Differential and Integral Equations, Volume 8, Number 6, July 1995, pp. 1385 – 1394.

# GLOBAL EXISTENCE OF SUBMANIFOLDS OF SOLUTIONS OF NONLINEAR SECOND ORDER DIFFERENTIAL SYSTEMS

Giancarlo Cantarelli[1]
Dipartimento di Matematica, Università degli studi di Parma, Via M. D'Azeglio 85, 43100 Parma, Italy

(Submitted by: James Serrin)

**Abstract.** Several works, quoted in the introduction, provide criteria which establish global existence in the future of all solutions of nonlinear second order differential systems. The aim of the present paper is to establish further sufficient conditions for global existence in the future for submanifolds of solutions.

**1. Introduction.** We consider scalar functions $G$, $F$ and an $N$-vector $Q$ ($N \geq 1$) defined as follows: $G = G(u,p)$ *is of class* $C^1(\mathbb{R}^N \times \mathbb{R}^N)$, *strictly convex in the variable* $p$ *for every* $u \in \mathbb{R}^N$, *and satisfies the conditions* $G(u,0) = 0$, $G_p(u,0) = 0$ *on* $\mathbb{R}^N$; $F = F(t,u)$ *is of class* $C^1(\mathbb{R}^N \times \mathbb{R}^N)$; $Q = Q(t,u,p)$ *is continuous on* $\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N$. In the present paper we establish sufficient conditions for global existence in the future of solutions $u = u(t)$ of the second order nonlinear differential system (see, e.g., [9])

$$(G_p(u,u'))' - G_u(u,u') + F_u(t,u) = Q(t,u,u'), \tag{1.1}$$

where $(\,\cdot\,)'$ denotes differentiation with respect to the independent variable $t \in \mathbb{R}^+$. A function $u(t)$ defined on an interval $I\,(\subset \mathbb{R}^+)$ is called a solution of (1.1) if $u(t)$ and $G_p(u(t),u'(t))$ are functions of class $C^1(I)$ such that

$$(G_p(u(t),u'(t)))' - G_u(u(t),u'(t)) + F_u(t,u(t)) = Q(t,u(t),u'(t)) \quad \text{on } I.$$

We denote by $H = H(u,p)$ the partial Legendre transform of the function $G(u,p)$, defined

$$H(u,p) = (G_p(u,p),p) - G(u,p) \quad \text{on } \mathbb{R}^N \times \mathbb{R}^N, \tag{1.2}$$

where $(G_p,p)$ denotes the scalar product of vectors $G_p$, $p \in \mathbb{R}^N$. For every fixed $u \in \mathbb{R}^N$ the function $H(u,p)$ is positive definite with respect to $p$, and along any solution $u(t)$ of (1.1) the following identity holds [12, Theorem 8]:

$$[H(u(t),u'(t)) + F(t,u(t))]' = (Q(t,u(t),u'(t)),u'(t)) + F_t(t,u(t)). \tag{1.3}$$

Sufficient conditions for global existence in the future of solutions of the system (1.1) were first established in [4] and [5], and later in [11], in the particular case

---

Received March 1994.
This work was reported on at the International Conference on Differential Equations in August, 1993.
[1]Partially supported by the Italian Ministero dell'Università e della Ricerca Scientifica e Tecnologica, fondi 40%.
AMS Subject Classification: 34A15, 34A34, 70H35.

when the function $G$ depends on the vector $p$ alone, and in [3], [7], [8], [10] when $G$ is a quadratic form in $p$ (in this case (1.1) is the Lagrange system for a holonomic mechanical system). It should be noted that in the above mentioned papers the criteria established guarantee global existence in the future of *all* solutions of the system (1.1), even though differential systems of the type (1.1) generally have some solutions existing globally in the future and some which have finite escape time.

Consider, for instance, the scalar equation

$$u'' - u' - 2e^{-4t}u^3 = 0, \tag{1.4}$$

which is of the form (1.1) with $G(p) = \frac{1}{2}p^2$, $F(t,u) = -\frac{1}{2}e^{-4t}u^4$, $Q(p) = p$. The solution satisfying the initial conditions $u(0) = \frac{1}{2}$, $u'(0) = \frac{3}{4}$ is $u(t) = e^t/(1+e^{-t})$, and is defined for each $t \in \mathbb{R}^+$, whereas the solution satisfying $u(0) = 2$, $u'(0) = 6$ is $u(t) = 2e^t/(2e^{-t} - 1)$, defined on the right maximal interval of existence $[0, \log 2)$.

In Sections 3 and 4 of the present work, two classes of differential systems of type (1.1) are studied, where, as in the preceding example, not all solutions exist globally in the future. By constructing appropriate Liapunov functions (obtained by perturbing the "total energy" $H + F$) and using the comparison method [1], we obtain sufficient conditions for global existence in the future of submanifolds of solutions of (1.1).

The author finally wishes to thank Professor J. Serrin for a number of helpful suggestions.

**2. A criterion for global existence of solutions of a particular comparison equation.** In Sections 3 and 4, we construct appropriate Liapunov functions and utilize scalar differential inequalities (comparison method), to arrive at differential comparison equations of the type

$$v' = \alpha(t)v + \beta(t)v^{1+\lambda}, \tag{2.1}$$

where $v \in \mathbb{R}^+$, $\alpha : \mathbb{R}^+ \to \mathbb{R}$, $\beta : \mathbb{R}^+ \to \mathbb{R}_0^+$ $(= (0,\infty))$ are continuous functions and $\lambda$ $(\geq -1)$ is a real constant. Putting

$$A(t) = \exp\Big\{ \int_0^t \alpha(s)\, ds \Big\} \tag{2.2}$$

the following result holds.

**Lemma 1.** *Consider the differential equation* (2.1). *Then the following dichotomy holds.*

(a) *If* $-1 \leq \lambda \leq 0$ *then all maximal*(*) *solutions* $v(t, t_0, v_0)$ *exist globally in the future.*

(b) *If* $\lambda > 0$ *then a solution* $v(t, t_0, v_0)$ *exists globally in the future if and only if*

$$A_0^\lambda \geq \lambda v_0^\lambda \int_{t_0}^\infty \beta(t) A^\lambda(t)\, dt, \tag{2.3}$$

*where* $A_0 = A(t_0)$.

---

*If $v_0 > 0$, the Cauchy problem (2.1) satisfying the initial condition $v(t_0) = v_0$ admits exactly one solution only if $\lambda \geq -1$, whereas it admits infinitely many solutions if $v_0 = 0$ and $\lambda \in (-1, 0)$.

**Proof.** Case 1: $-1 < \lambda < 0$ (if $\lambda = -1$ or $\lambda = 0$, the conclusion follows immediately since equation (2.1) becomes *linear*). Let $v(t) = v(t, t_0, v_0)$ be any solution of (2.1). Putting $z(t) = v(t)/A(t)$ and differentiating with respect to $t$, we get the equation, equivalent to (2.1),

$$z' = \beta(t)A^{\lambda}(t)z^{1+\lambda}. \tag{2.4}$$

The maximal solutions of (2.4) are, for each $t_0 \geq 0$, $z_0 \geq 0$,

$$z(t, t_0, z_0) = \big[z_0^{-\lambda} - \lambda \int_{t_0}^{t} \beta(s)A^{\lambda}(s)\, ds\big]^{-1/\lambda}, \tag{2.5}$$

from which we obtain the maximal solutions of (2.1)

$$v(t, t_0, v_0) = \big[\big(\frac{v_0}{A_0}\big)^{-\lambda} - \lambda \int_{t_0}^{t} \beta(s)A^{\lambda}(s)\, ds\big]^{-1/\lambda}; \tag{2.6}$$

all these solutions exist globally in the future.

Case 2: $\lambda > 0$. Proceeding as in the previous case we see that

$$v(t, t_0, v_0) = v_0 \frac{A(t)}{[A_0^{\lambda} - \lambda v_0^{\lambda} \int_{t_0}^{t} \beta(s)A^{\lambda}(s)\, ds]^{1/\lambda}}. \tag{2.7}$$

It is easy to verify that this function exists globally in the future if and only if inequality (2.3) is satisfied.

**3. A class of systems (1.1) characterized by a hypothesis on function $H(u, p)$.** Both in the present section and in the following, vectors formed with the first $\ell$ $(1 \leq \ell \leq N)$ components of $u$ and $p$, respectively, are indicated by the symbols $x$ and $y$; that is,

$$x = (u_1, \dots, u_\ell), \quad y = (p_1, \dots, p_\ell), \tag{3.1}$$

with their Euclidean norms denoted by $|x|$ and $|y|$.

In addition, for each real number $\omega > 0$, the symbol $\Omega$ denotes the set

$$\Omega = \{u \in \mathbb{R}^N : |x| \leq \omega\}. \tag{3.2}$$

Here we consider systems of type (1.1) for which the following condition is satisfied:

$$H(u, p) \geq [a(|x|)|y|]^m \quad \text{on} \quad \mathbb{R}^N \times \mathbb{R}^N, \tag{3.3}$$

where $m > 1$ is a real constant, and $a : \mathbb{R}^+ \to \mathbb{R}_0^+$ is a continuous function not belonging to $L^1$. A similar condition was introduced by Risito in [3] in his study of Lagrange equations. The function

$$\tau(|x|) = \int_0^{|x|} a(s)\, ds \tag{3.4}$$

is of class $K_*$, that is, of class $K$ according to Hahn [2] (continuous and strictly increasing, with $\tau(0) = 0$) and moreover $\tau(s) \to \infty$ as $s \to \infty$.

If $u(t) = u(t, t_0, u_0, u_0')$ is any solution of the system (1.1) obeying the initial conditions $u(t_0) = u_0$, $u'(t_0) = u_0'$, then the following results hold, in which we put

$$H_0 = H(u_0, u_0'), \quad F_0 = F(t_0, u_0), \quad \tau_0 = \tau(|x_0|). \tag{3.5}$$

**Theorem 1.** *Suppose that there exist*

(i) *a real constant* $m > 1$ *and a continuous function* $a : \mathbb{R}^+ \to \mathbb{R}_0^+$, *with* $\int^\infty a(t)\,dt = \infty$ *such that*

$$H(u,p) \geq [a(|x|)|y|]^m \quad on \quad \mathbb{R}^N \times \mathbb{R}^N;$$

(ii) *a real constant* $n \geq 1$ *and a function* $\sigma : \mathbb{R}^+ \to \mathbb{R}_0^+$ *of class* $C^1$ *such that*

$$F(t,u) + \sigma(t)\tau^n(|x|) \geq 0 \quad on \quad \mathbb{R}^+ \times \mathbb{R}^N;$$

(iii) *a continuous function* $\alpha : \mathbb{R}^+ \to \mathbb{R}$ *satisfying the inequality* $\sigma'(t)/\sigma(t) \leq \alpha(t)$ *and such that*

$$\begin{aligned}&(Q(t,u,p),p) + F_t(t,u)\\ &\leq \alpha(t)[H(u,p) + F(t,u) + \sigma(t)\tau^n(|x|)] - \sigma'(t)\tau^n(|x|)\end{aligned}$$

*on* $\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N$;

(iv) *a function* $\varphi_\omega \in K_*$ *corresponding to each constant* $\omega > 0$, *such that*

$$H(u,p) \geq \varphi_\omega(|p|) \quad on \quad \Omega \times \mathbb{R}^N.$$

*Then all solutions of* (1.1) *exist globally in the future if* $n \leq m$.

In case $n > m$, we consider the second order algebraic equation for the unknown $\xi$,

$$n\sigma_0\tau_0^n(\xi - m)(\xi - 1) + m(H_0 + F_0 + \sigma_0\tau_0^n)(\xi - n) = 0, \tag{3.6}$$

where $\sigma_0 = \sigma(t_0)$. Denote by $\rho$ its (unique) solution on the interval $[m,n]$, and put

$$\begin{aligned}&\lambda = (n-m)/nm, \quad V_0 = H_0 + F_0 + \rho\sigma_0\tau_0^n,\\ &\beta(t) = [n\rho(\rho-1)^{(1-n)/n}]\sqrt[n]{\sigma(t)}.\end{aligned} \tag{3.7}$$

**Theorem 1′.** *Suppose* $n > m$. *Then under the previous conditions* (i)–(iv) *a solution* $u(t) = u(t,t_0,u_0,u_0')$ *of* (1.1) *exists globally in the future if the following additional condition holds:*

(v) $$A_0^\lambda \geq \lambda V_0^\lambda \int_{t_0}^\infty \beta(t)A^\lambda(t)\,dt,$$

*where* $A(t)$ *is defined in* (2.2) *and* $A_0 = A(t_0)$.

**Proof of Theorem 1.** If, for the sake of contradiction, there exists a solution $u = u(t)$ defined on a right maximal interval of existence $I = [t_0, T)$, with $t_0 < T < \infty$, we would have

$$\lim_{t\to T^-}\{|u(t)| + |u'(t)|\} = \infty. \tag{3.8}$$

Choosing as Liapunov function

$$V(t,u,p) = H(u,p) + F(t,u) + \rho\sigma(t)\tau^n(|x|), \tag{3.9}$$

where $\rho > 1$ is an arbitrary real constant, we define

$$V_*(t) = V(t, u(t), u'(t)) \quad \text{on } I. \tag{3.10}$$

Owing to condition (i) and (ii), the following inequalities hold for every $t \in I$:

$$\begin{aligned} &\frac{d}{dt}\tau(|x(t)|) \le a(|x(t)|)|x'(t)| \le \sqrt[n]{V_*(t)}, \\ &\tau(|x(t)|) \le \sqrt[n]{V_*(t)/(\rho-1)\sigma(t)}. \end{aligned} \tag{3.11}$$

Therefore, because of the identity (1.3) and condition (iii), we obtain the estimate

$$\begin{aligned} V_*'(t) &= (Q(t, u(t), u'(t)), u'(t)) + F_t(t, u(t)) \\ &\quad + \rho\sigma'(t)\tau^n(|x(t)|) + n\rho\sigma(t)\tau^{n-1}(|x(t)|)\frac{d}{dt}\tau(|x(t)|) \\ &\le \alpha(t)V_*(t) + \beta(t)[V_*(t)]^{1+\lambda} \quad \text{on } I, \end{aligned} \tag{3.12}$$

where $\lambda$ and $\beta$ are given by (3.7).

Consider the comparison equation

$$v' = \alpha(t)v + \beta(t)v^{1+\lambda} \tag{3.13}$$

and denote by $v(t, t_0, V_0)$ its maximal solution satisfying the initial condition $v(t_0) = V_0 = V_*(t_0)$ $(= V(t_0, u_0, u_0'))$. Then by comparison, from (3.12) it follows that

$$V_*(t) \le v(t, t_0, V_0) \quad \text{on } I \cap J, \tag{3.14}$$

where $J$ is the right maximal interval of existence of the solution $v(t, t_0, V_0)$.

Since $-1 < \lambda \le 0$, we see by Lemma 1 that all solutions of the comparison equation (3.13) exist globally in the future. Therefore there exists a constant $k > 0$ (which depends on $I$ and $V_0$) such that $v(t, t_0, V_0) \le k$ on $I$. Thus taking into account (3.14) we have

$$V_*(t) \le k \quad \text{on } I. \tag{3.15}$$

Denote by $\tau^{-1}$ the inverse function of $\tau$ and let $\sigma_* = \inf\{\sigma(t) : t \in I\}$. Putting $\omega = \tau^{-1}(\sqrt[n]{k/(\rho-1)\sigma_*})$, from $(3.11)_2$ and (3.15) it follows that

$$|x(t)| \le \omega \quad \text{on } I. \tag{3.16}$$

Using inequalities (3.15) and (3.16) and conditions (ii) and (iv), we obtain

$$\varphi_\omega(|u'(t)|) \le H(u(t), u'(t)) \le V_*(t) \le k \quad \text{on } I, \tag{3.17}$$

where $\varphi_\omega(s)$ is a function of class $K_*$. Next, denoting by $\varphi_\omega^{-1}$ the inverse function of $\varphi_\omega$, from (3.17) it follows that

$$|u'(t)| \le \varphi_\omega^{-1}(k) \quad \text{on } I. \tag{3.18}$$

Then integrating with respect to $t$ on $I$, we obtain

$$|u(t)| \leq |u_0| + \varphi_\omega^{-1}(k)(T - t_0) \quad \text{on } I. \tag{3.19}$$

Inequalities (3.18) and (3.19) are in contradiction with (3.8), and the theorem is proved.

**Proof of Theorem 1′.** Let $u(t) = u(t, t_0, u_0, u_0')$ be a solution whose initial conditions satisfy condition (v) of Theorem 1′. If, for the sake of contradiction, this solution is defined on a right maximal interval of existence $I = [t_0, T)$, with $t_0 < T < \infty$, then (3.8) holds.

We choose again the Liapunov function (3.9), where $\rho$ is now the (unique) root of equation (3.6) in $[m, n]$ (this value for the constant $\rho$ is chosen because it minimizes the right side of inequality (v), and is thus the most advantageous). Repeating the proof of Theorem 1, we come to (3.14), where $v(t, t_0, V_0)$ is a solution of the comparison equation (3.13). By virtue of Lemma 1 and of condition (v) of Theorem 1′, this solution exists globally in the future. We thus obtain inequalities (3.18) and (3.19), which are in contradiction with (3.8). This completes the proof of the Theorem.

**Remark 1.** Obvious corollaries of Theorem 1 and 1′ arise by considering condition (iv) under the following additional properties:

(a) $H = H(p)$,
(b) $H = H(x, p)$,
(c) $x = u$.

In the particular case where $F(t, x) = -\sigma(t)\tau^n(|x|)$ (with $n > m$), equation (3.6) becomes

$$n\sigma_0\tau_0^n(\xi - m)(\xi - 1) + mH_0(\xi - n) = 0. \tag{3.20}$$

From this it follows that if we limit ourselves to solutions with $u'(t_0) = u_0' = 0$ (which are of considerable physical interest), we must choose $\rho = m$ (since $H_0 = 0$). Thus the Liapunov function becomes

$$V(t, u, p) = H(u, p) + (m - 1)\sigma(t)\tau^n(|x|). \tag{3.21}$$

The following corollaries of Theorem 1′ then hold.

**Corollary 1.** *Let $n > m$. Suppose that conditions* (i) *and* (iv) *above are satisfied, together with*

(ii)′ $\sigma : \mathbb{R}^+ \to \mathbb{R}_0^+$ *is a decreasing function of class $C^1$, with $\sqrt[n]{\sigma(t)} \in L^1$, and*
(iii)′ $(Q(t, u, p), p) \leq 0$ *on $\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N$.*

*Then solutions $u(t) = u(t, t_0, u_0, 0)$ of* (1.1) *exist globally in the future if the following further condition is satisfied:*

(v)′ $$(m - 1)^{\frac{m-1}{m}} \geq (n - m)(\sigma_0\tau_0^n)^{\frac{n-m}{nm}} \int_{t_0}^{\infty} \sqrt[n]{\sigma(t)}\, dt.$$

This is demonstrated by choosing (3.21) as the Liapunov function, and verifying that all conditions of Theorem 1′ are satisfied with $\alpha(t) \equiv 0$.

**Corollary 2.** *Let $n > m$. Suppose that conditions* (i) *and* (iv) *above are satisfied, together with*

(ii)″ $\sigma : \mathbb{R}^+ \to \mathbb{R}_0^+$ *is a function of class* $C^1$, *with* $\sqrt[m]{\sigma(t)} \in L^1$, *and*

(iii)″ $(Q(t,u,p),p) \leq \frac{\sigma'(t)}{\sigma(t)} H(u,p)$ *on* $\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N$.

*Then solutions* $u(t) = u(t,t_0,u_0,0)$ *of* $(1,1)$ *exist globally in the future if the following further condition is satisfied:*

(v)″ $$(m-1)^{\frac{m-1}{m}} \geq (n-m)\tau_0^{\frac{n-m}{m}} \int_{t_0}^{\infty} \sqrt[m]{\sigma(t)}\, dt.$$

This is proved by choosing (3.21) as the Liapunov function, and verifying that all conditions of Theorem 1′ are satisfied with $\alpha(t) \equiv \sigma'(t)/\sigma(t)$.

**Example 1.** Consider the scalar differential equation

$$u'' + \frac{k}{t}u' - \frac{|u|^{n-2}u}{t^\nu} = 0, \tag{3.22}$$

where $n \geq 1$, $\nu > 0$, $k \geq 0$ are real constants. This equation is of the form (1.1) with $G = H = (1/2)p^2$, $F = -|u|^n/nt^\nu$, $Q = -kp/t$, $m = 2$.

By means of Theorem 1 and Corollaries 1 and 2, we obtain the following results:

(a) If $n \leq 2$, then all solutions $u(t,t_0,u_0,u_0')$ with $t_0 > 0$ exist globally in the future.

(b) If $\nu > n > 2$, then by virtue of Corollary 1 it follows that a solution $u(t,t_0,u_0,0)$ with $t_0 > 0$ exists globally in the future if

$$t_0^{\nu-2} \geq n\Big(\frac{n-2}{\nu-2}\Big)^2 |u_0|^{n-2}. \tag{3.23}$$

(c) If $n > 2$ and $2k > \nu > 2$, then by virtue of Corollary 2 we prove that a solution $u(t,t_0,u_0,0)$ with $t_0 > 0$ exists globally in the future if

$$t_0^{\nu-2} \geq \frac{4}{n}\Big(\frac{n-2}{\nu-2}\Big)^2 |u_0|^{n-2}. \tag{3.24}$$

**4. A class of dissipative systems.** In this section, we replace condition (3.3), which was fundamental in the previous section, with the assumption

$$(Q(t,u,p),p) \leq \alpha(t)H(u,p) - \gamma(t)|x|^h|y|^k \text{ on } \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N, \tag{4.1}$$

where $\alpha : \mathbb{R}^+ \to \mathbb{R}$, $\gamma : \mathbb{R}^+ \to \mathbb{R}_0^+$ are continuous functions, and $h \geq 0$, $k > 1$ are real constants.

Condition (4.1) was introduced by the author in [6]. Supposing that (1.1) are the Lagrange equations of a holonomic mechanical system, the scalar product $(Q,p)$ represents the power of particular dissipative forces. For this reason, the system dealt with in this section may be called dissipative.

The following theorem holds.

**Theorem 2.** *Suppose there exist three scalar functions: $\alpha : \mathbb{R}^+ \to \mathbb{R}$, $b : \mathbb{R}^+ \to \mathbb{R}_0^+$ continuous, and $\sigma : \mathbb{R}^+ \to \mathbb{R}_0^+$ of class $C^1$ with $\alpha(t) \geq \sigma'(t)/\sigma(t)$ on $\mathbb{R}^+$, such that*

(i) $(Q(t,u,p),p) \leq \alpha(t)H(u,p) - \frac{\sigma(t)}{[b(t)]^{k-1}}|x|^h|y|^k$ *on* $\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N$*, where* $h \geq 0$, $k > 1$ *are real constants;*

(ii) $F(t,u) + \sigma(t)|x|^n \geq 0$ *on* $\mathbb{R}^+ \times \mathbb{R}^N$*;*

(iii) $F_t(t,u) \leq \alpha(t)[F(t,u) + \sigma(t)|x|^n] - \sigma'(t)|x|^n$ *on* $\mathbb{R}^+ \times \mathbb{R}^N$*, where* $n \geq 1 + h/k$ *is a real constant;*

(iv) *for every* $\omega > 0$ *there exists a function* $\varphi_\omega$ *of class* $K_*$ *such that*

$$H(u,p) \geq \varphi_\omega(|p|) \quad \text{on } \Omega \times \mathbb{R}^N.$$

*Then all solutions of* (1.1) *exist globally in the future if* $n \leq h + k$.

In case $n > h+k$, we consider the second order algebraic equation for the unknown $\xi$,

$$n\sigma_0|x_0|^n(\xi - k)(\xi - 1) + (h+k)[H_0 + F_0 + \sigma_0|x_0|^n](\xi - \frac{kn}{h+k}) = 0, \tag{4.2}$$

and let us denote by $\rho$ its (unique) solution on the interval $[k, kn/(h+k)]$. Recalling (3.5) and putting

$$\begin{aligned} &\lambda = (n-h-k)/n(k-1), \quad V_0 = H_0 + F_0 + \rho\sigma_0|x_0|^n, \\ &\beta(t) = [(k-1)(n\rho/k)^{\frac{k}{k-1}}/(\rho-1)^{1+\lambda}][b(t)/\sigma^\lambda(t)], \end{aligned} \tag{4.3}$$

we have the following result.

**Theorem 2′.** *Suppose* $n > h + k$*. Then a solution* $u(t) = u(t, t_0, u_0, u_0')$ *of* (1.1) *exists globally in the future if the following additional condition holds:*

$$\text{(v)} \qquad A_0^\lambda \geq \lambda V_0^\lambda \int_{t_0}^{\infty} \beta(t) A^\lambda(t)\, dt,$$

*where* $A(t)$ *is defined in* (2.2) *and* $A_0 = A(t_0)$.

**Proof.** We argue as in Theorems 1 and 1′, but replacing the Liapunov function (3.9) with

$$V(t,u,p) = H(u,p) + F(t,u) + \rho\sigma(t)|x|^n, \tag{4.4}$$

where, as in (3.9), the constant $\rho > 1$ can be chosen arbitrarily if $1+h/k \leq n \leq h+k$, or coincides with the root of equation (4.2) if $n > h + k$.

Therefore, if we consider a solution $u(t)$ of the system (1.1) defined on a right maximal interval of existence $I \subset \mathbb{R}^+$, it is sufficient that the derivative of the function $V_*(t) = V(t, u(t), u'(t))$ (where $V(r,u,p)$ is given by (4.4)) satisfies the inequality

$$V_*'(t) \leq \alpha(t)V_*(t) + \beta(t)[V_*(t)]^{1+\lambda} \quad \text{on } I, \tag{4.5}$$

where $\lambda \geq -1$ is a real constant and $\beta : \mathbb{R}^+ \to \mathbb{R}_0^+$ is a continuous function. For this purpose we differentiate $V_*(t)$, taking into account identity (1.3), the hypothesis

$\alpha(t) \geq \sigma'(t)/\sigma(t)$, and conditions (i) and (iii) of Theorem 2. We obtain the following chain of inequalities, valid for every $t \in I$:

$$\begin{aligned} V'_*(t) =& (Q(t,u(t),u'(t)),\, u'(t)) + F_t(t,u(t)) \\ &\quad + \rho\sigma'(t)|x(t)|^n + n\rho\sigma(t)|x(t)|^{n-1}\frac{d}{dt}|x(t)| \\ \leq& \alpha(t)H(u(t),u'(t)) - \frac{\sigma(t)}{[b(t)]^{k-1}}|x(t)|^h|x'(t)|^k \\ &\quad + \alpha(t)[F(t,u(t)) + \sigma(t)|x(t)|^n] - \sigma'(t)|x(t)|^n \\ &\quad + \rho\sigma'(t)|x(t)|^n + n\rho\sigma(t)|x(t)|^{n-1}|x'(t)| \\ \leq& \alpha(t)V_*(t) + \sigma(t)|x(t)|^h\big[-\frac{|x'(t)|^k}{[b(t)]^{k-1}} + n\rho|x(t)|^{n-h-1}|x'(t)|\big]. \end{aligned} \tag{4.6}$$

Whatever the constant $k > 1$, we have for every $\eta \in \mathbb{R}^+$ that

$$\eta^k - k\eta + k - 1 \geq 0. \tag{4.7}$$

Putting $\eta = [|x'(t)|/b(t)]/[(n\rho/k)|x(t)|^{n-h-1}]^{\frac{1}{k-1}}$ in (4.7), it follows that, for every $t \in I$,

$$\frac{|x'(t)|^k}{[b(t)]^{k-1}} - n\rho|x(t)|^{n-h-1}|x'(t)| + (k-1)b(t)\big(\frac{n\rho}{k}\big)^{\frac{k}{k-1}}|x(t)|^{\frac{k(n-h-1)}{k-1}} \geq 0. \tag{4.8}$$

Moreover, due to condition (ii) of Theorem 2, it follows that

$$|x(t)| \leq [V_*(t)/(\rho-1)\sigma(t)]^{\frac{1}{n}} \quad \text{on } I. \tag{4.9}$$

Therefore, owing to (4.8) and (4.9), from (4.6) we obtain

$$\begin{aligned} V'_*(t) &\leq \alpha(t)V_*(t) + (k-1)b(t)\sigma(t)\big(\frac{n\rho}{k}\big)^{\frac{k}{k-1}}|x(t)|^{\frac{kn-k-h}{k-1}} \\ &\leq \alpha(t)V_*(t) + \beta(t)[V_*(t)]^{1+\lambda} \quad \text{on } I, \end{aligned} \tag{4.10}$$

where $\lambda$ and $\beta$ are given by (4.3). Thus Theorems 2 and 2′ are proved. □

In the particular case when $F(t,x) = -\sigma(t)|x|^n$, and considering only solutions satisfying the initial condition $u'_0 = 0$, we deduce from (4.2) that $\rho = k$. Thus the following corollaries of Theorem 2′ hold, where $\lambda = (n-h-k)/n(k-1)$.

**Corollary 3.** *Suppose that condition* (iv) *of Theorem* 2 *is satisfied, together with*

(i)′ $(Q(t,u,p),p) \leq -\frac{\sigma(t)}{[b(t)]^{k-1}}|x|^h|y|^k$ *on* $\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N$,

(ii)′ $F(t,x) = -\sigma(t)|x|^n$ *on* $\mathbb{R}^+ \times \mathbb{R}^\ell$,

*where* $n > h + k$ *is a real constant,* $b : \mathbb{R}^+ \to \mathbb{R}^+_0$ *is a continuous function,* $\sigma : \mathbb{R}^+ \to \mathbb{R}^+_0$ *is a function of class* $C^1$*, and* $[b(t)/\sigma^\lambda(t)] \in L^1(\mathbb{R}^+)$*. Then solutions* $u(t) = u(t,t_0,u_0,0)$ *of* (1.1) *exist globally in the future if the following additional condition holds:*

$$\lambda n^{\frac{k}{k-1}}(\sigma_0|x_0|^n)^\lambda \int_{t_0}^\infty [b(t)/\sigma^\lambda(t)]\,dt \leq 1.$$

Choosing as Liapunov function

$$V(t,u,p) = H(u,p) + (k-1)\sigma(t)|x|^n, \tag{4.11}$$

it is sufficient to note that all conditions of Theorem 2′ are satisfied with $\alpha(t) \equiv 0$.

**Corollary 4.** *Suppose that condition* (iv) *of Theorem* 2 *is satisfied, together with*

(i)″ $(Q(t,u,p),p) \leq \frac{\sigma'(t)}{\sigma(t)}H(u,p) - \frac{\sigma(t)}{[b(t)]^{k-1}}|x|^h|y|^k$ *on* $\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N$,

(ii)″ $F(t,x) = -\sigma(t)|x|^n$ *on* $\mathbb{R}^+ \times \mathbb{R}^\ell$,

*where* $n > h+k$ *is a real constant,* $b : \mathbb{R}^+ \to \mathbb{R}_0^+$ *is a continuous function of class* $L^1$*, and* $\sigma : \mathbb{R}^+ \to \mathbb{R}_0^+$ *is a function of class* $C^1$*. Then solutions* $u(t) = u(t,t_0,u_0,0)$ *of* (1.1) *exist globally in the future if the following additional condition holds:*

$$\lambda n^{\frac{k}{k-1}}|x_0|^{n\lambda}\int_{t_0}^{\infty} b(t)\,dt \leq 1.$$

In fact, choosing (4.11) as Liapunov function, all the conditions of Theorem 2′ are satisfied with $\alpha(t) \equiv \sigma'(t)/\sigma(t)$.

## REFERENCES


[1] R. Conti, *Sulla prolungabilità delle soluzioni di un sistema di equazioni differenziali ordinarie,* Boll. Un. Mat. Ital., 11 (1956), 510–514.

[2] W. Hahn, "Stability of Motion," Springer, Berlin, 1967.

[3] C. Risito, *Sull'esistenza globale e sulla limitatezza dei moti dei sistemi olonomi scleronomi,* Ann. Mat. Pura Appl., (IV), 139 (1985), 341–348.

[4] P. Pucci and J. Serrin, *Continuation and limit properties for solutions of strongly nonlinear second order differential equations,* Asymptotic Anal., 4 (1991), 97–160.

[5] P. Pucci and J. Serrin, *Continuation and limit behavior for damped quasi-variational systems,* "Degenerate Diffusion," I.M.A., vol. 47 in Math. and its Applications, Springer-Verlag, W.-M. Ni, L.A. Peletier and J.L. Vasquez, eds., (1993), 157–174.

[6] G. Cantarelli, *Stabilizzazione dell'equilibrio dei sistemi olonomi mediante forze dissipative dipendenti dal tempo,* Boll. Un. Mat. Ital., (7) 6–A (1992), 193–196.

[7] G. Cantarelli and C. Risito, *On the continuation of the solutions of Lagrange equations for holonomic scleronomic systems,* Proceed. Equadiff. 91, C. Perelló, C. Simó and J. Solà-Norales, eds., (1993), 358–362.

[8] G. Cantarelli and C. Risito, *Criteri di esistenza globale e di limitatezza per i sistemi olonomi scleronomi,* Ann. Mat. Pura Appl., (IV), 162 (1992), 393–394.

[9] P. Pucci and J. Serrin, *Precise damping conditions for global asymptotic stability for nonlinear second order systems,* Acta Math., 170 (1993), 275–307.

[10] G. Cantarelli, *Nuovi criteri per l'esistenza globale in futuro dei moti dei sistemi olonomi scleronomi,* Ann. Mat. Pura Appl., (IV), 163 (1993), 247–264.

[11] G. Cantarelli, *Global existence of solutions of second order differential systems,* Asymptotic Anal., 8 (1994), 93–101.

[12] P. Pucci and J. Serrin, *On the derivation of Hamilton's equations,* Archive Rational Mechanics and Analysis, to appear.