi_{(a,\infty)}\|\big].\tag{4.45}$$

Estimating the norm, we have

$$\|e^{-2\rho/k}\rho^{-5/4}\chi_{(a,\infty)}\|^2 = \int_a^\infty e^{-4t/k}t^{-5/2}\,dt = \mathcal{C}_3 k^{-3/2}\int_{4a/k}^\infty e^{-z}z^{-5/2}dz$$
$$\le \tilde{\mathcal{C}}_3 k^{-3/2}e^{-z}z^{-3/2}\big|_{4a/k}^\infty + \mathcal{C}_4 k^{-3/2}\int_0^\infty e^{-z}z^{-3/2}dz$$
$$= \mathcal{C}_5 + \mathcal{C}_4 k^{-3/2}\Gamma(-0.5) < \infty.$$

Due to the latter estimate we arrive at the following

$$\|F_0(\ell,k,\cdot)\chi_{(a,\infty)}\| \le D\|F_0(\ell_0,k,\cdot)\chi_{(a,\infty)}\|.$$

Since ℓ and ℓ_0 are equivalent we have the relation (4.35) proved. To complete the proof of the lemma we choose $\ell_0 = 0$ and estimate $\|F_0(0,k,\cdot)\chi_{(a,\infty)}\|$. Due to the explicit formula for $F_0(0,k,\rho)$ we have the representation

$$F_0(0,k,\rho) = e^{-\rho/k}\big[\rho k(1+\rho^{-1})\Gamma(-k)/\Gamma(1-k) - M(1-k,2,2\rho/k)\ell n(2\rho/k)$$
$$-\sum_{n=1}^\infty (2\rho/k)^n(1-k)_n[(n+1)!n!]^{-1}[\psi(1+n-k)-\psi(1+n)-\psi(2+n)],$$

where $\psi(z)$ is the digamma function. Since $\lim_{k\to\infty} k\Gamma(-k)/\Gamma(1-k) = 1$ and $|\lim_{k\to\infty}\{M(1-k,2,2\rho/k)\ell n(2\rho)\ +\sum_{n=1}^\infty(1-k)_n[(n+1)!n!]^{-1}[\psi(1+n-k)-\psi(1+n)-\psi(2+n)-\ell nk]\}| \le \mathcal{G}_0$, where $\mathcal{G}_0$ is a constant, we have to estimate only the norm of the function $e^{-\rho/k}\rho\chi_{(a,\infty)}$. We have

$$\|e^{-\rho/k}\rho\chi_{(a,\infty)}\|^2 \le \int_0^\infty e^{-2\rho/k}\rho^2\,d\rho = k^3/8\int_0^\infty e^{-z}z^2dz = (1/4)k^3.$$

Due to this estimate we have (4.34). Lemma 4.3 is proved.

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