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ON A SEMILINEAR ELLIPTIC SYSTEM

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1. Introduction. In [9, 7, 11] the following system was studied:

$$\text{(I)} \quad \begin{cases} -\Delta v = H_u(u,v), & \text{in } \Omega, \qquad (1.1)\\ -\Delta u = H_v(u,v), & \text{in } \Omega, \qquad (1.2)\\ u = v = 0, & \text{on } \partial\Omega, \qquad (1.3) \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ (to be specified later) and $H \in C^1(\mathbb{R}^2;\mathbb{R})$ satisfies appropriate growth conditions. Solutions were obtained by means of a variational principle. Indeed (1.1) and (1.2) are the Euler-Lagrange equations of the Lagrangian

$$\mathcal{L}(z) = \int_\Omega \nabla u \nabla v - \int_\Omega H(u,v). \tag{1.4}$$

Suppose H_u and H_v satisfy the growth conditions

$$|H_u(u,v)| \le c_1 + c_2|u|^p + c_3|v|^{(q+1)\frac{p}{p+1}}, \quad |H_v(u,v)| \le c_4 + c_5|v|^q + c_6|u|^{(p+1)\frac{q}{q+1}}, \tag{1.5}$$

with

$$p,q>1, \quad \frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}, \quad N \ge 1, \tag{1.6}$$

with c_1-c_6 positive constants. Then the functional $\mathcal{L}$ is of class C^1 on

$$D((-\Delta)^\alpha) \times D((-\Delta)^{1-\alpha}), \tag{1.7}$$

for some $\alpha \in (0,1)$, depending on p and q, where $(-\Delta)^\alpha$ is the fractional power of the selfadjoint operator $-\Delta$ with domain $W^{2,2} \cap W_0^{1,2}(\Omega) \subset L^2(\Omega)$. The shortcoming of this approach is the fact that one is not able to formulate the problem variationally

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in the case $q < 1$ and $p > \frac{N+4}{N-4}$ (also with p and q interchanged); that means not all pairs (p, q) in the region

$$1 > \frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}, \quad p, q > 0, \quad N \geq 1 \tag{1.8}$$

(see figure 1) can be reached in this way. The first inequality in (1.8) is the superlinearity of H_z and the second inequality the subcriticality. It is clear that one cannot use the spaces given in (1.7) (see for instance [11], [9]). In [7], however, Hilbert spaces were used in the case $\Omega = B_R$ and the solutions are radially symmetric.

In this paper we shall define $\mathcal{L}$ on a Banach space so that $\mathcal{L}$ is a C^1-function for Hamiltonians H satisfying (1.5) and (1.8). We use the product Sobolev spaces

$$E_r = W_0^{1,r}(\Omega) \times W_0^{1,s}(\Omega), \quad \frac{1}{r} + \frac{1}{s} = 1, \tag{1.9}$$

which are compactly embedded into $L^\alpha(\Omega) \times L^\beta(\Omega)$, with $1 < \alpha < \frac{rN}{N-r}$ and $1 < \beta < \frac{sN}{N-s}$, $N > \max(r, s)$. If either $N \leq r$ or $N \leq s$ we have that the embeddings are compact for $1 < \alpha < \infty$ or $1 < \beta < \infty$. Thus for any p and q satisfying (1.6), one can find an $r \in [\frac{N}{N-1}, N]$, $N \geq 2$, such that the embedding of E_r into $L^{p+1}(\Omega) \times L^{q+1}(\Omega)$ is compact (in the case $N = 1$ one uses $(W_0^{1,2}(\Omega))^2$).

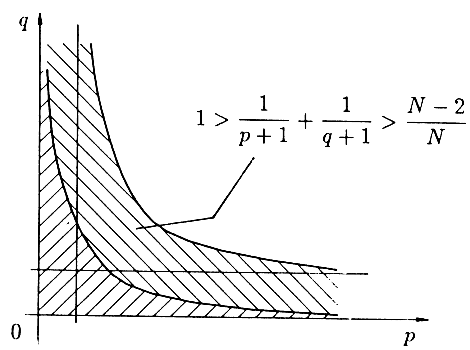

Figure 1. The subcritical region.

A disadvantage of using E_r instead of (1.7) is that the critical point theorem of Benci and Rabinowitz (The Indefinite Functional Theorem) [5] is no longer applicable. If we assume that H is a strictly convex function we can use a dual method for finding weak solutions of Problem (I) as critical points of some functional (see Benci and Fortunato [4] and Ekeland and Temam [8]). In order to find critical points of the dual functional we prove an abstract critical point theorem in Section 3, which is a slight extension of a theorem by Benci and Fortunato [4]. In Section 4 we shall use this theorem to prove an existence result concerning weak solutions of Problem (I). Before stating our main result we first introduce the concept of weak solutions of Problem (I).

Suppose H satisfies (1.5) and (1.8). Then a pair $(u, v)^t \in E_r$ is called a weak solution of Problem (I) if

$$\text{(W)} \qquad \int_\Omega \nabla\phi \nabla v = \int_\Omega H_u(u, v)\phi, \quad \int_\Omega \nabla u \nabla\psi = \int_\Omega H_v(u, v)\psi,$$

for all pairs $(\phi, \psi)^t \in E_r$. Our main existence result follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain with boundary $\partial\Omega$ of class C^1 and suppose H satisfies the hypotheses*

(a1) $c_1|u|^p \leq |H_u(u,v)| \leq c_2|u|^p + c_3 \sum_k |u|^{\alpha_k - 1}|v|^{\beta_k}$, $H(0,0) = 0$,

$c_4|v|^q \leq |H_v(u,v)| \leq c_5 \sum_k |u|^{\alpha_k}|v|^{\beta_k - 1} + c_6|v|^q$, $k = 1, \dots, m$, *with*

$1 > \frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}$, $p, q > 0$, $\frac{\alpha_k}{p+1} + \frac{\beta_k}{q+1} = 1$, $\alpha_k, \beta_k > 1$;

(a2) $H_z(z)$ *is strictly monotone;*

(a3) $\theta_1 u H_u(u,v) + \theta_2 v H_v(u,v) - H(u,v) \geq c_7|u|^{p+1} + c_8|v|^{q+1} - c_9$, *with* $\theta_1 + \theta_2 = 1$, $\theta_1, \theta_2 > 0$,

for some positive constants c_1-c_9. *Then Problem* (I) *has at least one nontrivial weak solution* $(u,v)^t \in E_r$, *for some* $r \in [\frac{N}{N-1}, N]$, *depending on* p *and* q.

If we assume additional regularity on the boundary $\partial\Omega$ we even have strong solutions of Problem (I).

Corollary 1.2. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$ be a bounded domain with boundary $\partial\Omega$ of class C^2 and let H satisfy the same hypotheses as in Theorem* 1.1. *Then Problem* (I) *has a nontrivial strong solution* $(u,v)^t \in \big(W^{2,\frac{p+1}{p}}(\Omega) \cap W_0^{1,\frac{p+1}{p}}(\Omega)\big) \times \big(W^{2,\frac{q+1}{q}}(\Omega) \cap W_0^{1,\frac{q+1}{q}}(\Omega)\big)$.

Theorem 1.1 and Corollary 1.2 concern the superlinear linear case. One can also obtain an existence result in the sublinear case. This means that $p, q > 0$ satisfy

$$\frac{1}{p+1} + \frac{1}{q+1} > 1. \tag{1.10}$$

We therefore introduce a variant of Hypothesis (a1):

(a1′) $c_1|u|^p \leq |H_u(u,v)| \leq c_2|u|^p$, $H(0,0) = 0$, $c_3|v|^q \leq |H_v(u,v)| \leq c_4|v|^q$, with $\frac{1}{p+1} + \frac{1}{q+1} > 1$, $p, q > 0$, $c_1, \dots, c_4$ positive constants.

In this case we have the following result.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain with boundary $\partial\Omega$ of class C^1 and suppose H satisfies hypotheses* (a1′) *and* (a2). *Then Problem* (I) *has at least one nontrivial weak solution* $(u,v)^t \in E_r$, *for some* $r \in [\frac{N}{N-1}, N]$, *depending on* p *and* q.

Remark 1.4. In the case $\Omega \subset \mathbb{R}^N$ is a bounded domain, with boundary $\partial\Omega$ if class $C^{2,\lambda}$, for some $\lambda \in (0,1)$ and $H \in C^{1,\mu}(\mathbb{R}^2, \mathbb{R})$, for some $\mu \in (0,1)$ satisfying (a1)–(a3), with $p, q > 1$ and $\alpha_k, \beta_k > 2$, $k = 1, \dots, m$, solutions are classical, i.e., $(u,v)^t \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^2$. The proof can be found in [11].

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2. Preliminaries. As briefly indicated in Section 1, the functional $\mathcal{L}$ is a C^1-function on $E_r = W_0^{1,r}(\Omega) \times W_0^{1,s}(\Omega)$, for some $r \in [\frac{N}{N-1}, N]$, depending on the growth conditions on H. Because r and s are conjugate to each other we have, related to $Q(z) = \int_\Omega \nabla u \nabla v$, the equivalent norm on E_r,

$$c_1 \|z\|_{E_r} \le \sup_{\|w\|_{E_r}=1} |Q'(z)w| \le c_2 \|z\|_{E_r}, \tag{2.1}$$

where $\|z\|^2_{E_r} = \|u\|^2_{W_0^{1,r}} + \|v\|^2_{W_0^{1,s}}$. Indeed from [15] we have

$$c\|u\|_{W_0^{1,r}} \le \sup_{\|\phi\|_{W_0^{1,s}}=1} \Big| \int_\Omega \nabla u \nabla \phi \Big| \le c' \|u\|_{W_0^{1,r}},$$

which gives

$$\begin{aligned} c^2\|u\|^2_{W_0^{1,r}} + c^2\|v\|^2_{W_0^{1,s}} &\le \Big(\sup_{\|w_2\|_{W_0^{1,s}}=1} \Big| \int_\Omega \nabla u \nabla w_2 \Big| \Big)^2 + \Big(\sup_{\|w_1\|_{W_0^{1,r}}=1} \Big| \int_\Omega \nabla w_1 \nabla v \Big| \Big)^2 \\ &= \Big(\sup_{\substack{w=(0,w_2)\\ \|w\|_{E_r}=1}} |Q'(z)w| \Big)^2 + \Big(\sup_{\substack{w=(w_1,0)\\ \|w\|_{E_r}=1}} |Q'(z)w| \Big)^2 \le 2 \Big(\sup_{\|w\|_{E_r}=1} |Q'(z)w| \Big)^2. \end{aligned}$$

The right hand side of (2.1) is proved similarly. The estimate in the book of Simader [15] requires that $\partial\Omega$ is of class C^1. Furthermore $-\Delta : W_0^{1,r}(\Omega) \to W^{-1,r}(\Omega)$ is an isomorphism (see [15]). From the Rellich-Kondrachov Embedding Theorem [1] we also have that the embeddings

$$i_r : W_0^{1,r}(\Omega) \hookrightarrow L^{p+1}(\Omega), \quad i_s : W_0^{1,s}(\Omega) \hookrightarrow L^{q+1}(\Omega) \tag{2.2}$$

are compact if $1 < p+1 < \frac{rN}{N-r}$, $N > r$ and $1 < q+1 < \frac{sN}{N-s}$, $N > s$. We obtain the scheme

$$L^{1+1/q}(\Omega) \hookrightarrow W^{-1,r}(\Omega) \overset{(-\Delta_0)^{-1}}{\longrightarrow} W_0^{1,r}(\Omega) \hookrightarrow L^{p+1}(\Omega), \tag{2.3}$$

which shows that the composition operator

$$T_s := i_r \cdot (\Delta_0)^{-1} \cdot i_s^* : \ L^{1+1/q}(\Omega) \longrightarrow L^{p+1}(\Omega) \tag{2.4}$$

is compact, i.e., $T_s \in K(L^{1+1/q}, L^{p+1})$. From (2.4) it follows that

$$\partial_T = \begin{pmatrix} 0 & T_r^* \\ T_r & 0 \end{pmatrix} = \begin{pmatrix} 0 & T_s \\ T_r & 0 \end{pmatrix} : X_r \to X_r^* \tag{2.5}$$

is also compact, where $X_r = L^{1+1/p}(\Omega) \times L^{1+1/q}(\Omega)$.

Remark 2.1. Concerning the embedding of $W_0^{1,r}(\Omega)$ into $L^{p+1}(\Omega)$ we also have that $W_0^{1,r}(\Omega) \hookrightarrow L^{p+1}(\Omega)$, compactly if $1 < p+1 < \infty$, when $N \le r$ (see for instance [1]). Thus for $N \ge 2$, when p, q satisfies (1.8), one can always find an $r \in [\frac{N}{N-1}, N]$ such that $E_r \hookrightarrow L^{p+1} \times L^{q+1}$ is compact. In the case $N = 1$ one uses $(W_0^{1,2}(\Omega))^2$, which is compactly embedded into $(L^\infty(\Omega))^2$. The dimension N is therefore no restriction.

As mentioned also in Section 1, we shall use a dual method in order to obtain solutions of Problem (I). This can be best illustrated by means of the special case

$$\text{(II)} \quad \begin{cases} -\Delta v = u|u|^{p-1} = \phi_p(u), & \text{in } \Omega, \quad (2.6)\\ -\Delta u = v|v|^{q-1} = \phi_q(v), & \text{in } \Omega, \quad (2.7)\\ u = v = 0, & \text{on } \partial\Omega. \quad (2.8)\end{cases}$$

By setting $-\Delta v = f$ and $-\Delta u = g$ equations (2.6) and (2.7) can be inverted as

$$\phi_p^{-1}(f) = (-\Delta_0)^{-1} g, \qquad \phi_q^{-1}(q) = (-\Delta_0)^{-1} f.$$

A straightforward computation shows that the latter equations are the Euler-Lagrange equations associated with the functional

$$F(f,g) = \int_\Omega \Phi^*(f,g) - \int_\Omega f(-\Delta_0)^{-1} g,$$

where $\Phi(u,v) = \frac{1}{p+1}|u|^{p+1} + \frac{1}{q+1}|v|^{q+1} = H(u,v)$ and Φ^* its Legendre-Fenchel transform [10, 8]. This inversion works because of the special form of Φ.

In Section 4 we consider more general Hamiltonians H (convex however), by studying the functional

$$F(f,g) = \int_\Omega H^*(f,g) - \int_\Omega f(-\Delta_0)^{-1} g.$$

Using a variant of the well known Mountain Pass Theorem we shall prove the existence of a critical point h. Finally, by setting $z = \partial_T h$, we find a weak solution of Problem (I).

3. A critical point theorem. In this section we establish a critical point theorem for functionals F on a real Banach space

$$X = X_1 \times X_2, \tag{3.1}$$

where X_1 and X_2 are two real reflexive Banach spaces. Elements of X are denoted by $h = (f,g)^t$, with $f \in X_1$, $g \in X_2$ and $\|h\|_X^2 = \|f\|_{X_1}^2 + \|g\|_{X_2}^2$. Let $T : X_1 \to X_2^*$ be a compact linear operator, not identically zero and X_2^* the norm dual of X_2. The functional F is of the form

$$F(h) = A(h) + B(h), \quad F : X \to \mathbb{R}. \tag{3.2}$$

The functional A satisfies
(H1)
(a) $A \in C^1(X,\mathbb{R})$ is a strictly convex functional and $A' = \mathbf{a}$, with $\mathbf{a}(0) = 0$, is a bounded homeomorphism from X to X^* (see Remark 3.1).

(b) If $\{z_n\} \subset X^*$, $z \in X^*$ is such that $\langle \mathbf{a}^{-1}(z_n) - \mathbf{a}^{-1}(z), z_n - z\rangle \to 0$ as $n \to \infty$, then $\{z_n\}$ has a convergent subsequence converging to z in X^*.

(c) There are strictly convex functions $\phi_1, \dots, \phi_4$ and positive constants $c_1, \dots, c_4$ satisfying $\phi_1(t)/t^a \to c_1$, $\phi_2(t)/t^b \to c_2$, as $t \to \infty$, $\phi_1(0) = \phi_2(0) = 0$, $\phi_3(t)/t^a \to c_3$, $\phi_4(t)/t^b \to c_4$, as $t \to 0$, for some $a, b > 0$, $\frac{1}{a} + \frac{1}{b} < 1$, such that $\phi_1(\|u\|_{X_1^*}) + \phi_2(\|v\|_{X_2^*}) \le A^*(z) \le \phi_3(\|u\|_{X_1^*}) + \phi_4(\|v\|_{X_2^*})$, $\forall z \in X^*$.

(d) $|\langle \mathbf{a}^{-1}(z), \Theta z\rangle - A^*(z)| \to \infty$ as $\|z\|_{X^*} \to \infty$, where $\Theta z = (\theta_1 u, \theta_2 v)^t$, with $\theta_1, \theta_2 > 0$, $\theta_1 + \theta_2 = 1$.

The functional B is of quadratic type and is of the special form

(H2) $B(h) = -\langle Tf, g\rangle = -\langle f, T^* g\rangle$, where T is as defined above and $T^* : X_2 \to X_1^*$ is the adjoint of T.

Remark 3.1. A homeomorphism $\mathbf{a}$ is called bounded if $\mathbf{a}$ and $\mathbf{a}^{-1}$ maps bounded sets into bounded sets. Furthermore, the elements of X^* will be denoted by $z = (u, v)^t$, with $u \in X_1^*$ and $v \in X_2^*$.

We present here a slight extension of the critical point theorem due to Benci and Fortunato [4]. The extension is twofold in the sense that the theorem is valid for Banach spaces and the functional A satisfies less restrictive hypotheses (see (c) and (d)), which is essential for our application to Problem (I).

Theorem 3.2. *Let the functional $F = A + B$ be as above where A and B satisfy respectively hypotheses* (H1) *and* (H2). *Then F possesses at least one nontrivial critical point in X.*

Proof of Theorem 3.2. In order to prove Theorem 3.2 we use a variant of the Mountain Pass Theorem of Ambrosetti and Rabinowitz [2] due to Bartolo, Benci and Fortunato (see [4, 3]). This variant allows a weaker form of the (PS) condition but requires the reflexivity of the underlying Banach space. This condition reads

$$(C)\quad \begin{cases} (i) & \text{every bounded sequence } \{h_n\} \subset F^{-1}((0, +\infty)), \text{ for which } \{F(h_n)\} \\ & \text{is bounded and } F'(h_n) \to 0, \text{ possesses a convergent subsequence;} \\ (ii) & \text{for all } c \in (0, +\infty), \text{ there is } \epsilon, R, \beta > 0, \text{ such that for all} \\ & h \in F^{-1}([c - \epsilon, c + \epsilon]),\ \|h\|_X \ge R : \|F'(h)\|_{X^*}\|h\|_X \ge \beta. \end{cases}$$

First we verify that (C) holds.

(i) Let $\{h_n\}$ be a bounded sequence in X, such that $\omega_n := F'(h_n) \to 0$, as $n \to \infty$. Set

$$A'(h_n) = \omega_n + \partial_T h_n, \tag{3.3}$$

where $\partial_T : X \to X^*$ is defined by

$$\partial_T = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}. \tag{3.4}$$

Since $\{h_n\}$ is bounded and the space X is reflexive, we may assume without loss of generality that $h_n \rightharpoonup h$[1] weakly in X. Hence by using the compactness of ∂_T we obtain

$$\langle h_n - h, A'(h_n) - A'(h)\rangle = \langle h_n - h, \omega_n\rangle + \langle h_n - h, \partial_T h_n\rangle \to 0 \quad \text{as} \quad n \to \infty, \tag{3.5}$$

[1] Subsequences of $\{h_n\}$ are again denoted by $\{h_n\}$.

and thus $\langle h_n - h, A'(h_n) - A'(h)\rangle \to 0$, as $n \to \infty$, or equivalently $\langle \mathbf{a}^{-1}(z_n) - \mathbf{a}^{-1}(z), z_n - z\rangle \to 0$, as $n \to \infty$, with $z_n := \mathbf{a}(h_n) = A'(h_n)$ and $z = \mathbf{a}(h)$. From (H1)(b) it follows then that a subsequence of $\{z_n\}$ converges (strongly) to z in X^* and thus a subsequence of $\{h_n\}$ converges to h in X by the continuity of $\mathbf{a}^{-1}$ (see (H1)(a)).

(ii) As for the second part of condition (C), let $c \in \mathbb{R}$ and take $h \in F^{-1}([c-\epsilon, c+\epsilon])$. Further choose $y = (2\sigma_1 f, 2\sigma_2 g)^t \in X$ with $1 > \sigma_1, \sigma_2 > 0$; then

$$\begin{aligned}2\|h\|_X\|F'(h)\|_{X^*} &\geq \|y\|_X\|F'(h)\|_{X^*} \geq |\langle y, F'(h)\rangle| \\ &= |\langle 2\sigma_1 f, \mathbf{a}_1(h)\rangle + \langle 2\sigma_2 g, \mathbf{a}_2(h)\rangle - \langle h, \partial_T h\rangle|,\end{aligned}$$

if we set $\sigma_1 + \sigma_2 = 1$, using

$$\langle \partial_T h, y\rangle = 2\sigma_1\langle Tf, g\rangle + 2\sigma_2\langle Tf, g\rangle = \langle Tf, g\rangle + \langle f, T^* g\rangle = \langle \partial_T h, h\rangle.$$

Now define

$$q(h) = A(h) - \tfrac{1}{2}\langle h, \partial_T h\rangle - c, \quad p(h) = \sigma_1\langle f, \mathbf{a}_1(h)\rangle + \sigma_2\langle g, \mathbf{a}_2(h)\rangle - A(h) - c,$$

where q satisfies $|q(h)| \leq \epsilon$ by assumption and $\|y\|_X \ \|F'(h)\|_{X'} \geq |\langle y, F'(h)\rangle| = |2(p(h) + q(h)) + 4c|$. We are done if we show that $p(h) \to \infty$,as $\|h\|_X \to \infty$. We have

$$\begin{aligned}|\tfrac{1}{2}\langle y, A'(h)\rangle - A(h)| &= |\tfrac{1}{2}\langle y, \mathbf{a}(h)\rangle - \langle h, \mathbf{a}(h)\rangle + A^*(\mathbf{a}(h))| \\ &= |(\sigma_1 - 1)\langle f, u\rangle + (\sigma_2 - 1)\langle g, v\rangle + A^*(z)| \qquad (3.6)\\ &= |\langle \mathbf{a}^{-1}(z), \Theta z\rangle - A^*(z)|, \quad z = \mathbf{a}(h).\end{aligned}$$

If $\|h\|_X \to \infty$ also $\|z\|_{X^*} \to \infty$ and consequently the right hand side of (3.6) tends to infinity by Hypothesis (H1)(d). This concludes the verification of condition (C). Next we show that the geometric assumption of Theorem 2.4 of [4] is fulfilled.

From (H1)(c) we have

$$\Phi_1^*(h) \geq A(h) \geq \Phi_2^*(h),$$

where $\Phi_1(z) = \phi_1(\|u\|_{X_1^*}) + \phi_2(\|v\|_{X_2^*})$ and $\Phi_2(z) = \phi_3(\|u\|_{X_1^*}) + \phi_4(\|v\|_{X_2^*})$ and its Legendre-Fenchel transforms

$$\Phi_1^*(h) = \phi_1^*(\|f\|_{X_1}) + \phi_2^*(\|g\|_{X_2}), \quad \Phi_2^*(h) = \phi_3^*(\|f\|_{X_1}) + \phi_4^*(\|g\|_{X_2}), \quad h \in X.$$

We consider the set $S = \{h_S = (\rho^{k-1} f, \rho^{l-1} g)^t : h = (f, g)^t \in X, \ \|h\|_X = \rho\}$, where we choose k and l such that $\frac{1}{a'} > \frac{k}{k+l}$, $\frac{1}{b'} > \frac{l}{k+l}$, with a' and b' the conjugates of a and b. For $h \in S$ we obtain

$$F(h_S) \geq A(h_S) - \tfrac{1}{2}\rho^{k+l-2}\|\partial_T\| \ \|h\|_X^2 \geq \phi_3^*(\rho^{k-1}\|f\|_{X_1}) + \phi_4^*(\rho^{l-1}\|g\|_{X_2}) - \tfrac{1}{2}\rho^{k+l}\|\partial_T\|.$$

Using (H1)(c) and assuming $\|h\|_X = \rho < 1$, we obtain for some C_1 and C_2, independent of h, that

$$\begin{aligned}F(h_S) &\geq C_1\rho^{a'(k-1)}\|f\|_{X_1}^{a'} + C_2\rho^{b'(l-1)}\|g\|_{X_2}^{b'} - \tfrac{1}{2}\rho^{k+l}\|\partial_T\| \\ &\geq C_1\rho^{a'k}|\sin\phi|^{a'} + C_2\rho^{b'l}|\cos\phi|^{b'} - \frac{1}{2}\rho^{k+l}\|\partial_T\| \\ &\geq \rho^m\big(C_1|\sin\phi|^{a'} + C_2|\cos\phi|^{b'}\big) - \frac{1}{2}\rho^{k+l}\|\partial_T\|\end{aligned}$$

for some ϕ depending on h and where m is defined by $m = \max(a'k, b'l)$, which is strictly less than $k+l$ by choice of a' and b'. Hence for ρ_* small enough we have $F(h_S) \geq \gamma > 0$. If we introduce the equivalent norm

$$\|h\|_*^2 = \rho_*^{2k-2}\|f\|_{X_1}^2 + \rho_*^{2l-2}\|g\|_{X_2}^2,$$

then we have $F_\Sigma \geq \gamma > 0$, where $\Sigma = \{h \in X : \|h\|_* = \rho_*\}$.

Finally we can find an $h \in X$ such that $F(h) < 0$ and $\|h\|_* > \rho_*$. Since T is not identically 0, there exist elements $f_+ \in X_1$ and $g_+ \in X_2$ such that $< Tf_+, g_+ > > 0$[2]. Define $h_r = (r^k f_+, r^l g_+)^t$, $r \in (0,\infty)$. If r is sufficiently large then clearly $\|h_r\|_* > \rho_*$ and from (H1)(c) we obtain, for some positive constant $C > 0$, that

$$\begin{aligned} F(h_r) &= A(r^k f_+, r^l g_+) - \tfrac{1}{2}Cr^{k+l} \leq \phi_1^*(r^k\|f_+\|) + \phi_2^*(r^l\|g_+\|) - \tfrac{1}{2}Cr^{k+l} \\ &\leq C_1 r^{a'k}\|f_+\|^{a'} + C_2 r^{b'l}\|g_+\|^{b'} - \tfrac{1}{2}Cr^{k+l} \leq C_1 r^{a'k} + C_2 r^{b'l} - \tfrac{C}{r}^{k+l} \to -\infty, \end{aligned}$$

as $r \to \infty$. This completes the verification of the condition of Theorem 2.4 of [4], hence F possesses a nontrivial critical point.

Remark 3.3. If the operator T in (H2) is identically zero, $z = 0$ is the only critical point (absolute minimum) of F.

Remark 3.4. The hypotheses (H1) of Theorem 3.2 are tailormade for our application to Problem (I). One can also obtain a critical point theorem for F under somewhat milder conditions on A:

(H1′)

(a) $A \in C^1(X, \mathbb{R})$, $A' = \mathbf{a} : X \to X^*$, with $\mathbf{a}(0) = 0$.

(b) If $\{h_n\} \subset X$, $h \in X$ such that $\langle \mathbf{a}(h_n) - \mathbf{a}(h), h_n - h\rangle \to 0$ as $n \to \infty$, then $\{h_n\}$ has a convergent subsequence converging to h in X.

(c) There are strictly convex functions $\phi_1, \dots, \phi_4$ and positive constants $c_1, \dots, c_4$ satisfying $\phi_1(t)/t^\alpha \to c_1$, $\phi_2(t)/t^\beta \to c_2$, as $t \to \infty$, $\phi_1(0) = \phi_2(0) = 0$, $\phi_3(t)/t^\alpha \to c_3$, $\phi_4(t)/t^\beta \to c_4$, as $t \to 0$, for some $\alpha, \beta > 0$, $\frac{1}{\alpha} + \frac{1}{\beta} > 1$, such that $\phi_1(\|f\|_{X_1}) + \phi_2(\|g\|_{X_2}) \leq A(h) \leq \phi_3(\|f\|_{X_1}) + \phi_4(\|g\|_{X_2})$, $\forall h \in X$.

(d) $|\langle \mathbf{a}(h), \Sigma h\rangle - A(h)| \to \infty$ as $\|h\|_X \to \infty$, where $\Sigma h = (\sigma_1 f, \sigma_2 g)^t$, with $\sigma_1, \sigma_2 > 0$, $\sigma_1 + \sigma_2 = 1$.

Because A need not be strictly convex the inversion procedure is not possible in general (see Section 4).

4. Proofs of Theorems 1.1 and 1.3. We consider the functional

$$F(f,g) = \int_\Omega H^*(f,g)dx - \int_\Omega Tf \cdot g dx, \tag{4.1}$$

where H^* is the Legendre-Fenchel transform of H (see for instance [8, 16]) and

$$T = T_r = i_s \cdot (-\Delta_0)^{-1} \cdot i_r^* : \ L^{1+1/p}(\Omega) \longrightarrow L^{q+1}(\Omega),$$

[2]This property is evident since the map $(f,g) \to (f,-g)$ reverses the sign of $B(h) = \langle Tf, g\rangle$.

with $p, q > 0$ satisfying

$$1 > \frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N}. \qquad (4.2)$$

It was shown in Section 2 that the operator T is compact. In order to prove Theorem 1.1 we shall apply Theorem 3.2, with $X_1 = L^{1+1/p}(\Omega)$ and $X_2 = L^{1+1/q}(\Omega)$. Clearly X_1 and X_2 are real reflexive Banach spaces and Hypothesis (H2) is satisfied. For the verification of (H1)(a) we need the following lemmas concerning the differentiability of potentials and the continuity of Nemytsky operators, which are slight extensions of results due to Krasnoselskii [12, 17, 8]. We state without proof.

Lemma 4.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ and $1 < p_1, p_2, q < \infty$. Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying*

$$|\mathbf{f}(u_1, u_2)| \le c_1 + c_2|u_1|^{p_1/q} + c_3|u_2|^{p_2/q}, \quad u_1, u_2 \in \mathbb{R}, \qquad (4.3)$$

for some constants $c_1, c_2, c_3 > 0$. Then the corresponding Nemytsky operator

$$(u_1, u_2) \longrightarrow \mathbf{f}(u_1, u_2),$$

is a well-defined, bounded[3] *and continuous mapping from $L^{p_1}(\Omega) \times L^{p_2}(\Omega)$ into $L^q(\Omega)$.*

Lemma 4.2. *Let Ω, p_1, p_2 be as in Lemma 4.1 and let $\mathbf{F} \in C^1(\mathbb{R}^2; \mathbb{R})$ be such that the partial derivatives $\mathbf{F}_{u_1}$ and $\mathbf{F}_{u_2}$ satisfy (4.3) with respectively $q = \frac{p_1}{p_1-1}$ and $q = \frac{p_2}{p_2-1}$. Then the potential $\mathcal{F} : (u_1, u_2) \to \int_\Omega \mathbf{F}(u_1, u_2)dx$ is well-defined on $L^{p_1}(\Omega) \times L^{p_2}(\Omega)$. Moreover, the operator $\mathcal{F}$ is continuously Fréchet differentiable, with bounded partial derivatives*

$$\mathcal{F}_{u_i}(u_1, u_2)\phi_i = \int_\Omega \mathbf{F}_{u_i}(u_1, u_2)\phi_i dx, \qquad (4.4)$$

for $u_i, \phi_i \in L^{p_i}(\Omega)$, $i = 1, 2$.

Now we proceed with the verification of (H1). We have the following lemma.

Lemma 4.3. *Let $H : \mathbb{R}^2 \to \mathbb{R}$ satisfy assumptions* (a1)–(a3). *Then the functional*

$$A(h) := \int_\Omega H^*(f(x), g(x))dx, \quad h = (f, g)^t, \qquad (4.5)$$

is well-defined on $X = L^{1+1/p}(\Omega) \times L^{1+1/q}(\Omega)$ and satisfies hypotheses (H1).

Proof of Lemma 4.3. We start with the verification of (a). Observe that from (a1) and (a2) it follows that $H_z : \mathbb{R}^2 \to \mathbb{R}^2$ is an homeomorphism. From lemmas 4.1 and

[3] An operator $\mathbf{f} : X \to Y$ is called bounded if it maps bounded sets in X into bounded sets in Y.

4.2 we have that $\mathcal{H} = \int_\Omega H(z(x))dx$ is a C^1-function from $X^* = L^{p_1+1}(\Omega)\times L^{p_2+1}(\Omega)$ to $\mathbb{R}$ and the Legendre-Fenchel transform (see for instance [8, 16, 10]) is given by

$$\mathcal{H}^*(h) = \int_\Omega H^*(h(x))dx = A(h). \tag{4.6}$$

It follows also that $\mathcal{H}'$ is a bijection from X^* to X, which is continuous and bounded. Furthermore, we have that $\mathcal{H}^* = A$ is everywhere Gâteaux differentiable and $(\mathcal{H}')^{-1}(h) = (\mathcal{H}^*)'(h)$, for every $h \in X$. Moreover,

$$\langle(\mathcal{H}^*)'(h), \phi\rangle = \int_\Omega (H_h^*(h(x)), \phi(x))dx,$$

for all $\phi \in X$ and $h \in X$ and the operator $(\mathcal{H}^*)' : X \to X^*$ given by $h \to H_w^*(h)$ is continuous and bounded by lemmas 4.1 and 4.2. This implies that $\mathcal{H}^*$ is continuously Fréchet differentiable.

Next we proceed with the verification of (c). From (a1)–(a2) it follows that

$$d_1|u|^{p+1} + d_2|v|^{q+1} \le H(z) \le d_3|u|^{p+1} + d_4|v|^{q+1}, \quad p, q > 0, \tag{4.7}$$

for some positive constants $d_1 - d_4$. Indeed we have the identity

$$H(z) = \int_0^1 \frac{d}{ds}H(sz)ds = \int_0^1 (\nabla H(sz), sz)\frac{ds}{s} = \int_0^1 H_z(sz)sz\frac{ds}{s}, \quad z \in \mathbb{R}^2.$$

Using the strict monotonicity of H_z (condition (a2)) and (a1) we obtain

$$\begin{aligned} H(z) &= \int_0^1 H_u(sz)su\frac{ds}{s} + \int_0^1 H_v(sz)sv\frac{ds}{s} \\ &= \int_0^1 |H_u(sz)||su|\frac{ds}{s} + \int_0^1 |H_v(sz)||sv|\frac{ds}{s}. \end{aligned}$$

Substituting the estimates of (a1) and using the equivalence of norms in $\mathbb{R}^2$ yield (4.7) and consequently (H1)(c) taking $\phi_1(t) = d_1|t|^{p+1}$, $\phi_2(t) = d_2|t|^{q+1}$, $\phi_3(t) = d_3|t|^{p+1}$, $\phi_4(t) = d_4|t|^{q+1}$ with $a = p+1$, $b = q+1$. This is possible by virtue of (a1).

As for condition (d) we have

$$\begin{aligned} |\langle \mathbf{a}^{-1}(z), \Theta z\rangle - A^*(z)| &= \Big|\int_\Omega \big(\theta_1 uH_u(u,v) + \theta_2 vH_v(u,v) - H(u,v)\big)dx\Big| \\ &\ge c_7\int_\Omega |u|^{p+1} + c_8\int_\Omega |v|^{q+1} - c_9', \end{aligned}$$

by (a3), which completes the verification of (H1)(d).

Finally we verify (b). Suppose

$$\langle \mathbf{a}^{-1}(z_n) - \mathbf{a}^{-1}(z), z_n - z\rangle = \int_\Omega \big(H_z(z_{n_k}) - H_z(z)\big)\cdot\big(z_{n_k} - z\big)dx$$

tends to zero as $n \to \infty$. Since the integrand is non-negative (condition (a2)), there is a nonnegative function χ_1 and a subsequence, still denoted by $\{z_n\}$, such that

$$\big(H_z(z_{n_k}) - H_z(z)\big) \cdot \big(z_{n_k} - z\big) \to 0, \quad \text{a.e. in } \Omega$$

and

$$\big(H_z(z_{n_k}) - H_z(z)\big) \cdot \big(z_{n_k} - z\big) \le \chi_1, \quad \text{in } \Omega.$$

The strict monotonicity of H_z (condition (a2)) now yields $z_n \to z$ almost everywhere in Ω. We are done if we show that

$$|u_n|^{p+1} \le \chi_2 \in L^1, \quad |v_n|^{q+1} \le \chi_3 \in L^1,$$

for some functions χ_2, χ_3. By (a1) we have

$$\begin{aligned} &c_1|u_n|^{p+1} + c_4|v_n|^{q+1} \le |H_u(z_n)||u_n| + |H_v(z_n)||v_n| = H_u(z_n) \cdot z_n \\ &\le \chi_1 + H_z(z) \cdot z_n + H_z(z_n) \cdot z - H_z(z) \cdot z \le \chi_1 + H_z(z) \cdot z_n + H_z(z_n) \cdot z \\ &= \chi_1 + |H_u(z)||u_n| + |H_v(z)||v_n| + |H_u(z_n)||u| + |H_v(z_n)||v| \\ &\le \chi_1 + c_2|u|^p|u_n| + c_3\sum_k |u|^{\alpha_k - 1}|u_n||v|^{\beta_k} + c_5\sum_k |u|^{\alpha_k}|v|^{\beta_k - 1}|v_n| + c_6|v|^q|v_n| \\ &\quad + c_2|u_n|^p|u| + c_3\sum_k |u_n|^{\alpha_k - 1}|u||v_n|^{\beta_k} + c_5\sum_k |u_n|^{\alpha_k}|v_n|^{\beta_k - 1}|v| + c_6|v_n|^q|v| \\ &\le \chi_1 + \epsilon|u_n|^{p+1} + \epsilon|v_n|^{q+1} + C_\epsilon|u|^{p+1} + C_\epsilon|v|^{q+1}. \end{aligned} \tag{4.8}$$

Taking ϵ small enough in (4.8) yields

$$|u_n|^{p+1} + |v_n|^{q+1} \le C\chi_1 + C'|u|^{p+1} + C''|v|^{q+1}.$$

This concludes the verification of (H1)(b) and thus the proof of Lemma 4.3. □

For the functional (4.1) hypotheses (H1) and (H2) are satisfied by virtue of lemma 4.3 and section 2. We can therefore apply Theorem 3.2 and we obtain a nontrivial critical point $h \in L^{1+1/p}(\Omega) \times L^{1+1/q}(\Omega)$ of F, which satisfies the equation

$$\partial_T h = \mathbf{a}(h), \quad \text{in } \ L^{p+1}(\Omega) \times L^{q+1}(\Omega). \tag{4.9}$$

Proof of Theorem 1.1. To prove Theorem 1.1 it is enough to show that h, satisfying (4.9), is a weak solution of Problem (I). Using the properties of A and $\mathbf{a}$ (see (H1)) we can rewrite (4.9) as

$$(\mathcal{H}')^{-1}(h) - \partial_T h = 0, \quad \text{in } \ L^{p+1}(\Omega) \times L^{q+1}(\Omega). \tag{4.10}$$

Define $z = \partial_T h \in L^{p+1}(\Omega) \times L^{q+1}(\Omega)$. Then

$$(\mathcal{H}')^{-1}(h) - z = 0, \quad \text{in } \ L^{p+1}(\Omega) \times L^{q+1}(\Omega). \tag{4.11}$$

From lemma 4.3 we know that $(\mathcal{H}')^{-1}$ is a bounded homeomorphism between $L^{p+1}(\Omega) \times L^{q+1}(\Omega)$ and $L^{1+1/p}(\Omega) \times L^{1+1/q}(\Omega)$, which yields

$$h = \mathcal{H}'(z), \quad \text{in } \ L^{1+1/p}(\Omega) \times L^{1+1/q}(\Omega). \tag{4.12}$$

Because ∂_T^{-1} is a (bounded) isomorphism between $W_0^{1,r}(\Omega)\times W_0^{1,s}(\Omega)$ and $W^{-1,s}(\Omega)$ $\times W^{-1.r}(\Omega)$ and $L^{1+1/p}(\Omega)\times L^{1+1/q}(\Omega)\hookrightarrow W^{-1,s}(\Omega)\times W^{-1.r}(\Omega)$ we have $z\in E_r$ and

$$\langle \partial_T^{-1}z,\phi\rangle = \mathcal{H}'(z)\phi, \quad \forall\ \phi\in W_0^{1,r}(\Omega)\times W_0^{1,s}(\Omega). \tag{4.13}$$

This proves that $z=\partial_T h$ is a weak solution of Problem (I) and a critical point of $\mathcal{L}$. $\square$

For the proof of Corollary 1.2 we only need to prove the additional regularity of weak solutions under the assumption that $\partial\Omega$ is of class C^2.

Proof of Corollary 1.2. From (a1) and the fact that $E_r\hookrightarrow X_r$ we have that $z\in L^{p+1}(\Omega)\times L^{q+1}(\Omega)$ and $H_z(z)\in L^{\frac{p+1}{p}}(\Omega)\times L^{\frac{q+1}{q}}(\Omega)$. By the elliptic regularity this yields that $z\in \big(W^{2,\frac{p+1}{p}}(\Omega)\cap W_0^{1,\frac{p+1}{p}}(\Omega)\big)\times\big(W^{2,\frac{q+1}{q}}(\Omega)\cap W_0^{1,\frac{q+1}{q}}(\Omega)\big)$.

Proof of Theorem 1.3. Observe that it is sufficient to prove that in case H satisfies (a1'), (a2), F, given by (4.1), has a nontrivial critical point $z\in E_r$, for some $r\in[\frac{N}{N-1},N]$ depending on p and q. We have that $F(h)$ is bounded from below on X_r. Indeed from the previous,

$$\begin{aligned} F(h) &\geq C\int_\Omega |f|^{1+1/p}dx + C\int_\Omega |g|^{1+1/q}dx - \int_\Omega (-\Delta_0)^{-1}f\cdot g dx\\ &\geq C\|f\|_{L^{1+1/p}}^{\frac{p+1}{p}} + C\|g\|_{L^{1+1/q}}^{\frac{q+1}{q}} - C\|f\|_{L^{1+1/p}}\|g\|_{L^{1+1/q}}\\ &\geq C\|f\|_{L^{1+1/p}}^{\frac{p+1}{p}} + C\|g\|_{L^{1+1/q}}^{\frac{q+1}{q}} - C\|f\|_{L^{1+1/p}}^{\alpha} - C\|g\|_{L^{1+1/q}}^{\beta}, \end{aligned} \tag{4.14}$$

with $\frac{1}{\alpha}+\frac{1}{\beta}=1$. Since $\frac{p}{p+1}+\frac{q}{q+1}<1$, one can choose $\alpha<\frac{p+1}{p}$ and $\beta<\frac{q+1}{q}$. This yields that $F(h)\geq c$, for some $c\in\mathbb{R}$. It also follows from (4.14) that if $\|h\|_{X_r}\to\infty$ also $F(h)\to\infty$ (coercive). This gives that minimizing sequences $\{h_n\}$, i.e., $F(h_n)\to\inf_{X_r}F$, are bounded in X_r. By the reflexivity of X_r a subsequence $h_n\rightharpoonup h\in X_r$, weakly in X_r. If we prove that F is sequentially weakly lower semicontinuous on X_r, the proof follows, because then

$$F(h)\leq \liminf_{n\to\infty}F(h_n)=\inf_{X_r}F$$

and by the definition of $\inf F$, $F(h)\geq\inf F$, which gives

$$F(h)=\inf_{X_r}F=\min_{X_r}F.$$

By the compactness of ∂_T (see Section 2), $B(h)$ is weakly continuous on X_r. The functional $A(h)$ is strictly convex and continuously Fréchet differentiable on X_r (for the proper r depending on p and q). This implies that A is sequentially weakly lower semicontinuous on X_r (see [10]). This proves also that F is sequentially weakly lower semicontinuous on X_r.

Finally, by choosing numbers k,l such that $\frac{p}{p+1}<\frac{k}{k+l}$ and $\frac{q}{q+1}<\frac{l}{k+l}$, and h^+ as in the proof of Theorem 3.2, we obtain

$$F(r^kf^+,r^lg^+)\leq C_1r^{\frac{p+1}{p}k}+C_2r^{\frac{q+1}{q}l}-C_3r^{k+l},$$

for some positive constants C_1-C_3. By assumption $\frac{p+1}{p}k > k+l$ and $\frac{q+1}{q}l > k+l$, and thus $F(r^k f^+, r^l g^+) < 0$ provided r is small and $\min_{X_r} F < 0$. This completes the proof.

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