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SLOW DECAY AND THE HARNACK INEQUALITY FOR POSITIVE SOLUTIONS OF $\Delta u + u^p = 0$ in $\mathbf{R}^n$

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1. Introduction. In a recent paper [9], the author investigated the question of symmetry of positive solutions of the Lane-Emden equation

$$\Delta u + u^p = 0, \quad \text{in } \mathbf{R}^n,\ p > 1,\ n > 2. \tag{I}$$

Let

$$l = \frac{n+2}{n-2}, \quad m = \begin{cases} \infty, & n = 3, \\ \frac{n+1}{n-3}, & n > 3, \end{cases}$$

the Sobolev exponent for dimensions n and $n-1$ respectively. The following result was proved in [9], using the Alexandroff-Serrin moving-plane method and an asymptotic expansion at infinity.

Theorem 1.1. *Let u be a positive solution of* (I). *Suppose that there exists a constant $M = M(u) > 0$ such that*

$$|x|^\alpha u(x) \le M, \tag{1.1}$$

where

$$\alpha = \frac{2}{p-1}, \qquad \lambda = \Big(\alpha(n-2-\alpha)\Big)^{\alpha/2}.$$

Then u is necessarily radially symmetric about some point $x_0 \in \mathbf{R}^n$, provided that

$$l < p < m. \tag{1.2}$$

Remark. Equation (I) admits infinitely many solutions satisfying (1.1); see [1].

In this paper, our main purpose is to weaken the *slow decay* assumption (1.1). In fact, a large class of solutions satisfies (1.1). Consider the function class

$$Z = \Big\{u > 0,\ u \in C^1(\mathbf{R}^n) : \nabla u(x) \cdot x = ru'(x) \le \Lambda u,\ |x| \ge \Lambda\Big\}$$

for some $\Lambda = \Lambda(u) > 0$. For solutions of (I) in Z, a local Harnack inequality at infinity is obtained, which implies the needed slow decay estimates. We have the following result.

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Theorem 1.2. *Let $n = 3$ and u a solution of* (I) *in Z. Then* (1.1) *holds.*

The exponent α in (1.1) is best possible for $p > l$. In fact, u is identically zero if

$$u = o(|x|^{-\alpha}), \quad \text{as } |x| \to \infty.$$

Note that all positive solutions of (I) have *fast decay* ($|x|^{-(n-2)}$) at infinity when $p = l$.

The proof of Theorem 1.2 is based on a reduction of dimension. We write $u(\theta) = u(r, \theta) = u(x)$ on S^2 and view r as a parameter, where (r, θ) are the spherical coordinates. We first estimate the functions $u(\theta)$ on S^2 to avoid the difficulties on the entire space $\mathbf{R}^3$. For $q > 0$, $s > 0$ and $K > 0$, we introduce a function class $Z_{K,q,s}$ on S^2 (see the precise definition in Section 3). Observing of a useful feature of $Z_{K,q,s}$, we are able to establish L^q-estimates for any $q > 0$ in terms of L^p-norm in the class $Z_{K,p,q}$. Using a boot-strap type technique, we then extend the estimates on S^2 to the desired supremum decay estimate (1.1).

An interesting by-product is the following local Harnack inequality.

Theorem 1.3. *Suppose $n = 3$ and u is a solution of* (I) *in Z. Then there exists a constant $M = M(\Lambda)$ such that for any $x \in \mathbf{R}^n$,*

$$\sup_{y \in B_R(x)} u(y) \le M \inf_{y \in B_R(x)} u(y), \quad R > M, \tag{1.3}$$

where $4R = |x|$ and $B_R(x)$ is the ball centered at x with radius R.

Remark. It is easy to see that the slow decay (1.1) implies the Harnack inequality (1.3) by standard elliptic theory and vice versa (see the estimate (2.4)). Therefore the slow decay (1.1), the Harnack inequality (1.3), the Z condition (plus (1.4) if $n > 3$) and the symmetry are equivalent if $p < m$. However, it seems that the Z condition (plus (1.4) if $n > 3$) is more general among solutions of (I) when $p \ge m$. Indeed, (I) admits solutions in Z (plus (1.4) if $n > 3$) which do not decay at all when $p \ge m$ but all known solutions of (I) are in Z (plus (1.4) if $n > 3$).

For $n \ge 4$, an additional condition is needed. Assume that there exists a constant $\Lambda_1 = \Lambda_1(u)$ such that

$$\int_{B_r} u^{p+1}(y)dy \le \Lambda_1 r^{n-2(p+1)/(p-1)}, \quad r \ge \Lambda_1. \tag{1.4}$$

The supremum estimates and the local Harnack inequality then read

Theorem 1.4. *Let $n > 3$ and* (1.2) *hold. Suppose that u is a solution of* (I) *in Z satisfying* (1.4)*. Then* (1.1) *and* (1.3) *necessarily hold.*

On the other hand, (I) *admits infinitely many solutions in Z satisfying* (1.4) *for which neither* (1.1) *nor* (1.3) *holds if $p \ge m$.*

Both the results and the arguments reflect that the hypothesis (1.2) is fundamental. In fact, Theorem 1.4 implies that there exist infinitely many non-symmetric solutions of (I) in Z satisfying (1.4) if $p \ge m$.

Finally, we would like to point out that extensions and generalizations are possible to the more general quasilinear equations

$$\mathrm{div}\,(A(\nabla u)) + f(x, u, \nabla u) = 0;$$

see, for instance, a forthcoming paper [10].

For simplicity, we shall mainly treat the case of $n = 3$, the case of $n > 3$ being essentially the same though an additional condition is needed. The paper is organized as follows. In Section 2, we derive upper bounds of L^p-norm on S^{n-1}, including some preliminaries. In Section 3, we introduce the function class $Z_{K,q,s}$ on S^2 and a useful feature is given. An auxiliary lemma is proved in Section 4. Section 5 is then devoted to develop the supremum decay estimate and the local Harnack inequality. The case of $n > 3$ is discussed in Section 6. An appendix is included at the end of the paper for the convenience of the readers.

2. A priori estimates in L^p norm. For $p > 1$, consider the problem

$$\Delta u + u^p = 0, \quad u > 0, \quad x \in \mathbf{R}^n. \tag{2.1}$$

We are only interested in C^2-solutions (hence smooth). In this section, some preliminary backgrounds for (2.1) will be discussed.

Throughout the paper, we shall assume that all solutions considered are in Z. Put

$$\lambda = \Big(\alpha(n-2-\alpha)\Big)^{\alpha/2}, \qquad \alpha = \frac{2}{p-1}. \tag{2.2}$$

In the sequel, we denote $M = M(\cdots)$ positive constants, besides the arguments inside the parenthesis, depending on the structural numbers n and p, which may vary line from line.

Let $\gamma > 0$ and u a positive function on $\mathbf{R}^n$. We say that u decays (pointwise) at infinity with rate γ if there exists a constant $M = M(u) > 0$ such that

$$|x|^\gamma u(x) \le M, \qquad |x| \ge M.$$

A positive solution u of (I) need not decay at infinity pointwise (see Section 6), however, it must decay in a certain norm. For instance, the average $\overline{u}(r)$ on S^{n-1} decays with rate $-2/(p-1)$ (i.e., in L^1-norm). If one can show that the decay happens in L^q-norm for suitably large q, the pointwise decay estimate then follows from the standard boot–strap arguments.

The condition $u \in Z$ guarantees uniform upper bounds of L^p-norm on S^{n-1}.

Lemma 2.1. *Let u be a solution of* (2.1) *in Z. Then there exists a constant $M = M(u) > 0$ such that*

$$\int_{S^{n-1}} u^p(r,\theta) \le M r^{-2p/(p-1)}, \qquad r \ge M, \tag{2.3}$$

where (r, θ) are the spherical-coordinates.

Proof. Clearly the average $\overline{u}(r)$ of u on S^{n-1} satisfies the equation

$$\overline{u}'' + \frac{n-1}{r}\overline{u}' + \frac{1}{\omega_n}\int_{S^{n-1}} u^p(r,\theta) = 0.$$

Applying Jensen's inequality, we obtain

$$\overline{u}'' + \frac{n-1}{r}\overline{u}' + \overline{u}^p \le 0.$$

Multiply by r^{n-1} and integrate from 0 to $r > 0$ to obtain

$$r^{n-1}\overline{u}' + \int_0^r s^{n-1}\overline{u}^p \le 0.$$

Therefore $\overline{u}' < 0$ for $r > 0$, and in turn $\overline{u}$ is monotonically decreasing. It follows that

$$-\overline{u}' \ge \frac{r}{n}\overline{u}^p, \quad \text{or} \quad \frac{-\overline{u}'}{\overline{u}^p} \ge \frac{r}{n}.$$

Integrating from 0 to $r > 0$ immediately yields

$$\overline{u} \le \Big(\frac{2n}{p-1}\Big)^{1/(p-1)} r^{-\alpha}. \tag{2.4}$$

Using integration by parts and (2.4), we have

$$\begin{aligned}\frac{R}{\omega_n}\int_{B_R} u^p &\le \frac{1}{\omega_n}\int_R^{2R}\int_{B_r} u^p = -\int_R^{2R} r^{n-1}\overline{u}'(r)dr \\ &= R^{n-1}\overline{u}(R) - (2R)^{n-1}\overline{u}(2R) + (n-1)\int_R^{2R} r^{n-2}\overline{u}(r)dr \\ &\le R^{n-1}\overline{u}(R) + (n-1)M\int_R^{2R} r^{n-2-\alpha} = MR^{n-1-\alpha}.\end{aligned}$$

It follows that

$$\begin{aligned}R^n\int_{S^{n-1}} u^p(R,\theta) &= p\int_0^R r^n\int_{S^{n-1}} u^{p-1}(r,\theta)u'(r,\theta) + n\int_0^R r^{n-1}\int_{S^{n-1}} u^p \\ &\le M\int_0^R r^{n-1}\int_{S^{n-1}} u^p \le MR^{n-2p/(p-1)},\end{aligned}$$

where we have used the fact $u \in Z$. The proof is now complete.

3. The class $Z_{K,q,s}$. Let q, $s > 1$ and K be three positive numbers, and $H^1(S^2)$ the usual Hilbert space. We introduce the function class on S^2,

$$Z_{K,q,s} = Z_{K,q,s}(S^2) = \{\ z \in H^1(S^2):\ ||\nabla_\theta(|z|^{s/2})||_2^2 \le K||z||_{q+s-1}^{q+s-1}\},$$

where $||z||_{q+s-1}$ is the standard L^{q+s-1}-norm. We have the following lemma.

Lemma 3.1. *Let $q > 1$, $s > 1$, $K > 0$ and $v(\theta) \in Z_{K,q,s}(S^2)$. Then there exists a positive constant $M = M(q, s, K)$ such that*

$$||\nabla_\theta(|v|^{s/2})||_2^2 \le M\big(||v||_q^{q+s-1} + ||v||_q^{qs}\big). \tag{3.1}$$

Proof. By the Gagliardo-Nirenberg inequality ([4]), there exists a constant $M > 0$ such that

$$\begin{aligned} ||v||_{q+s-1}^{q+s-1} = ||v^{s/2}||_{2(q+s-1)/s}^{2(q+s-1)/s} &\le M\big(||\nabla_\theta(v^{s/2})||_2^a \cdot ||v^{s/2}||_{2q/s}^{1-a} + ||v^{s/2}||_{2q/s}\big)^{2(q+s-1)/s} \\ &\le M||\nabla_\theta(v^{s/2})||_2^{2a(q+s-1)/s} \cdot ||v||_q^{(1-a)(q+s-1)} + M||v||_q^{q+s-1}, \end{aligned}$$

where

$$\frac{s}{2(q+s-1)} = (1-a)\frac{s}{2q}, \quad \text{i.e.,} \quad a = \frac{s-1}{q+s-1},$$

and so

$$a \cdot \frac{q+s-1}{s} = \frac{s-1}{s} < 1.$$

It follows, by the definition of $Z_{K,q,s}(S^2)$ and the Hölder inequality, that

$$||\nabla_\theta(v^{s/2})||_2^2 \le K||v||_{q+s-1}^{q+s-1} \le \frac{1}{2}||\nabla_\theta(v^{s/2})||_2^2 + M\big(||v||_q^{q+s-1} + ||v||_q^{qs}\big),$$

and (3.1) follows.

4. An auxiliary lemma. In this section, we prove an auxiliary lemma. Let u be a positive solution of (2.1). We introduce the functions

$$v(t,\theta) = r^\alpha u(r,\theta), \qquad t = \ln r,$$

and

$$w(t) = w_q(t) = \int_{S^{n-1}} v^q(t,\theta), \qquad q > 0.$$

We shall establish an upper bound for $w(t)$ which depends on V_p via the boot-strap arguments, where ($d > 0$)

$$V_d = \sup_{t>0} ||v(t)||_d.$$

Lemma 4.1. *Let $v(t,\theta)$ and w be given as above. Then v satisfies the equation*

$$v'' + \mu v' + \Delta_\theta v + v^p - \tau v = 0, \tag{4.1}$$

where $\mu = n - 2 - 2\alpha$, $\tau = \alpha(n - 2 - \alpha)$, and $w(t)$ satisfies the equation

$$w'' + \mu w' - \tau w = q(q-1)\int_{S^{n-1}} v^{q-2}(|\nabla_\theta v|^2 + |v'|^2) - q\int_{S^{n-1}} v^{q+p-1}. \tag{4.2}$$

The proof of this lemma is by direct calculations.

Using equations (4.1) and (4.2), we can prove the main results of this section.

Lemma 4.2. *Let $n = 3$, $q > 1$ and $p \geq 3$. Suppose that there exists a positive number T_0 such that*

$$w'(t) \geq 0, \qquad t \geq T_0. \tag{4.3}$$

Then there exists a positive constant $M = M(q, V_p)$ such that

$$w(t) \leq M, \qquad t \geq T_0. \tag{4.4}$$

Proof. First, by (2.3), $V_p < \infty$ so that (4.4) is meaningful. By (4.3), it suffices to show that there exist a sequence $\{t_j\} \to \infty$ and a positive constant $M = M(q, V_p)$ such that

$$w(t_j) \leq M, \qquad j = 1, 2, \cdots. \tag{4.5}$$

We first claim that there exists a sequence $\{t_j\} \to \infty$ such that

$$w''(t_j) + \mu w'(t_j) - \tau w(t_j) \leq 0, \qquad j = 1, 2, \cdots. \tag{4.6}$$

Suppose for contradiction that (4.6) is false, that is, there exists a number $T = T(w) > 0$ such that

$$w''(t) + \mu w'(t) - \tau w(t) > 0, \qquad t \geq T.$$

The corresponding homogeneous linear ordinary equation has two characteristic values, $\kappa_1 = \alpha > 0$, $\kappa_2 = -(n-2-\alpha) \leq 0$. It follows that

$$\left[e^{-\alpha t}(w' + (n-2-\alpha)w)\right]' > 0, \qquad t > T,$$

and in turn (without loss of generality, assuming that $c = w'(T)e^{-\alpha T} > 0$)

$$w' + (n-2-\alpha)w > (w'(T) + (n-2-\alpha)w(T))e^{-\alpha T}e^{\alpha t} \geq ce^{\alpha t}, \qquad t > T.$$

Integrating immediately yields

$$we^{(n-2-\alpha)t} > \frac{c}{n-2}e^{(n-2)t} - c', \qquad t > T$$

for some constant c'. Therefore there exist two positive constants which we still denote by c and T (we shall continue to do so and it should not cause confusion) such that

$$w > ce^{\alpha t}, \qquad t > T. \tag{4.7}$$

Now we claim that there is a positive number T such that

$$(q-1)\int_{S^{n-1}} v^{q-2}(|\nabla_\theta v|^2 + |v'|^2) > 4\int_{S^{n-1}} v^{q+p-1}, \qquad t > T. \tag{4.8}$$

For otherwise, there exists a sequence $\{t_j\} \to \infty$ such that

$$(q-1)\int_{S^{n-1}} v^{q-2}(|\nabla_\theta v|^2+|v'|^2) \le 4\int_{S^{n-1}} v^{q+p-1}, \qquad j=1,2,\cdots,$$

and so $v_j(\theta)=v(t_j,\theta)\in Z_{q^2/(q-1),p,q}$. It now follows by the Gagliardo-Nirenberg inequality and (3.1) that there exists a positive constant $M=M(q)>0$ such that

$$\begin{aligned}||v_j||_q = ||v_j^{q/2}||_2^{2/q} &\le M\Big[||\nabla_\theta v_j^{q/2}||_2^{(q-p)/q}\cdot ||v_j^{q/2}||_{2p/q}^{p/q} + ||v_j^{q/2}||_{2p/q}\Big]^{2/q}\\ &\le M\Big[||v_j||_p^{p/2+p(q-p)/2} + ||v_j||_p^{q/2}\Big]^{2/q} \le M||v_j||_p^{p/2+p(q-p)/q},\end{aligned}$$

and in turn $w(t_j)\le M$, $j=1,2,\dots$ for some $M=M(q,V_p)>0$, which contradicts (4.7). Thus (4.8) holds. Combining (4.2) and (4.8), with the aid of the Hölder inequality (since $p>1$), there exists a constant $M=M(p,q,n)>0$ such that

$$\begin{aligned}w''+\mu w'-\tau w &= q(q-1)\int_{S^{n-1}} v^{q-2}(|\nabla_\theta v|^2+|v'|^2) - q\int_{S^{n-1}} v^{q+p-1}\\ &> 3q\int_{S^{n-1}} v^{q+p-1} \ge M\Big(\int_{S^{n-1}} v^q\Big)^{(q+p-1)/q} \ge Mw^{(q+p-1)/q}.\end{aligned} \tag{4.9}$$

Multiply (4.9) by w' and integrate from T to t to get

$$\frac{w'^2}{2}\Big|_T^t + \mu\int_T^t w'^2 - \frac{\tau w^2}{2}\Big|_T^t \ge Mw^{(2q+p-1)/q}\Big|_T^t,$$

since $w'\ge 0$. It follows that

$$w'^2 + 2\mu\int_T^t w'^2 \ge Mw^{2(1+\delta)} - C, \tag{4.10}$$

since $\delta=(p-1)/2q>0$. Similarly, multiplying (4.9) by w and integrating from T to t, we deduce that

$$\int_T^t w'^2 \le ww' + \frac{\mu w^2}{2} + C. \tag{4.11}$$

Combining (4.10) and (4.11), we infer that

$$w'^2 + 2\mu\Big(ww' + \frac{\mu w^2}{2}\Big) \ge Mw^{2(1+\delta)} - C.$$

It follows that for large t,

$$w' \ge Mw^{1+\delta},$$

which implies that w blows up at a finite time, a contradiction. Therefore (4.6) holds and hence $v_j(\theta)=v(t_j,\theta)\in Z_{q^2/4(q-1),p,q}$. Using (3.1) once more, it is easy to see that there exists $M = M(q, V_p) > 0$ such that $w(t_j) \le M$, $j = 1, 2, \dots$. Thus (4.5) follows and so does (4.4).

5. A local Harnack inequality. The classical Harnack inequality fails for equation (2.1) when p is critical or supercritical, i.e, $p \ge l$. Here we shall show that a local Harnack inequality remains valid for solutions in Z near infinity when $n = 3$. The case of $n > 3$ will be discussed in Section 6.

For $R > 0$ and $x \in \mathbf{R}^n$, let $B_R(x)$ denote the ball centered at x with radius R and u a positive solution of (2.1). By a local Harnack inequality of u near infinity, we mean that there exists a constant $M = M(u) > 0$ such that for any $x \in \mathbf{R}^n$ with $2R = |x|$,

$$\sup_{y \in B_R(x)} u(y) \le M \inf_{y \in B_R(x)} u(y), \qquad R \ge M.$$

We first establish, for any $q > 0$, a L^q-norm estimate on S^2.

Theorem 5.1. *Let $n = 3$, $q > 0$, $p > 3$ and $u \in Z$ a solution of* (2.1). *Then there exists a constant $M = M(q, u) > 0$ such that*

$$||u(r)||_q = \Big(\int_{S^{n-1}} u^q(r, \theta) \Big)^{1/q} \le M r^{-\alpha}, \qquad r \ge M. \tag{5.1}$$

Proof. Let $v(t, \theta)$ and $w(t)$ be as given in Section 4. Then it is equivalent to show that

$$w(t) \le M, \qquad t \ge T_0 \tag{5.2}$$

for some $M = M(q, u) > 0$ and $T_0 = T_0(q, w) > 0$.

We first recall, from Lemma 2.1, that there exists a constant $M = M(q, u) > 0$ such that

$$V_p = \sup_{t \ge \ln R_0} ||v(t)||_p \le M.$$

Consider the derivative $w'(t)$. Its behavior has three possibilities near infinity, namely:

(i) $w'(t) \ge 0$ ultimately;
(ii) $w'(t) \le 0$ ultimately;
(iii) $w'(t)$ changes sign ultimately.

If (i) occurs, estimate (5.2) follows directly from Lemma 4.2 by taking

$$T_0 = T_0(q, w) = \inf\{T \ge 0 \ : \ w'(t) \ge 0, \ t \in (T, \infty)\}.$$

Next suppose that (ii) happens. Set

$$T_0 = T_0(q, w) = \inf\{T \ge 0 \ : \ w'(t) \le 0, \ t \in (T, \infty)\}.$$

Thus (5.2) follows with $M = w(T_0)$ since $w(t)$ is monotonically decreasing for $t \ge T_0$.

Finally, consider case (iii). Clearly, there exists a sequence $\{t_j\} \to \infty$ such that $w(t)$ assumes (local) maximum values at t_j ($j = 1, 2, \cdots$). This implies at each t_j that

$$\begin{aligned} q(q-1)\int_{S^{n-1}} & v^{q-2}(|\nabla_\theta v|^2 + |v'|^2) - q\int_{S^{n-1}} v^{q+s-1} \\ &= w''(t_j) + \mu w'(t_j) - \tau w(t_j) = w''(t_j) - \tau w(t_j) \le w''(t_j) \le 0, \end{aligned}$$

since $\tau > 0$, and so

$$v(\theta) = v(t_j, \theta) \in Z_{q^2/4(q-1),p,q}(S^2).$$

We immediately infer that $w(t_j) \le M(V_p)$ by (3.1) and hence (5.2) follows. □

Now we are ready to prove the local Harnack inequality.

Theorem 5.2. *Let $n = 3$ and u a positive solution of* (2.1) *in Z. Then there exists a constant $M = M(u)$ such that*

$$\sup_{y \in B_R(x)} u(y) \le M \inf_{y \in B_R(x)} u(y), \qquad R \ge M, \tag{5.3}$$

where $4R = |x|$ and $B_R(x)$ is the ball centered at x with radius R.

Proof. The proof is standard according to a result in [8]. Following the notation of [8], u satisfies a quasilinear elliptic equation (of divergence structure) on $\Omega = B_{2R}(x)$ with the structural coefficients

$$a_1(x) = a_2(x) = a_3(x) = a_4(x) = b_0(x) = b_1(x) = b_3(x) = 0,$$

and

$$a_0 = 1, \qquad b_2(x) = u^{(p-1)/2}(x).$$

Thus, for $0 < \delta < 1$, (5.3) holds with M depending boundedly on the quantity

$$R^\delta ||b_2||_{n,\rho^\delta,\Omega} = R^\delta \sup_{\rho > 0, x \in \Omega} \frac{||b_2||_{n,B_\rho(x)\cap\Omega}}{\rho^\delta}.$$

A simple use of Hölder inequality yields

$$\begin{aligned} \rho^{-\delta}||b_2||_{n,B_\rho(x)\cap\Omega} &= \rho^{-\delta}\big(\int_{B_\rho(x)\cap\Omega} u^{n(p-1)/2}(x)\big)^{1/n} \\ &\le \rho^{-\delta}\big(\int_{B_\rho(x)\cap\Omega} u^{n(p-1)/2(1-\delta)}(x)\big)^{(1-\delta)/n}\big(\int_{B_\rho(x)\cap\Omega}\big)^{\delta/n} \\ &\le c_n\big(\int_\Omega u^{n(p-1)/2(1-\delta)}(x)\big)^{(1-\delta)/n} \\ &\le c_n\big(\int_{2R}^{6R} r^{n-1}\int_{S^{n-1}} u^{n(p-1)/2(1-\delta)}(r,\theta)\big)^{(1-\delta)/n} \\ &\le c_n\big(\int_{2R}^{6R} r^{n-1-n/(1-\delta)}\big)^{(1-\delta)/n} = c_n R^{-\delta}, \end{aligned}$$

where we have used Theorem 5.1. Thus

$$R^\delta ||b_2||_{n,\rho^\delta,\Omega} \le c_n$$

is uniformly bounded and so is M. This finishes the proof. □

The following result gives the desired supremum (decay) estimate Theorem 1.3.

Theorem 5.3. *Let $n = 3$ and u a positive solution of* (2.1) *in Z. Then there exists a constant $M = M(q, u)$ such that*

$$||u||_{L^\infty, B_R(x)} \le M|x|^{-\alpha}, \qquad R \ge M,$$

where $4R = |x|$ and $B_R(x)$ is the ball centered at x with radius R.

This theorem, in fact, is half of Theorem 5.2.

6. The case of $n > 3$. In this section, we consider the case of $n > 3$. The results here are parallel to the previous ones, though an additional condition is needed.

First, let us recall that Lemma 2.1 remains valid with the exponent $p + 1$ if we assume that there exists a constant $M = M(u)$ such that

$$\int_{B_r} u^{p+1}(y)dy \le Mr^{n-2(p+1)/(p-1)}, \qquad r \ge M. \tag{6.1}$$

Lemma 6.1. *Let u be a solution of* (2.1) *in Z. Suppose* (6.1) *holds. Then there exists a constant $M = M(u) > 0$ such that*

$$\int_{S^{n-1}} u^{p+1}(r, \theta) \le Mr^{-2(p+1)/(p-1)}.$$

One can introduce the $Z_{K,q,s}$ class on S^{n-1} $(n > 3)$ similarly. The following result is an analogue of Lemma 3.1.

Lemma 6.2. *Let $n > 3$. Suppose that $q > 1$, $s > 1$ and $K > 0$ are three numbers satisfying*

$$q < \frac{n+1}{n-3}, \qquad s > \frac{n-3}{n-1} \cdot (q+1).$$

Then for $v(\theta) \in Z_{K,q,s}$, there exists a constant $M = M(q, K) > 0$ such that

$$||\nabla_\theta(|v|^{s/2})||_2^2 \le M\big(||v||_{q+1}^{\rho} + ||v||_{q+1}^{q+s-1}\big),$$

where

$$\rho = \frac{(q+1)(2s+n-1-(n-3)q)}{n+1-(n-3)q}.$$

The proof of Lemma 6.2 is exactly the same as that of Lemma 3.1. However, the result here is weaker because the embedding

$$H^1(S^{n-1}) \hookrightarrow L^q(S^{n-1})$$

is compact if and only if $q < m$.

We now prove Theorem 1.4.

Proof of Theorem 1.4. It is clear that two key ingredients in the proofs of Theorems 1.2 and 1.3 for $n = 3$ are the uniform L^p-estimates on S^2 (Lemma 2.1) and the embedding in $Z_{K,q,s}$ (Lemma 3.1) with $q = p$. Since Lemma 6.1 gives the uniform L^{p+1}-estimates on S^{n-1} and Lemma 6.2 gives the embedding in $Z_{K,q,s}$ with $q = p+1$, we immediately conclude the first part of the theorem.

To prove the second part, consider

$$\Delta u + u^p = 0, \qquad x \in \mathbf{R}^{n-1}. \tag{6.2}$$

By a result due to Fowler, equation (6.2) possesses infinitely many positive (radial) C^2–solutions (see the appendix) since

$$p \geq \frac{n+1}{n-3} = \frac{(n-1)+2}{(n-1)-2}, \qquad n > 3.$$

Let $U(x) = U(|x|)$ be a non-trivial radial solution of (6.2) which is symmetric about the origin (for simplicity). Set

$$u(y) = U(x) = U(|x|), \qquad y = (x, x_n).$$

Obviously u is a (non-symmetric) solution of (2.1). We need to show that $u \in Z$ and (1.4) (or (6.1)) is satisfied.

It is easy to see that

$$|y|u'(y) = y \cdot \nabla u = y \cdot \nabla U(|x|) = y \cdot \frac{(x,0)}{|x|} U'(|x|) = |x|U'(|x|) \leq 0,$$

that is, $u \in Z$.

To show (1.4), we first consider $p = (n+1)/(n-3)$. Then U has the form

$$U(|x|) = (a + b|x|^2)^{-(n-3)/2}. \tag{6.3}$$

Using (6.3), we obtain for any $R > 0$ that

$$\begin{aligned} \int_B u^{p+1}(y)dy &= \int_B (a+b|x|^2)^{-(n-1)} dy \\ &\leq 2R \int_{B'} (a+b|x|^2)^{-(n-1)} dx \leq c_n R = c_n R^{n-2(p+1)/(p-1)}, \end{aligned}$$

where $B = B_R(0) \subset \mathbf{R}^n$ and $B' = B'_R(0) \subset \mathbf{R}^{n-1}$ are balls centered at the origin with radius R in $\mathbf{R}^n$ and $\mathbf{R}^{n-1}$ respectively.

If $p > (n+1)/(n-3)$, we again obtain

$$\begin{aligned} \int_B u^{p+1}(y)d &\leq M \int_B |x|^{-2(p+1)/(p-1)} dy \leq 2MR \int_{B'} |x|^{-2(p+1)/(p-1)} dx \\ &\leq c_n MR \cdot R^{n-1-2(p+1)/(p-1)} = c_n M R^{n-2(p+1)/(p-1)}, \end{aligned}$$

since $n-1 > 2(p+1)/(p-1)$ (p supercritical for $n-1$).

Finally we need to show that neither (1.1) nor (1.3) holds for $u(y)$. Take a sequence $\{x_n^j\} \to \infty$ and denote $4R_j = |x_n^j|$. Take any $x^j \in \mathbf{R}^{n-1}$ with $|x^j| = R_j$ for each j. Put

$$y_0^j = (0, x_n^j), \qquad y^j = (x^j, x_n^j).$$

Then

$$y^j \in B_{R_j}(y_0^j);$$

but,

$$u(y_0^j) = U(0) > 0, \qquad u(y^j) = U(|x^j|) \le |x^j|^{-2/(p-1)} \to 0.$$

Thus neither (1.1) nor (1.3) can hold and the proof is complete now.

7. Appendix. In this appendix, for the sake of completeness, we include a simple proof of a classic existence result of Fowler. Specifically, we shall prove existence of positive solutions of (2.1) and obtain some asymptotic estimates for radial solutions of (2.1) when p is supercritical. The tools here are the Pohozaev identity and a shooting method of ordinary differential equations.

Before stating the existence theorem, we introduce the initial value problem

$$u''(r) + \frac{n-1}{r}u'(r) + u^p(r) = 0, \quad u(0) = \xi, \quad u'(0) = 0. \tag{IVP}$$

The local existence and uniqueness of solutions of (IVP) is assured by standard theory. Moreover, any solution u of (IVP) must be decreasing as long as $u > 0$. We now give the existence theorem.

Theorem 7.1. *Let $n > 2$ and $p \ge (n+2)/(n-2)$. Then for each $\xi > 0$ equation (2.1) has a unique radial ground state $u(r)$ with central value $u(0) = \xi$ (that is, $|u|_{L^\infty} = \xi$).*

Proof. The proof is a combination of the shooting arguments and the Pohozaev identity ([5]). Consider solutions of (IVP). Our main goal is to show that, for all initial values $\xi > 0$, the solution $u(r)$ of (IVP) can never reach zero at a finite value of r, i.e., $u(r)$ exists and stays positive for all $r > 0$. If this is done, then the theorem will be proved, since u must be a positive solution of (2.1) with $|u|_{L^\infty} = \xi$.

To prove that $u(r)$ cannot reach zero at a finite point, suppose for contradiction that $u(r)$ attains zero at a finite value of r. Put

$$R = R(\xi) = \inf\{\rho > 0 : \ u(0) = \xi, \ u(\rho) = 0\}.$$

Then u is a positive solution of (2.1) with $\Omega = B_R(0)$. By the standard Pohozaev identity, we have

$$R\int_{\partial\Omega} |\nabla u|^2 = \left[\frac{2n}{p+1} - (n-2)\right]\int_\Omega u^{p+1}.$$

Obviously the right side is nonpositive since $p \geq (n+2)/(n-2)$, while the left is strictly positive by strong maximal principle. This is a contradiction and the theorem is proved.

Positive radial solutions of (2.1) are well understood now, including their asymptotic behaviors. Here we display a global estimate for radial solutions, which is needed in Section 7, with a sketch of proof. Apparently, any such solution is a positive solution of (IVP).

Lemma 7.1. *Let u be a positive solution of* (IVP). *Then we have*

$$u' < 0, \qquad \text{for all } r > 0$$

and

$$u(r) \leq \left(\frac{2n}{p-1}\right)^{\alpha/2} r^{-\alpha}. \tag{7.1}$$

Proof. The reader may prove the first part using the fact that any critical point of u for $r > 0$ must be a strict maximum.

To prove the second part, since $u' < 0$ (so u is decreasing), we get

$$r^{n-1}|u'(r)| \geq \int_0^r s^{n-1} u^p \geq u^p(r) \int_0^r s^{n-1} = \frac{u^p(r) r^n}{n}.$$

Therefore

$$\frac{-u'}{u^p} \geq \frac{r}{n}, \qquad r > 0.$$

Now integrate from 0 to $r > 0$ to obtain

$$\frac{u^{1-p}(t)}{p-1} \Big|_0^r \geq \frac{t^2}{2n} \Big|_0^r,$$

which yields

$$u^{1-p}(r) \geq \frac{p-1}{2n} r^2$$

and (7.1) follows. The lemma is proved.

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