NONAUTONOMOUS INTEGRO-DIFFERENTIAL EQUATIONS
OF HYPERBOLIC TYPE

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Abstract. This paper is devoted to two problems for the nonautonomous integrodifferential
equation in a Banach space $X$ of hyperbolic type

$$u'(t) = A(t)u(t) + \int_0^t B(t, s)u(s)ds + f(t) \quad \text{for } t \in [0, T], \text{ and } u(0) = u_0.$$ 

One is the problem of existence and uniqueness of classical solutions without assuming that the
common domain of $A(t)$ is dense in $X$, and the other is the regularity problem in the case where
the common domain of $A(t)$ is dense in $X$. The regularity result will play a role in developing
an abstract theory which can be applied to the second-order integrodifferential equations with
the third kind boundary conditions.

1. Introduction. In this paper we consider the nonautonomous integrodifferential
equation in a Banach space $X$ with norm $\| \cdot \|$,

$$\begin{cases}
  u'(t) = A(t)u(t) + \int_0^t B(t, s)u(s)ds + f(t) & \text{for } t \in [0, T] \\
  u(0) = u_0.
\end{cases}$$

(IE; $u_0, f$)

Here $\{ A(t) : t \in [0, T] \}$ is a given family of closed linear operators in $X$ satisfying all
conditions which are usually referred to as the “hyperbolic” case except for the density
of the common domain of $A(t)$, and $\{ B(t, s) : 0 \leq s \leq t \leq T \}$ is a family of linear
operators in $X$.

There are a number of papers dealing with the hyperbolic case of (IE; $u_0, f$) (see
[1], [3] and [7]). In all of those papers, the density of the common domain of $A(t)$
is required, but there are some cases where such condition is not satisfied. The first
purpose of this paper is to study the problem of existence and uniqueness of classical
solutions to (IE; $u_0, f$) without assuming that the common domain of $A(t)$ is dense in

Received for publication March 1994.
AMS Subject Classifications: 45J05, 47G20.
$X$ (see Theorem 2.1). This object will be attained in the following way: The results in [8] for the inhomogeneous initial value problem

$$
\begin{aligned}
u'(t) &= A(t)u(t) + f(t) \quad \text{for } t \in [0, T] \\
u(0) &= u_0
\end{aligned}
$$

(CP; $u_0, f$)

will enable us to construct a sequence $\{u_n\}$ of approximate solutions to problem (IE; $u_0, f$) by treating the integral term of (IE; $u_0, f$) as a perturbation of problem (CP; $u_0, f$). The convergence of $\{A(\cdot)u_n(\cdot)\}$ in $X$ will be proved by using the integral equation concerning the unique classical solution to problem (CP; $u_0, f$) (see (2.2)).

The second purpose is to obtain the regularity result for problem (IE; $u_0, f$) in the case where the common domain of $A(t)$ is dense in $X$. It plays an important role in developing an abstract theory which is applicable to the second-order integrodifferential equations with the so-called third kind boundary conditions (see Theorem 3.2).

2. Classical solutions to problem (IE; $u_0, f$). This section is devoted to the problem of existence and uniqueness of classical solutions to (IE; $u_0, f$) of “hyperbolic” type. Throughout this section, let $D$ be another Banach space with norm $\| \cdot \|_D$ which is continuously imbedded in $X$ and assume the following three conditions on a family $\{A(t) : t \in [0, T]\}$ of closed linear operators in $X$.

(A1) $D(A(t)) = D$ is independent of $t$, and there exists a $c > 0$ such that

$$c^{-1}\|y\|_D \leq \|y\| + \|A(t)y\| \leq c\|y\|_D$$

for $y \in D$ and $t \in [0, T]$.

(A2) There are constants $M \geq 1$ and $\omega \in (-\infty, \infty)$ such that

$$(\omega, \infty) \subset \rho(A(t)) \quad \text{for } t \in [0, T]$$

and

$$\left\| \prod_{j=1}^{k} (\lambda I - A(t_j))^{-1} \right\| \leq M(\lambda - \omega)^{-k} \quad \text{for } \lambda > \omega$$

and every finite sequence $\{t_j\}_{j=1}^{k}$ with $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq T$ and $k = 1, 2, \ldots$.

(A3) For $y \in D$, $A(t)y$ is continuously differentiable in $X$.

We also impose the following assumption on a family $\{B(t, s) : 0 \leq s \leq t \leq T\}$ of bounded linear operators on $D$ to $X$.

(B1) For $y \in D$, $B(t, s)y$ is continuous on the set $\Delta = \{(t, s) : 0 \leq s \leq t \leq T\}$ and differentiable with respect to $t$, and $(\partial/\partial t)B(t, s)y$ is continuous on $\Delta$ for $y \in D$.

Our main result in this section is given by the following theorem.
Theorem 2.1. Let $f \in W^{1,1}(0, T : X)$ and suppose that $u_0 \in D$ satisfies the compatibility condition that $A(0)u_0 + f(0) \in \overline{D}$. Then, $(IE; u_0, f)$ has a unique classical solution $u \in C([0, T] : D) \cap C^1([0, T] : X)$.

We shall prove this theorem after recalling the results in [8] for problem $(CP; u_0, f)$. By [8, Theorem 4.2] we obtain the following existence and uniqueness result of classical solutions to the problem $(CP; u_0, f)$, since the norm $\| \cdot \|_D$ is equivalent to the norm $\| \cdot \|_Y$ defined by $\|y\|_Y = \|y\| + \|A(0)y\|$ for $y \in D$, by condition $(A_1)$.

Theorem 2.2. Let $f \in W^{1,1}(0, T : X)$ and suppose that $u_0 \in D$ satisfies the compatibility condition that $A(0)u_0 + f(0) \in \overline{D}$. Then, $(CP; u_0, f)$ has a unique classical solution $u \in C([0, T] : D) \cap C^1([0, T] : X)$ given by

$$u(t) = \lim_{\lambda \to 0+} \left( U_\lambda(t, 0)u_0 + \int_0^{[t/\lambda]} U_\lambda(t, r)f(r)dr \right)$$

for $t \in [0, T]$, where $U_\lambda(t, s) = \prod_{i=[s/\lambda]+1}^{[t/\lambda]} (I - \lambda A(i\lambda))^{-1}$ for $0 \leq s \leq t \leq T$.

Remark. Combining (4.2) and (4.5) in [8], we see that the unique classical solution $u$ to $(CP; u_0, f)$ satisfies the integral equation

$$(A(t) - \lambda_0)u(t) + f(t) = U(t, 0)((A(0) - \lambda_0)u_0 + f(0))$$

$$+ \lim_{\lambda \to 0+} \int_0^{[t/\lambda]} U_\lambda(t, r)(\dot{A}(r)u(r) + f'(r) - \lambda_0 f(r))dr$$

for $t \in [0, T]$, where $\lambda_0 > \omega$ and $U(t, 0)z = \lim_{\lambda \to 0+} U_\lambda(t, 0)z$ for $z \in \overline{D}$ and $t \in [0, T]$.

Proof of Theorem 2.1. By condition $(B_1)$ we can define a mapping $F : C([0, T] : D) \cap C^1([0, T] : X) \to W^{1,1}(0, T : X)$ by

$$(Fu)(t) = \int_0^t B(t, s)u(s)ds + f(t)$$

for $t \in [0, T]$. Set $u_0(t) = u_0$ on $[0, T]$. By the compatibility condition, Theorem 2.2 enables us to define $u_n \in C([0, T] : D) \cap C^1([0, T] : X)$ inductively by the unique classical solution to the problem

$$\begin{cases} u'_n(t) = A(t)u_n(t) + (Fu_{n-1})(t) & \text{for } t \in [0, T] \\ u_n(0) = u_0. \end{cases}$$

Since $(Fu)(0) = f(0)$ for $u \in C([0, T] : D) \cap C^1([0, T] : X)$, we see by (2.2) that each $u_n$ satisfies the integral equation

$$(A(t) - \lambda_0)u_n(t) + (Fu_{n-1})(t) = U(t, 0)((A(0) - \lambda_0)u_0 + f(0))$$

$$+ \lim_{\lambda \to 0+} \int_0^{[t/\lambda]} U_\lambda(t, s)(\dot{A}(s)u_n(s) + (d/ds)(Fu_{n-1})(s) - \lambda_0(Fu_{n-1})(s))ds$$
for $t \in [0, T]$. Since
\[
(d/dt)(Fu)(t) = B(t,t)u(t) + \int_0^t \frac{\partial}{\partial t} B(t,s)u(s)ds + f'(t)
\]
for almost every $t \in (0, T)$, it follows from the principle of uniform boundedness together with condition (B₁) that there exists a $K > 0$ such that
\[
\|u_{n+1}(t) - u_n(t)\|_D \leq K \int_0^t (\|u_{n+1}(s) - u_n(s)\|_D + \|u_n(s) - u_{n-1}(s)\|_D)ds
\]
for $t \in [0, T]$ and $n \geq 1$. Here we have used the fact that there exists an $M' > 0$ such that $\|y\|_D \leq M'\|(A(t) - \lambda_0)y\|$ for $y \in D$ and $t \in [0, T]$ which follows from condition (A₁). Putting $\phi_n(t) = \|u_n(t) - u_{n-1}(t)\|_D$, we have by Gronwall’s inequality that
\[
\phi_{n+1}(t) \leq K \int_0^t e^{K(t-s)}\phi_n(s)ds
\]
for $t \in [0, T]$ and $n \geq 1$. A standard argument shows that the series $\sum_{n=1}^{\infty} \phi_n(t)$ converges uniformly on $[0, T]$, which implies that there exists an element $u \in C([0, T] : D)$ such that $\sup\{\|u_n(t) - u(t)\|_D : t \in [0, T]\}$ converges to zero as $n \to \infty$. It then follows that $u'_n(t) (= A(t)u_n(t) + (Fu_{n-1})(t))$ converges to $A(t)u(t) + (Fu)(t)$ uniformly on $[0, T]$ as $n \to \infty$, and so we see that $u$ is a classical solution to (IE; $u_0, f$).

Finally we shall prove the uniqueness of classical solutions to (IE; $u_0, f$). To this end, let $u_i$ ($i = 1, 2$) be classical solutions to (IE; $u_0, f$) and let $w = u_1 - u_2$. Then, since $w$ is a classical solution to problem (CP; 0, $(Fu_1)(\cdot) - (Fu_2)(\cdot)$), we have by (2.2) the integral equation
\[
(A(t) - \lambda_0)w(t) + \int_0^t B(t,s)w(s)ds
\]
\[
= \lim_{\lambda \to 0^+} \int_0^{[t/\lambda]} U_\lambda(t,s)\Big(\dot{A}(s)w(s) + B(s,s)w(s)\Big) + \int_0^s \frac{\partial}{\partial s} B(s,r)w(r)dr - \lambda_0 \int_0^s B(s,r)w(r)dr\Big)ds
\]
for $t \in [0, T]$. Estimating this equality we find a $K' > 0$ such that
\[
\|w(t)\|_D \leq K' \int_0^t \|w(s)\|_D ds
\]
for $t \in [0, T]$, which implies $w = 0$ by Gronwall’s inequality. □

We shall deal with an application of Theorem 2.1 to the following first-order hyperbolic integrodifferential equation in one space variable:
\[
\begin{cases}
    u_t(t, x) + a(t, x)u_x(t, x) = \int_0^t b(t, s, x)u_x(s, x)ds + f(t, x) & \text{for } (t, x) \in [0, T] \times [0, 1], \\
    u(t, 0) = u(t, 1) & \text{for } t \in [0, T], \\
    u(0, x) = u_0(x) & \text{for } x \in [0, 1].
\end{cases}
\]
(2.3)
We denote by $X$ the Banach space $C[0,1]$ with norm $\|u\|_\infty = \sup\{|u(x)| : x \in [0,1]\}$ and by $D$ the Banach space $\{ u \in C^1[0,1] : u(0) = u(1) \}$ with norm $\|u\|_\infty + \|u'\|_\infty$. Define two families $\{A(t) : t \in [0,T]\}$ and $\{B(t,s) : (t,s) \in \Delta\}$ of linear operators in $X$ by

\[
\begin{align*}
D(A(t)) &= D \\
(A(t)u)(x) &= -a(t,x)u'(x) \text{ for } u \in D,
\end{align*}
\]
and

\[
\begin{align*}
D(B(t,s)) &= D \\
(B(t,s)u)(x) &= b(t,s,x)u'(x) \text{ for } u \in D \text{ and } (t,s) \in \Delta,
\end{align*}
\]
respectively. If $a$ is a positive function of class $C^1$, then the family $\{A(t) : t \in [0,T]\}$ of closed linear operators satisfies three conditions (A₁) through (A₃) (see [2, Theorem 6.1]). Condition (B₁) is satisfied if $b$ is of class $C^1$. A simple computation gives $D = \{ u \in C[0,1] : u(0) = u(1) \}$. Therefore, Theorem 2.1 asserts that if $f \in W^{1,1}(0,T : C[0,1])$ and $u_0 \in C^1[0,1]$ satisfies the condition that $u_0(0) = u_0(1)$ and $-\alpha(0,0)u_0(0) + f(0,0) = -\alpha(0,1)u_0(1) + f(0,1)$, then problem (2.3) has a unique solution $u \in C([0,T] : C^1[0,1] \cap C^1([0,T] : C[0,1])$.

We shall conclude this section with a result on the continuous dependence of classical solutions of $(IE;u_0,f)$ with respect to the initial data $u_0$ and the forcing terms $f$.

**Theorem 2.3.** Suppose that for $y \in D$, $B(t,s)y$ is differentiable with respect to $s$ and $(\partial/\partial s)B(t,s)y$ is continuous on $\Delta$ for $y \in D$, in addition to condition (B₁). Then, the classical solution $u$ to $(IE;u_0,f)$ satisfies the estimate

\[
\|u(t)\| \leq C\left(\|u_0\| + \int_0^t \|f(s)\|ds\right)
\]
for $t \in [0,T]$, where $C$ is some constant not depending on $u_0$ nor $f$.

**Proof.** Let $\lambda_0 > \omega$. Putting

\[
K(t,s) = B(t,s)(A(s) - \lambda_0)^{-1}
\]
for $(t,s) \in \Delta$, we see that $K(t,s) \in B(X)$ for $(t,s) \in \Delta$ and $K(t,s)x$ is continuous on $\Delta$ for $x \in X$, where $B(X)$ is the set of all bounded linear operators on $X$ to $X$. It is then known that a resolvent kernel $R(\cdot,\cdot)$ of $K(\cdot,\cdot)$ exists, namely the following equation holds:

\[
R(t,s)x = K(t,s)x - \int_s^t R(t,r)K(r,s)xdr
\]
for $x \in X$ and $(t,s) \in \Delta$. Since $K(t,s)x$ is differentiable with respect to $s$ and $(\partial/\partial s)K(t,s)x$ is continuous on $\Delta$ for $x \in X$, so is $R(t,s)x$ for $x \in X$.

Now, we denote by $u$ the classical solution to $(IE;u_0,f)$. By (2.5) and (2.6) we have

\[
B \ast u = (K - R \ast K) \ast Au + R \ast KA \ast u - \lambda_0 K \ast u
\]
\[
= R \ast Au + R \ast (\lambda_0 K + B) \ast u - \lambda_0 K \ast u
\]
\[
= R \ast (Au + B \ast u) + \lambda_0 (R \ast K - K) \ast u = R \ast u' - \lambda_0 R \ast u - R \ast f.
\]
Here we have used the following notion for convenience: if $V$ and $W$ are $B(X)$-valued functions on $\Delta$ and $h$ is an $X$-valued function on $[0,T]$, then we define

$$(V*W)(t,s)x = \int_s^t V(t,r)W(r,s)x\,dr \text{ for } x \in X \text{ and } (t,s) \in \Delta$$

and

$$(V*h)(t) = \int_0^t V(t,s)h(s)\,ds \text{ for } t \in [0,T].$$

Integration by parts yields

$$(R*u')(t) = R(t,t)u(t) - R(t,0)u_0 - \int_0^t \frac{\partial}{\partial s}R(t,s)u(s)\,ds$$

for $t \in [0,T]$. By this and (2.7), $B*u$ is represented in terms of $R$, $u$ and $f$. Rewriting the equation $u' = Au + B*u + f$ by this representation of $B*u$, and then applying (2.1) to the resulting equation, we find the integral equation

$$u(t) = U(t,0)u_0 + \lim_{\lambda \to 0^+} \int_0^{[t/\lambda]} U_\lambda(t,s) \left( R(s,s)u(s) - R(s,0)u_0 \right)$$

$$- \int_0^s \frac{\partial}{\partial r}R(s,r)u(r)\,dr - \lambda_0 \int_0^s R(s,r)u(r)\,dr - \int_0^s R(s,r)f(r)\,dr + f(s) \right) ds$$

for $t \in [0,T]$. The desired estimate (2.4) is obtained by Gronwall’s inequality.

3. Applications to second-order integrodifferential equations. In this section we shall give an abstract theory which can be applied to the second-order integrodifferential equations with the third kind boundary conditions. For this purpose, we start with the regularity result for problem (IE; $u_0, f$).

Theorem 3.1. Suppose that $D$ is dense in $X$ and two families $\{A(t) : t \in [0,T]\}$ and $\{B(t,s) : (t,s) \in \Delta\}$ satisfy the following conditions in addition to conditions (A$_1$), (A$_2$) and (B$_1$):

(A\textsubscript{3}') for $y \in D$, $A(t)y$ is twice continuously differentiable in $X$;
(B$_2$) there exists a $\delta_0 > 0$ such that $B(t,s)(\delta_0 I - A(s))^{-1}(D) \subset D$ for $(t,s) \in \Delta$;
(B$_3$) for $y \in D$, $B(t,s)(\delta_0 I - A(s))^{-1}y$ is continuous in $D$ on $\Delta$ and differentiable in $D$ with respect to $t$, and $(\partial/\partial t)B(t,s)(\delta_0 I - A(s))^{-1}y$ is continuous in $D$ on $\Delta$ for $y \in D$.

If $u_0 \in D(A(0)^2)$ and $f \in W^{1,1}(0,T : D)$ then the unique classical solution $u$ to problem (IE; $u_0, f$) satisfies $A(t)u(t) \in D$ and $A(\cdot)u(\cdot) \in C([0,T] : D)$.

Proof. Set $\hat{A}(t) = A(t) - \hat{A}(t)(\delta_0 I - A(t))^{-1}$ for $t \in [0,T]$ and $\hat{B}(t,s) = (\delta_0 I - A(t))B(t,s)(\delta_0 I - A(s))^{-1}$ for $(t,s) \in \Delta$. Clearly, $\hat{A}(t)$ and $\hat{B}(t,s)$ are closed linear operators in $X$, and the closed graph theorem implies that $\hat{B}(t,s)$ is a bounded linear operator on $D$ to $X$, by condition (B$_2$). From conditions (A\textsubscript{3}') and (B$_3$) it follows that
\( \hat{A}(t) \) and \( \hat{B}(t, s) \) satisfy conditions \((A_1)\) through \((A_3)\) and \((B_1)\) with \( A(t) \) and \( B(t, s) \) replaced by \( \hat{A}(t) \) and \( \hat{B}(t, s) \), respectively. Here we have used the well known fact that the stability condition is preserved under the perturbation of a uniformly bounded family of bounded linear operators on \( X \) (see [4] or \([6, \text{Theorem 5.2.3}]\)). Therefore, Theorem 2.1 asserts that the problem

\[
\begin{align*}
\dot{u}'(t) &= \hat{A}(t)\dot{u}(t) + \int_0^t \hat{B}(t, s)\dot{u}(s)\,ds + (\delta_0I - A(t))f(t) \quad \text{for } t \in [0, T] \\
\dot{u}(0) &= (\delta_0I - A(0))u_0
\end{align*}
\]

has a unique classical solution \( \dot{u} \in C([0, T] : D) \cap C^1([0, T] : X) \). Setting \( v(t) = (\delta_0I - A(t))^{-1}\dot{u}(t) \), we see by simple computation that \( v \) is a classical solution to problem \((\text{IE}; u_0, f)\), and so \( v(t) = u(t) \) by the uniqueness of classical solutions to \((\text{IE}; u_0, f)\). This fact therefore proves that \( A(t)u(t) \in D \) and \( A(\cdot)u(\cdot) \in C([0, T] : D) \). \( \square \)

We now consider the second-order integrodifferential equation

\[
u''(t) + A(t)u(t) + \int_0^t B(t, s)u(s)\,ds = g(t) \quad \text{for } t \in [0, T].
\]

On the basis of Kato’s idea [5, Chap. II], we convert this second-order problem into the first-order system

\[
\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}' = A(t)\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} + \int_0^t B(t, s)\begin{pmatrix} u(s) \\ \dot{u}(s) \end{pmatrix}ds + f(t) \quad \text{for } t \in [0, T],
\]

where

\[
A(t) = \begin{pmatrix} 0 & 1 \\ -A(t) & 0 \end{pmatrix}, \quad B(t, s) = \begin{pmatrix} 0 & 0 \\ -B(t, s) & 0 \end{pmatrix} \quad \text{and} \quad f(t) = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}.
\] (3.1)

Here \( \dot{u} \) is just a symbol for an unknown function.

In order to apply Theorem 3.1, we shall express the conditions for \( \{A(t) : t \in [0, T]\} \) and \( \{B(t, s) : (t, s) \in \Delta\} \) in terms of \( \{A(t) : t \in [0, T]\} \) and \( \{B(t, s) : (t, s) \in \Delta\} \).

Let \( V \subset H \subset V' \) be a triplet of Banach spaces with inclusions being continuous and dense.

(i) \( \{A(t) : t \in [0, T]\} \) is a family of closed linear operators in \( V' \) such that \( \left( \begin{smallmatrix} 0 & 1 \\ -A(t) & 0 \end{smallmatrix} \right) \)

forms a stable family in \( \left( \begin{smallmatrix} H \\ V \end{smallmatrix} \right) \) with stability constants \( M \) and \( \omega \).

(ii) \( D(A(t)) = V \) is independent of \( t \), and there exists a \( c_0 > 0 \) such that

\[
c_0^{-1}\|v\|_V \leq \|v\|_{V'} + \|A(t)v\|_{V'} \leq c_0\|v\|_V
\]

for \( v \in V \) and \( t \in [0, T] \).

(iii) For \( v \in V \), \( A(t)v \) is twice continuously differentiable in \( V' \).

(iv) \( \{B(t, s) : (t, s) \in \Delta\} \) is a family of bounded linear operators on \( V \) to \( V' \).
(v) For $v \in V$, $B(t, s)v$ is continuous in $V'$ on $\Delta$ and differentiable in $V'$ with respect to $t$, and $(\partial/\partial t)B(t, s)v$ is continuous in $V'$ on $\Delta$ for $v \in V$.

(vi) $D(A_0(s)) \subset D(B_0(t, s))$ for $(t, s) \in \Delta$, where $A_0(s)$ and $B_0(t, s)$ are the parts of $A(t)$ and $B(t, s)$ in $H$, respectively.

(vii) There exists $\delta_0 > \omega$ such that for $u \in H$, $B_0(t, s)(\delta_0^2 I + A_0(s))^{-1}u$ is continuous in $H$ on $\Delta$ and differentiable in $H$ with respect to $t$, and $(\partial/\partial t)B(t, s)(\delta_0^2 I + A_0(s))^{-1}u$ is continuous in $H$ on $\Delta$ for $u \in H$.

By a simple translation it is seen that under these conditions, the families $\{A(t) : t \in \Delta\}$ and $\{B(t, s) : (t, s) \in \Delta\}$ given by (3.1) satisfy all conditions in Theorem 3.1.

In fact, conditions (B2) and (B3) follow from conditions (vi) and (vii), respectively. Therefore, Theorem 3.1 is immediately translated into

**Theorem 3.2.** Suppose that two families $\{A(t) : t \in \Delta\}$ and $\{B(t, s) : (t, s) \in \Delta\}$ satisfy conditions (i) through (vii). If $\phi \in D(A_0(0))$, $\hat{\phi} \in V$ and $g \in W^{1,1}(0, T ; H)$, then the second-order integrodifferential equation in $H$,

$$
\begin{align*}
\begin{cases}
  u''(t) + A_0(t)u(t) + \int_0^t B_0(t, s)u(s)ds = g(t) & \text{for } t \in [0, T], \\
  u(0) = \phi, & u'(0) = \hat{\phi},
\end{cases}
\end{align*}
$$

(3.2)

has a unique solution $u$ such that $u$ is continuously differentiable in $V$ and twice continuously differentiable in $H$ on $[0, T]$, $u(t) \in D(A_0(t))$ for $t \in [0, T]$ and $A_0(\cdot)u(\cdot)$ is continuous in $H$ on $[0, T]$, and (3.2) is satisfied in $H$.

Let us consider the mixed problem for the second-order integrodifferential equation of hyperbolic type,

$$
\begin{align*}
\begin{cases}
  u_{tt}(t, x) &= (a(t, x)u_x(t, x))_x + \int_0^t (b(t, s, x)u_x(s, x))_xds + g(t, x) \\
  u_x(t, 0) - \alpha(t)u(t, 0) &= u_x(t, 1) + \beta(t)u(t, 1) = 0 & \text{for } (t, x) \in [0, T] \times [0, 1], \\
  u(0, x) &= \phi(x), & u_t(0, x) = \hat{\phi}(x) & \text{for } x \in [0, 1].
\end{cases}
\end{align*}
$$

(3.3)

Let $H = L^2(0, 1)$ and $V = H^1(0, 1)$ with the duality $\langle \cdot, \cdot \rangle$ between $V' = H^1(0, 1)^*$ and $V$. We then introduce two families $\{a[t; u, v] : t \in [0, T]\}$ and $\{b[t, s; u, v] : (t, s) \in \Delta\}$ of continuous bilinear forms on $V$ defined by

$$
a[t; u, v] = a(t, 1)\beta(t)u(1)v(1) + a(t, 0)\alpha(t)u(0)v(0) + \int_0^1 a(t, x)u_x(x)v_x(x)dx
$$

and

$$
b[t, s; u, v] = b(t, s, 1)\beta(s)u(1)v(1) + b(t, s, 0)\alpha(s)u(0)v(0) + \int_0^1 b(t, s, x)u_x(x)v_x(x)dx
$$
for $u, v \in V$, respectively. Assume that $a$ is a positive function of class $C^2$, and

$\alpha$ and $\beta$ are nonnegative functions of class $C^2$. Then, it is seen that the family

$\{A(t) : t \in [0, T]\}$, defined by $\langle A(t)u, v \rangle = a[t; u, v]$ for $u, v \in V$, satisfies conditions (i)

through (iii) (cf. [5, pp. 50-51]). If $b$ is of class $C^1$ then the family $\{B(t, s) : (t, s) \in \Delta\}$,

defined by $\langle B(t, s)u, v \rangle = b[t, s; u, v]$ for $u, v \in V$, satisfies conditions (iv) and (v). Let

$A_0(t)$ and $B_0(t, s)$ be the parts of $A(t)$ and $B(t, s)$ in $H$, respectively. Then, we have

$$
\begin{align*}
(A_0(t)u)(x) & = -(a(t, x)u_x(x))_x \quad \text{for} \quad u \in D(A_0(t)), \\
(D(A_0(t)) & = \{u \in H^2(0, 1) : u_x(0) - \alpha(t)u(0) = u_x(1) + \beta(t)u(1) = 0\}
\end{align*}
$$

and

$$
\begin{align*}
(B_0(t, s)u)(x) & = -(b(t, s, x)u_x(x))_x \quad \text{for} \quad u \in D(A_0(s)) \subset D(B_0(t, s)).
\end{align*}
$$

Let $\delta_0 = (\min\{a(t, x) : (t, x) \in [0, T] \times [0, 1]\})^{1/2}$. Applying the fact that $(\delta_0^2I + A_0(s))^{-1}u \in C([0, T] : V)$ for $u \in H (\subset V')$ to the identity

$$
a(s, x)((\delta_0^2I + A_0(s))^{-1}u)_{xx} = -a_x(s, x)((\delta_0^2I + A_0(s))^{-1}u)_x + \delta_0^2((\delta_0^2I + A_0(s))^{-1}u) - u
$$

for $u \in H$ and $(s, x) \in [0, T] \times [0, 1]$, we have $(\delta_0^2I + A_0(s))^{-1}u \in C([0, T] : H^2(0, 1))$

for $u \in H$, since $a$ is positive and of class $C^2$. This fact implies that condition

(vii) is satisfied if $b$ is of class $C^2$. Therefore, by Theorem 3.2 we see that if $g \in W^{1, 1}(0, T; L^2(0, 1))$ and $\phi \in H^2(0, 1)$ satisfies the condition $\phi'(0) - \alpha(0)\phi(0) = \phi'(1) +

\beta(0)\phi(1) = 0$ and $\phi \in H^1(0, 1)$, then problem (3.3) has a unique solution $u$ in the class

$$
u \in C([0, T] : H^2(0, 1)) \cap C^1([0, T] : H^1(0, 1)) \cap C^2([0, T] : L^2(0, 1)).
$$

REFERENCES


