Differential and Integral Equations, Volume 8, Number 7, September 1995, pp. 1873 – 1883.

# T-PERIODIC SOLUTIONS FOR A SECOND ORDER SYSTEM WITH SINGULAR NONLINEARITY*

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(Submitted by: A.R. Aftabizadeh)

**Abstract.** We consider a system of the form

$$u'' + au' = H_v(u,v) - h(t)$$
$$v'' + bv' = H_u(u,v) - k(t),$$

where $h, k$ are locally integrable and $T$-periodic, and $H$ is a $C^1$ function defined on $(0,\infty)\times(0,\infty)$, for which a good model is given by

$$H(u,v) = -(\frac{1}{u^\alpha} + \frac{1}{v^\beta}), \quad \alpha, \beta > 0.$$

We state conditions under which existence of positive, $T$-periodic solutions for this system is ensured. We also study the problems of uniqueness and existence of multiple solutions in some special cases.

**1. Introduction.** In [6], Lazer and Solimini considered the problem of finding $T$-periodic, positive solutions of some semilinear second order o.d.e's with a singular nonlinearity of repulsive type, a model being

$$x'' = \frac{1}{x^\nu} - h(t), \tag{1.1}$$

where $\nu \geq 1$ and $h$ is locally integrable and $T$-periodic. In this reference, the authors establish that (1.1) possesses a $T$-periodic solution if and only if $\int_0^T h > 0$. This result was extended by Solimini [8] to the case of a potential system of the form

$$x'' = \nabla V(x) - h(t), \tag{1.2}$$

where a model for the potential $V$ is given by $V(x) = -1/|x|^\alpha$, $\alpha > 0$. Solimini proves existence of a $T$-periodic, nonvanishing solution $x(t)$ to (1.2) provided that $h$ is $T$-periodic and $\int_0^T h \neq 0$.

In [7], Theorem 6.2, Mawhin generalizes Lazer and Solimini's result allowing the presence of a linear term in the first derivative and further growth in the nonlinearity at infinity. We state next his result in full. Let us consider the problem

$$x'' + ax' = g(x) - h(t), \tag{1.3}$$

Then the following result holds.

Received April 1993.
*This work has been partially supported by grants Fondecyt 1209-91 and NSF DMS-9100383.
AMS Subject Classifications: 34B15, 34C25.

**Theorem 1.1.** *Assume that the function $g : (0,+\infty) \to \mathbb{R}$ is continuous and such that the following conditions hold.*

1. $g(x) \geq -Ax - D$, *where* $0 < 2A < (1/T^2)e^{-2|a|T}$, $D \geq 0$,
2. $\lim_{x \downarrow 0} g(x) = +\infty$,
3. $\limsup_{x \to +\infty} g(x) < \bar{h} := 1/T \int_0^T h$,
4. $\lim_{x \downarrow 0} \int_x^1 g(s)ds = +\infty$,

*then equation* (1.3) *possesses at least one positive $T$-periodic solution.*

In this paper we consider the extension of equation (1.1) to certain two by two singular systems enjoying a *Hamiltonian* rather than a *potential* structure in its nonlinearity.

Let $H : (0,+\infty) \times (0,+\infty) \to \mathbb{R}$ be a function of class $C^1$. Then we consider a system of the form

$$u'' + au' = H_v(u,v) - h(t) \tag{1.4}$$

$$v'' + bv' = H_u(u,v) - k(t). \tag{1.5}$$

Here $h, k$ are locally integrable and $T$-periodic, $a, b \in \mathbb{R}$.

A model of the kind of $H$'s we would like to consider is given by

$$H(u,v) = -(\frac{1}{u^\alpha} + \frac{1}{v^\beta}), \quad \alpha, \beta > 0.$$

In fact, for this $H$ and as a consequence of our first result, Theorem 1.2 below, system (1.4)–(1.5) possesses at least one $T$-periodic solution $(u(t), v(t))$ with $u(t), v(t) > 0$, provided that $\int_0^T h$, $\int_0^T k > 0$. More generally, we shall fully extend the result of Theorem 1.1 to this class of systems, assuming the following conditions on $H$.

1. Let us denote $\bar{h} := \frac{1}{T}\int_0^T h$, $\bar{k} := \frac{1}{T}\int_0^T k$. Then
$$\limsup_{y \to +\infty} H_v(x,y) < \bar{h}, \quad \limsup_{x \to +\infty} H_u(x,y) < \bar{k} \tag{H1}$$
respectively uniformly on $x > 0$ and $y > 0$.
2. There exist nonnegative constants $A, B, D$, such that $2\sqrt{AB} < (1/T^2)e^{-(|a|+|b|)T}$ and
$$H_v(x,y) \geq -Ay - D, \quad H_u(x,y) \geq -Bx - D, \tag{H2}$$
for all $x, y > 0$.
3. $$\lim_{x \to 0^+} H_u(x,y) = +\infty, \quad \lim_{y \to 0^+} H_v(x,y) = +\infty \tag{H3}$$
respectively uniformly on $y > 0$ and $x > 0$.
4. $$\lim_{x \to 0^+} H(x,y) = -\infty, \quad \lim_{y \to 0^+} H(x,y) = -\infty \tag{H4}$$
respectively uniformly on $y > 0$ and $x > 0$.

**Theorem 1.2.** *Assume that $H$ satisfies conditions* (H1)–(H4). *Then system* (1.4)–(1.5) *has at least one $T$-periodic solution $(u,v)$ with $u(t), v(t) > 0$ for all $t$.*

Fundamental in our approach is the existence of suitable *a priori* estimates for the solutions of this system, which are found in §2. After these bounds are found, we complete the proof in §3 making use of degree theory.

In §4 we study the problems of uniqueness and multiplicity of $T$-periodic solutions for a special case of system (1.4)–(1.5). Here we consider the system

$$du'' = f(v) - h(t) \tag{1.6}$$

$$dv'' = g(u) - k(t), \tag{1.7}$$

where $d > 0$ and $H(u,v) = \int_1^v f(s)ds + \int_1^u g(s)ds$ satisfies assumptions (H1)–(H4).

As a consequence of the results of that section, we have that if we additionally assume that $f$, $g$ are of class $C^1$ and strictly decreasing, then system (1.6)–(1.7) possesses a unique $T$-periodic solution for all sufficiently large $d$. On the other hand, if we assume instead that $f$ takes the value $\bar{h}$ at least at $n$ points where $f'$ is positive, then system (1.6)–(1.7) will possess at least $2n+1$ solutions for all large $d$. How large $d$ should be taken in both results can also be explicitly estimated.

We end this section by mentioning that several other works exist in the literature concerning this kind of singular behavior of the nonlinearity at zero in the scalar case. We refer the reader to [7], Chapter 6 for further references. See also [3], [4] where other kinds of growths at $+\infty$ are considered, and [2] for the extension of Lazer and Solimini's result to a semilinear Neumann problem in higher dimensions. In [5] existence of subharmonic-type solutions when more than one singularity appears is studied. Semilinear elliptic systems with a *Hamiltonian* nonlinearity have been considered by some authors, see for example [1] and its references.

**2. A priori estimates.** In this section we will find a priori estimates for the positive, $T$-periodic solutions of a system of the form

$$d(u'' + au') = H_v(u,v) - h(t) \tag{2.1}$$

$$d(v'' + bv') = H_u(u,v) - k(t). \tag{2.2}$$

Here $h, k$ are locally integrable and $T$-periodic, $d \geq 1$, $a, b \in \mathbb{R}$.

We want to prove the following result.

**Proposition 2.1.** *Assume that $H$ satisfies assumptions* (H1)–(H4). *Then there exist positive constants $\alpha, \beta$ such that for all $d \geq 1$ and any $T$-periodic solution $(u,v)$ of* (2.1)–(2.2), *one has*

$$\alpha < u(t), v(t) < \beta \quad \textit{for all } t \in [0,T].$$

**Proof.** Clearly, we do not lose generality in assuming $D = 0$ in (H2). Then, from this assumption we have that

$$|H_u(x,y)| \leq H_u(x,y) + 2Bx, \quad |H_v(x,y)| \leq H_v(x,y) + 2Ay. \tag{2.3}$$

Let $(u,v)$ be a positive solution of (2.1)–(2.2). Integrating the equations we obtain

$$\frac{1}{T}\int_0^T H_v(u,v)=\bar{h},\quad \frac{1}{T}\int_0^T H_u(u,v)=\bar{k}. \tag{2.4}$$

(2.1)–(2.2) also imply the relations

$$d\int_0^T |u''+au'|dt \le \int_0^T |H_v(u,v)| + \int_0^T |h(t)|dt$$

and

$$d\int_0^T |v''+bv'|dt \le \int_0^T |H_u(u,v)| + \int_0^T |k(t)|dt,$$

so that using (2.3) and (2.4) we obtain

$$d||u''+au'||_1 \le \int_0^T H_v(u,v)dt + 2A\int_0^T vdt + \|h\|_1 \le 2AT||v||_\infty + C. \tag{2.5}$$

Here and in the rest of the proof we denote by the letters $C$ or $c$ positive constants independent of $d$ and the particular solution $(u,v)$ we are considering, whose values may vary from step to step. We also denote $\|z\|_1=\int_0^T |z(t)|dt$.

A similar argument applied to equation (2.2) yields

$$d\|v''+bv'\|_1 \le 2BT\|u\|_\infty + C. \tag{2.6}$$

The above relations will help us to derive a uniform upper estimate for $u$ and $v$. First we note that since $u$ is periodic, there is a number $\tau_1\in[0,T]$ such that $u'(\tau_1)=0$. On the other hand, since $\frac{1}{T}\int_0^T H_u(u,v)=\bar{k}$, it follows from assumption (H1) that for a certain number $\tau_2\in[0,T]$, one has

$$u(\tau_2)\le C. \tag{2.7}$$

Thus, we have

$$de^{at}u'(t)=\int_{\tau_1}^t e^{at}[u''(s)+au'(s)]ds \tag{2.8}$$

and hence,

$$u(t)=u(\tau_2)+d^{-1}\int_{\tau_2}^t e^{-a\gamma}\int_{\tau_1}^{\gamma} e^{as}[u''(s)+au'(s)]dsd\gamma,$$

so that, using (2.7),

$$|u(t)|\le Te^{2|a|T}\|u''+au'\|_1 + C$$

since $d \geq 1$. Then we conclude from (2.5) that

$$\|u\|_\infty \leq 2AT^2 e^{2|a|T} \|v\|_\infty + C. \tag{2.9}$$

Symmetrically, the above argument provides a relation of the form

$$\|v\|_\infty \leq 2BT^2 e^{2|b|T} \|u\|_\infty + C. \tag{2.10}$$

Combining (2.9) and (2.10) we obtain the inequality

$$\|u\|_\infty \leq 4ABT^4 e^{2(|a|+|b|)T} \|u\|_\infty + C,$$

and therefore $\|u\|_\infty \leq C$, thanks to assumption (H2). An upper bound for $\|v\|_\infty$ is also immediately derived.

Note that relations (2.8) and (2.5) also imply

$$d\|u'\|_\infty \leq 2e^{2|a|T} AT \|v\|_\infty + C,$$

hence a uniform bound for $d\|u'\|_\infty$ follows. Similarly, we also obtain a bound for $d\|v'\|_\infty$. Summarizing, we have shown

$$\|u\|_\infty + \|v\|_\infty + d\|u'\|_\infty + d\|v'\|_\infty \leq C. \tag{2.11}$$

for $d \geq 1$. Thus, to conclude the result of the proposition it only remains to find a uniform lower bound for $u$ and $v$. To do this, we multiply equation (2.1) by $v'$, equation (2.2) by $u'$ and add up the results to obtain

$$d\frac{d}{dt}(u'v') + d(a+b)u'v' = \frac{d}{dt}H(u,v) - (hv' + ku').$$

After integrating this relation between $\tilde{t}$ and $t \in [0,T]$, we get

$$-H(u(t), v(t)) \leq -H(u(\tilde{t}), v(\tilde{t})) + C(d\|u'v'\|_\infty + \|u'\|_\infty + \|v'\|_\infty + 1). \tag{2.12}$$

We claim that we can choose $\tilde{t}$ such that $H(u(\tilde{t}), v(\tilde{t}))$ is uniformly bounded. To see this, it clearly suffices to find $\tilde{t}$ such that both $u(\tilde{t})$ and $v(\tilde{t})$ are bounded below by a uniform positive constant.

From relation (2.4) and assumption (H3) we see that there are points $t_0, \tilde{t}_0$ such that $u(t_0), v(\tilde{t}_0) \geq c > 0$ . Let us assume that $\min u < c/2$ or $\min v < c/2$ (otherwise we are done). Suppose the first case holds; if not, the argument is similar. Let $t_1 > t_0$ be the first point such that $u(t_1) \leq c/2$. Then $u'(t_1) \leq 0$. Also, since $u'$ is uniformly bounded, we have that $t_1 - t_0 > \delta$ for a certain (uniform) positive number $\delta$. From equation (1.4) we obtain

$$u'(t_1) = \int_{t_0}^{t_1} (H_v(u(s), v(s)) - h(s))ds$$

and hence

$$\int_{t_0}^{t_1} (H_v(u(s), v(s)) - \frac{1}{\delta}\|k\|_1) ds \leq 0.$$

(H3) thus implies that there is a number $\tilde{t} \in (t_0, t_1)$ such that $v(\tilde{t})$ is greater than a uniform positive constant. The definition of $t_1$ gives that, also, $u(\tilde{t}) \geq c/2$ and the proof of the claim is complete.

Finally, the existence of a uniform lower bound for $u$ and $v$ follows immediately from this choice of $\tilde{t}$. In fact, inequality (2.12) and the estimates (2.11) imply that

$$-H(u(t), v(t)) \leq C \quad \text{for all } t \in [0, T],$$

and the result follows from assumption (H4). This concludes the proof of the proposition.

**3. Existence result.** In this section we prove our existence result,Theorem 1.2, for system (1.4)–(1.5). The proof we present here follows similar lines to that of Theorem 2.4 in [7], where a more general operator-valued situation is considered. We will carry it out in detail in our specific setting for the reader's convenience.

**Proof of Theorem 1.2.** Let us set

$$x = \begin{bmatrix} u \\ v \end{bmatrix}, \quad F(x) = \begin{bmatrix} H_v(u, v) \\ H_u(u, v) \end{bmatrix}, \quad \text{and} \quad e(t) = \begin{bmatrix} h(t) \\ k(t) \end{bmatrix}.$$

Then system (1.1)–(1.2) can be rewritten as

$$x'' + mx' = F(x) - e(t) \tag{3.1}$$

with $m = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$.

We shall denote in what follows by $C_T^0$ (resp. $L_T^1$) the Banach space of all real, continuous (resp. locally integrable) $T$-periodic functions of the real line endowed with their natural norms. Let us set

$$W := \{w \in C_T^0 \times C_T^0 : \textstyle\int_0^T w = 0\}, \quad \text{and} \quad Q := \{q \in L_T^1 \times L_T^1 : \textstyle\int_0^T q = 0\}.$$

We will also shorten $X := C_T^0 \times C_T^0$. We shall rewrite the problem of finding $T$-periodic solutions to system (3.1) as that of finding fixed points of a certain compact operator on $X$. To define this operator, we denote by $L(q)$, for $q \in Q$, the unique solution $\gamma \in W$ to the equation

$$\gamma'' + m\gamma' = q(t),$$

where $q \in Q$. It is standard to check that $L : q \in Q \mapsto L(q) \in W$ defines a compact linear map. Also, it is easy to see that $x$ is a $T$-periodic solution of (3.1) if and only if

$$0 = \int_0^T (F(x) - e(t)) \quad \text{and} \quad x - \frac{1}{T}\int_0^T x = L(F(x) - e(t)).$$

For $z \in L^1_T \times L^1_T$, let us denote $Pz = z - 1/T \int_0^T z$ and consider for $0 \le \tau \le 1$, the one-parameter family of operators on $X$

$$T_\tau(x) = \tau LP(F(x) - e(t)) + \frac{1}{T}\int_0^T x + \int_0^T (F(x) - e(t))dt. \tag{3.2}$$

Then finding $T$-periodic solutions to system (3.1) is clearly equivalent to finding fixed points for the operator $T_1$. To do this, we will apply degree theory by deforming homotopically $T_\tau$ to $\tau = 0$ on some suitable open subset in $X$.

We note that Proposition 1.1 can be restated by saying that there exist $0 < \alpha < \beta$ such that if $x = (u, v) \in X$ is any solution of

$$x = T_\tau(x) \tag{3.3}$$

for $0 < \tau \le 1$, then

$$\alpha < u(t), v(t) < \beta, \quad t \in [0, T]. \tag{3.4}$$

From the assumptions on $H$, we easily check that $\alpha, \beta$ can also be assumed to satisfy (3.4) when $\tau = 0$. Then we fix such numbers $\alpha, \beta$ and define $\Lambda$ to be the set of all elements $x = (u, v)$ in $X$ which satisfy the restrictions (3.4). Note that $\Lambda$ is an open, bounded set in $X$ and that the map $(x, \tau) \in \bar{\Lambda} \times [0, 1] \mapsto T_\tau(x) \in X$ defines a completely continuous operator.

Since, by definition of $\alpha$ and $\beta$, equation (3.3) does not have solutions on $\partial\Lambda$ for all $\tau \in [0, 1]$, the invariance of the degree under compact homotopies yields

$$\deg_X(I - T_\tau, \Lambda, 0) = c = \text{constant}, \quad \tau \in [0, 1], \tag{3.5}$$

where $\deg_X$ denotes the Leray-Schauder degree in $X$ and $I$ the identity operator. We will find the above degree by computing its value for $\tau = 0$.

Let us denote by $V$ the two-dimensional subspace of all constant functions in $X$. We observe that for $\tau = 0$, equation (3.3) has a solution $x$ if and only if $x \in V$ and $x = \lambda \in \mathbb{R}^2$ satisfies $F(\lambda) - \bar{e} = 0$ where $\bar{e} = 1/T \int_0^T e$.

Since all fixed points of $T_0$ lie on $V$ and $T_0$ applies $V \cap \Lambda$ into $V$, the definition of the degree gives us

$$\deg_X(I - T_0, \Lambda, 0) = \deg_V(I - T_0, \Lambda \cap V, 0). \tag{3.6}$$

Let $\Gamma = (\alpha, \beta) \times (\alpha, \beta)$ . We clearly have

$$\deg_V(I - T_0, \Lambda \cap V, 0) = \deg_{\mathbb{R}^2}(F - \bar{e}, \Gamma, 0). \tag{3.7}$$

Finally, we compute the latter degree. Let us set $\Phi(\lambda_1, \lambda_2) = (\phi_1(\lambda_2), \phi_2(\lambda_1))$, where $\phi_1, \phi_2$ are continuous functions such that $\phi_i(\alpha) > 0 > \phi_i(\beta)$, $i = 1, 2$. A convex homotopy between $F - \bar{e}$ and $\Phi$ immediately yields, after reducing $\alpha$ and enlarging $\beta$ if necessary,

$$\deg_{\mathbb{R}^2}(F - \bar{e}, \Gamma, 0) = \deg_{\mathbb{R}^2}(\Phi, \Gamma, 0).$$

But, setting $\tilde{\Phi}(\lambda_1, \lambda_2) = (\phi_2(\lambda_1), \phi_1(\lambda_2))$, we find

$$\begin{aligned}\deg_{\mathbb{R}^2}(\Phi, \Gamma, 0) &= -\deg_{\mathbb{R}^2}(\tilde{\Phi}, \Gamma, 0) = -\deg_{\mathbb{R}}(\phi_2, (\alpha, \beta), 0)\deg_{\mathbb{R}}(\phi_1, (\alpha, \beta), 0) \\ &= -(-1)(-1) = -1.\end{aligned}$$

Thus, we conclude from this and (3.5)–(3.7) that

$$\deg_X(I - T_1, \Lambda, 0) = -1 \neq 0,$$

and the existence of a fixed point of $T_1$ in $\Lambda$ follows. This concludes the proof.

**4. Multiplicity and uniqueness.** In this section we restrict ourselves to the case of a potential $H$ in problem (1.4)–(1.5) of the form $H(u, v) = F(v) + G(u)$ such that $H$ still satisfies assumptions (H1)–(H4). Thus we consider the system

$$du'' = f(v) - h(t) \tag{4.1}$$

$$dv'' = g(u) - k(t), \tag{4.2}$$

where $h, k$ are locally integrable on the real line and periodic with period $T > 0$. As before, we look for $T$-periodic, positive solutions to system (4.1)–(4.2) when $d > 0$. Our purpose in this section is to show that under certain conditions, additional to those we already impose to obtain existence, we can generate multiplicity or uniqueness results for sufficiently large values of the parameter $d$.

We assume henceforth that $f, g$ are continuous functions defined on $(0, +\infty)$ which additionally satisfy the following conditions.

$$f(s), g(s) \geq 0 \quad \text{for all} \quad s > 0, \tag{4.3}$$

$$\lim_{s \downarrow 0} \int_s^1 f(x)dx = +\infty, \quad \lim_{s \downarrow 0} \int_s^1 g(x)dx = +\infty. \tag{4.4}$$

Setting as before $\bar{h} = (1/T)\int_0^T h(t)dt$, $\bar{k} = (1/T)\int_0^T k(t)dt$, we assume that $f, g$ also satisfy

$$\limsup_{s \to \infty} f(s) < \bar{h}, \quad \limsup_{s \to \infty} g(s) < \bar{k}, \tag{4.5}$$

$$\lim_{s \downarrow 0} f(s) = \lim_{s \downarrow 0} g(s) = +\infty. \tag{4.6}$$

Assumptions (4.3)–(4.6) imply the validity of (H1)–(H4) when

$$H(u, v) = \int_1^v f(s)ds + \int_1^u g(s)ds,$$

and therefore Theorem 1.2 ensures the existence of at least one $T$-periodic solution to (4.1)-(4.2) for all $d > 0$.

We begin by showing that an extra oscillatory assumption on the nonlinearity implies multiplicity of solutions, at least for all sufficiently large $d$. More precisely, we will prove:

**Theorem 4.1.** *Assume that $f, g$ satisfy* (4.3)–(4.6), *and additionally that there exist numbers $\delta > 0$ and $\delta < \alpha_1 < \alpha_2 < \cdots < \alpha_n$ with the property that*

$$f(\alpha_i - s) < \bar{h} < f(\alpha_i + s) \quad \text{for all } \ 0 < s \leq \delta, \ i = 1, \dots, n. \tag{4.7}$$

*Then, if $d > \bar{d}$, where*

$$\bar{d} = \frac{1}{\delta}(2T \int_0^T |k|),$$

*problem* (4.1)–(4.2) *admits at least $2n+1$ $T$-periodic solutions.*

**Proof.** Let $\alpha, \beta > 0$ be numbers such that $\alpha < u, v < \beta$ for any $T$-periodic solution $(u, v)$ to (4.1)–(4.2) with $d > \bar{d}$. We consider the following open subsets of $X := C^0_T \times C^0_T$.

$$\Gamma_i = \{(u, v) \in X : \alpha < u < \beta, \ |v - \alpha_i| < \delta\}, \quad i = 1, \dots, n,$$

$$\Lambda_i = \{(u, v) \in X : \alpha < u < \beta, \ \alpha < v < \alpha_i + \delta\}, \quad i = 1, \dots, n,$$

$$\Lambda = \{(u, v) \in X : \alpha < u, v < \beta\}.$$

We claim that if $d > \bar{d}$ then (4.1)-(4.2) does not possess any solution $(u, v) \in \partial\Gamma_i \cup \partial\Lambda_i$. Assume that such a solution exists. Then, by definition of $\alpha, \beta$, we must have that

$$\max_{[0,T]} v = \alpha_i + \delta \quad \text{ or } \quad \min_{[0,T]} v = \alpha_i - \delta. \tag{4.8}$$

Since from (4.1) we have $\int_0^T f(v) = \int_0^T h$, it follows from (4.8) and (4.7) that there is a number $\bar{t} \in [0, T]$ such that $v(\bar{t}) = \alpha_i$. On the other hand, since (4.2) holds and $\int_0^T |g(u)| = \int_0^T g(u) = \int_0^T k$, and $v'$ vanishes somewhere in $[0, T]$, it follows that

$$|v'(t)| \leq \frac{2}{d} \int_0^T |k|.$$

Hence,

$$|v(t) - \alpha_i| \leq \frac{2T}{d} \int_0^T |k| < \frac{2T}{\bar{d}} \int_0^T |k| = \delta,$$

and therefore (4.8) is impossible. This proves the validity of the claim.

We shall denote by $T_d$ the compact operator on $X$ obtained when one replaces $\tau$ by $1/d$ in $T_\tau$ given by (3.2). To prove the theorem, we need to show that $T_d$ possesses at least $2n+1$ fixed points provided that $d > \bar{d}$.

From the above claim, we have that the degrees $\deg(I - T_d, \Gamma_i, 0)$ and $\deg(I - T_d, \Lambda_i, 0)$ are well defined for all $d > \bar{d}$. Moreover, the argument leading to (3.8) in the proof of Theorem 3.1 shows that

$$\deg(I - T_d, \Gamma_i, 0) = -\deg(\phi_1, (\alpha_i - \delta, \alpha_i + \delta), 0) \deg(\phi_2, (\alpha, \beta), 0),$$

and

$$\deg(I - T_d, \Lambda_i, 0) = -\deg(\phi_1, (\alpha, \alpha_i + \delta), 0)\,\deg(\phi_2, (\alpha, \beta), 0),$$

where $\phi_1(s) = f(s) - \bar{h}$ and $\phi_2(s) = g(s) - \bar{k}$. Since $\phi_1(\alpha_i - \delta) < 0$ and $\phi_1(\alpha_i + \delta) > 0$, we have that $\deg(\phi_1, (\alpha_i - \delta, \alpha_i + \delta), 0) = 1$. Similarly, we also have $\deg(\phi_2, (\alpha, \beta), 0) = -1$ and $\deg(\phi_1, (\alpha, \alpha_i + \delta), 0) = 0$. Thus, we conclude,

$$\deg(I - T_d, \Gamma_i, 0) = 1, \qquad \deg(I - T_d, \Lambda_i, 0) = 0.$$

Set $\Lambda_0 = \emptyset$ and assume that $1 \leq i \leq n$ . Then the excision and additivity properties of the degree imply

$$\deg(I - T_d, \Lambda_i \setminus (\bar{\Lambda}_{i-1} \cup \bar{\Gamma}_i), 0) = 0 - 0 - 1 = -1.$$

We conclude, therefore, the existence of at least two solutions in $\Lambda_i \setminus \Lambda_{i-1}$ for each $i = 1, \dots, n$. Since, also, using (3.8)

$$\deg(I - T_d, \Lambda \setminus \bar{\Lambda}_n, 0) = (-1) - 0 = -1,$$

we finally obtain the existence of at least $2n+1$ solutions and the theorem is proven. $\square$

We end this section with a uniqueness result for equation (4.1)–(4.2).

**Theorem 4.2.** *Assume that* $f, g$ *satisfy* (4.3)–(4.6), *are continuously differentiable and additionally that there is only one pair* $(\bar{u}, \bar{v})$ *such that* $f(\bar{v}) = \bar{h}$, $g(\bar{u}) = \bar{k}$. *Assume also that for certain numbers* $a, b > 0$, $f$ *and* $g$ *are decreasing, respectively on the intervals* $(\bar{v} - b, \bar{v} + b)$ *and* $(\bar{u} - a, \bar{u} + a)$. *Then* (4.1)–(4.2) *possesses a unique* $T$*-periodic solution for all* $d > d_0$ *where* $d_0$ *can be defined as follows: Set* $B = \sup_{[\bar{v}-b, \bar{v}+b]} |f'|$, $C = \sup_{[\bar{u}-a, \bar{u}+a]} |g'|$. *Then define*

$$d_0 = \max\left\{\frac{2T}{a}\int_0^T |h|\,, \frac{2T}{b}\int_0^T |k|\,, \sqrt{BC}T^2\right\}.$$

**Proof.** Since existence follows from Theorem 1.2, we only need to establish uniqueness. From $\int_0^T (f(v) - h) = 0 = \int_0^T (g(u) - k)$, we see that there are points in $[0, T]$ where $u$ takes the value $\bar{u}$ and $v$ the value $\bar{v}$. Estimating as in the proof of Proposition 2.1 we find that

$$|u(t) - \bar{u}| \leq \frac{2T}{d}\int_0^T |h|, \quad |v(t) - \bar{v}| \leq \frac{2T}{d}\int_0^T |k|.$$

It follows that if $d > d_0$ then

$$(u(t), v(t)) \in (\bar{u} - a, \bar{u} + a) \times (\bar{v} - b, \bar{v} + b) \qquad \text{for all } t \in [0, T].$$

Now, let $(u_1, v_1)$, $(u_2, v_2)$ be two $T$-periodic solutions to (4.1)–(4.2) and assume $d > d_0$. Then

$$d|(v_1 - v_2)'| \leq \int_0^T |g(u_1) - g(u_2)| \leq CT\|u_1 - u_2\|_\infty \tag{4.9}$$

and

$$d|(u_1 - u_2)'| \leq \int_0^T |f(v_1) - f(v_2)| \leq BT\|v_1 - v_2\|_\infty \tag{4.10}$$

Since $f$ and $g$ are decreasing respectively on $(\bar{u} - a, \bar{u} + a)$, $(\bar{v} - a, \bar{v} + a)$ and we have

$$\int_0^T (f(v_1) - f(v_2)) = 0 = \int_0^T (g(u_1) - g(u_2)),$$

then $u_1 - u_2$ and $v_1 - v_2$ vanish somewhere in $[0, T]$. Therefore, from (4.9) and (4.10) we obtain

$$d\|v_1 - v_2\|_\infty \leq CT^2\|u_1 - u_2\|_\infty, \quad d\|u_1 - u_2\|_\infty \leq BT^2\|v_1 - v_2\|_\infty,$$

so that

$$(1 - \frac{BCT^4}{d^2})\|v_1 - v_2\|_\infty \leq 0,$$

and the conclusion of the theorem follows since $d > d_0$. This concludes the proof.

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